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Real hypersurfaces in complex two-plane Grassmannians related to the Ricci curvature

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Abstract. In this paper we introduce a new notion of the Ricci tensor derived from the curvature tensor of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Moreover, we give a characterization of real hypersurfaces of type A in $G_2(\mathbb{C}^{m+2})$, that is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in terms of integral formulas related to the Ricci curvature $\mathrm{Ric}(\xi,\xi)$ along the direction of the structure vector field ξ for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

0. Introduction

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms there have been many characterizations of model hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $\mathbb{C}P^m$, of type A_0, A_1, A_2 and B in complex hyperbolic space $\mathbb{C}H^m$ or A_1, A_2, B in quaternionic projective space $\mathbb{Q}P^m$, which are completely classified by Cecil and Ryan [4], Kimura [5], Berndt [1], Martinez and Pérez [6] respectively.

Among them there were some characterizations of homogeneous real hypersurfaces of type A_1, A_2 in complex projective space $\mathbb{C}P^m$ and of type A_0, A_1, A_2 in complex hyperbolic space $\mathbb{C}H^m$. As an example, we say

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that the shape operator A and the structure tensor ϕ commute with each other, that is $A\phi - \phi A = 0$, is a model characterization of hypersurfaces, which are tubes over a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^m$ (See Okumura [8]), a tube over a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^m$ or a horosphere in $\mathbb{C}H^m$ (See Montiel and Romero [7]).

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces M that $[\xi] = \text{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$, which are spanned by *almost contact* 3-*structure* vector fileds $\{\xi_1, \xi_2, \xi_3\}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, are invariant under the shape operator A of M (See [2] and [3]).

The almost contact structure vector field ξ mentioned above is defined by $\xi = -JN$, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ and the almost contact 3-structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, where $\{J_\nu\}$ denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$.

The first result in this direction is the classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying both conditions. Namely, Berndt and the first author [2] have proved the following

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In the paper [3] due to Berndt and the first author we have given a characterization of real hypersurfaces of type (A) in Theorem A when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor

ϕ. This is equivalent to the condition that the Reeb flow on M is isometric, that is $\mathcal{L}_\xi g = 0$, where $\mathcal{L}$(resp. g) denotes the Lie derivative(resp. the induced Riemannian metric) of M in the direction of the Reeb vector field ξ as follows:

Theorem B. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Now the purpose of this paper is to show non-existence properties related to the Ricci curvature along the direction of the structure vector ξ of a compact real hypersurface in $G_2(\mathbb{C}^{m+2})$. In order to do this we recall some integral formulas due to Watanabe [14] (See also Yano [15]) on a compact Riemannian manifold and give some relations between the Ricci curvature and the covariant derivative for the structure vector field ξ of a real hypersurface in $G_2(\mathbb{C}^{m+2})$ as follows:

$$\int_M \{\mathrm{Ric}(\xi,\xi) + \|\nabla\xi\|^2\} * 1 = 0$$

and

$$\int_M \Big\{\mathrm{Ric}(\xi,\xi) + \frac{1}{2}\|\mathcal{L}_\xi g\|^2 - \|\nabla\xi\|^2 - (\mathrm{div}\ \xi)^2\Big\} * 1 = 0.$$

By virtue of these formulas we are able to assert the following theorems respectively:

Theorem 1. *There does not exist any compact real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq3$, satisfying Ric $(\xi,\xi)\geq 0$ and*

$$TrA^2 \leq 4\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)^2 + 2\|A\xi\|^2 - TrA\ g(A\xi,\xi) - 4(m+1).$$

Theorem 2. *There does not exist any compact real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq3$, satisfying Ric$(\xi,\xi)\leq 0$ and*

$$TrA^2 \leq 4(m+1) - 4\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)^2 + TrA\ g(A\xi,\xi).$$

In this paper we also give a characterization of real hypersurfces of type A in $G_2(\mathbb{C}^{m+2})$ by the second integral formula mentioned above. Then if we use the expression of the shape operator A of a compact real hypersurface M in $G_2(\mathbb{C}^{m+2})$, we assert the following:

Theorem 3. *Let M be a compact real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$. If it satisfies*

$$\int_M \left\{4(m+1) - 4\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)^2 + TrAg(A\xi,\xi) - TrA^2\right\} * 1\geq 0.$$

Then M is congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2] and [3]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2)\times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m}\times\mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the three-dimensional complex projective space $\mathbb{C}P^3$ with constant holomorphic sectional curvature eight we will assume $m\geq 2$ from now on. Note that the isomorphism $Spin(6)\simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m)\oplus\mathfrak{su}(2)\oplus\mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$. This fact will be used in Section 2.

On the other hand, we introduce the Riemannian curvature tensor of $G_2(\mathbb{C}^{m+2})$ defined in such a way that

$$\begin{aligned}\bar{R}(X,Y)Z =& g(Y,Z)X - g(X,Z)Y \\ &+ g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &+ \sum_{\nu=1}^{3} \{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y \\ &- 2g(J_\nu X,Y)J_\nu Z\} \\ &+ \sum_{\nu=1}^{3} \{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\} ,\end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 denotes a canonical local basis of $\mathfrak{J}$ (See [2]).

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulas from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$.

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression (1.2) for $\bar{R}$, the Gauss and the Codazzi equations are respectively given by

$$\begin{aligned}
R(X,Y)Z =& g(Y,Z)X - g(X,Z)Y \\
&+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\
&+ \sum\nolimits_{\nu=1}^{3} \big\{ g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y \\
&\quad - 2g(\phi_\nu X,Y)\phi_\nu Z \big\} \\
&+ \sum\nolimits_{\nu=1}^{3} \big\{ g(\phi_\nu \phi Y,Z)\phi_\nu \phi X - g(\phi_\nu \phi X,Z)\phi_\nu \phi Y \big\} \\
&- \sum\nolimits_{\nu=1}^{3} \big\{ \eta(Y)\eta_\nu(Z)\phi_\nu \phi X - \eta(X)\eta_\nu(Z)\phi_\nu \phi Y \big\} \\
&- \sum\nolimits_{\nu=1}^{3} \big\{ \eta(X) g(\phi_\nu \phi Y,Z) - \eta(Y) g(\phi_\nu \phi X,Z) \big\} \xi_\nu \\
&+ g(AY,Z)AX - g(AX,Z)AY
\end{aligned}$$

and

$$\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X =& \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\
&+ \sum_{\nu=1}^{3} \big\{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X \\
&- 2g(\phi_\nu X,Y)\xi_\nu \big\} \\
&+ \sum_{\nu=1}^{3} \big\{ \eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X \big\} \\
&+ \sum_{\nu=1}^{3} \big\{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X) \big\} \xi_\nu \ ,
\end{aligned}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2}, \\
&\phi \xi_\nu = \phi_\nu \xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\
&\phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu, \\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}.
\end{aligned} \tag{2.1}$$

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.3}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y = &- q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX,Y)\xi_\nu.\end{aligned} \tag{2.4}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi) =&\nabla_X(\phi\xi_\nu) \\ =&(\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ =&q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &- g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{2.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.6}$$

3. Proof of the main theorem

Now let us contract Y and Z in the equation of Gauss in Section 2. Then the Ricci tensor S of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned}SX =&\sum_{i=1}^{4m-1} R(X,e_i)e_i \\ =&(4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &+ \sum_{\nu=1}^{3}\{(\mathrm{Tr}\phi_\nu\phi)\phi_\nu\phi X - (\phi_\nu\phi)^2X\} \\ &- \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(X)\phi_\nu\phi\xi_\nu\} \\ &- \sum_{\nu=1}^{3}\{(\mathrm{Tr}\ \phi_\nu\phi)\eta(X) - \eta(\phi_\nu\phi X)\}\xi_\nu \\ &+ hAX - A^2X,\end{aligned} \tag{3.1}$$

where h denotes the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. From the formula $JJ_\nu = J_\nu J$, Tr $JJ_\nu = 0$, $\nu = 1,2,3$, we calculate the following for any basis $\{e_1, \cdots, e_{4m-1}, N\}$ of the tangent space of $G_2(\mathbb{C}^{m+2})$

$$
\begin{aligned}
0 =& \text{Tr } JJ_\nu = \sum_{k=1}^{4m} g(JJ_\nu e_k, e_k) \\
=& \sum_{k=1}^{4m} g(JJ_\nu e_k, e_k) + g(JJ_\nu N, N) \\
=& \text{Tr } \phi\phi_\nu - \eta_\nu(\xi) - g(J_\nu N, JN) \\
=& \text{Tr } \phi\phi_\nu - 2\eta_\nu(\xi)
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
(\phi_\nu\phi)^2 X =& \phi_\nu\phi(\phi\phi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu) \\
=& \phi_\nu(-\phi_\nu X + \eta(\phi_\nu X)\xi) + \eta(X){\phi_\nu}^2\xi \\
=& X - \eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi + \eta(X)\{-\xi + \eta_\nu(\xi)\xi\}.
\end{aligned} \tag{3.3}
$$

Substituting (3.2) and (3.3) into (3.1), we have

$$
\begin{aligned}
SX =& (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&+ \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi\} \\
&+ hAX - A^2X \\
=& (4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\
&+ \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi\} \\
&+ hAX - A^2X.
\end{aligned} \tag{3.4}
$$

From this, substituting $X = \xi$, we have

$$
S\xi = 4(m+1)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi.
$$

Then the Ricci curvature $\text{Ric}(\xi,\xi)$ along the direction ξ is given by

$$
\begin{aligned}
\text{Ric}(\xi,\xi) =& g(S\xi,\xi) = 4(m+1) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 \\
&+ hg(A\xi,\xi) - g(A^2\xi,\xi).
\end{aligned} \tag{3.5}
$$

Now we want to introduce an integral formula due to Watanabe [14] as follows:

Theorem A. *Let M be a compact Riemannian manifold. Then for any vector field X defined on M we have*

$$\int_M (Ric(X,X) + \|\nabla X\|^2) * 1 \geq 0,$$

where $Ric(X,X)$ denotes the Ricci curvature along the direction of the vector X. Then the equality holds if and only if X is a harmonic vector field.

By applying Theorem A to the structure vector ξ of a compact real hypersurface M in $G_2(\mathbb{C}^{m+2})$ we know that

$$\begin{aligned}&\int_M (\text{Ric } (\xi,\xi) + \|\nabla \xi\|^2) * 1\\ &= \int_M \{4(m+1) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)^2 + \text{Tr}A\ g(A\xi,\xi) - g(A^2\xi,\xi)\\ &\quad + \text{Tr } A^2 - g(A\xi, A\xi)\} * 1 \geq 0.\end{aligned}$$

From this we know that if the trace of the shape operator A^2 satisfies

$$\text{Tr}A^2 \leq 4\sum_{\nu=1}^{3} \eta_\nu(\xi)^2 + 2\|A\xi\|^2 - \text{Tr}A\ g(A\xi,\xi) - 4(m+1), \tag{3.6}$$

then the equality holds and the structure vector ξ is a harmonic vector field.

Now by Theorem A on a compact real hypersurface in $G_2(\mathbb{C}^{m+2})$ we have the following:

Proposition 3.1. *Let M be a compact real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with the formula* (3.6)*. If the Ricci curvature satisfies $Ric(\xi,\xi) \geq 0$, then ξ is a harmonic vector field and has vanishing covariant derivative. Moreover, if the Ricci curvature is positive definite, then a harmonic vector field other than zero does not exist in M.*

By the assumption of Proposition 3.1 we know that $\text{Ric}(\xi,\xi) = 0$ and $\nabla\xi = 0$ when the Ricci curvature satisfy $\text{Ric}(\xi,\xi) \geq 0$. The latter part implies

$$AX = \eta(AX)\xi$$

for any tangent vector field X on M, that is, M is a totally η-umbilical real hypersurface in $G_2(\mathbb{C}^{m+2})$. From this we know that the structure vector

ξ is principal, that is, $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$ and the trace h of the shape operator is given by

$$\begin{aligned} h =& \mathrm{Tr}\ A = \sum_{i=1}^{4m-1} g(Ae_i, e_i) \\ =& \sum_{i=1}^{4m-1} g(\eta(Ae_i)\xi, e_i) = \eta(A\xi) = \alpha. \end{aligned}$$

From this, together with (3.5),we have

$$\mathrm{Ric}(\xi,\xi) = 4(m+1) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)^2.$$

Then on such a compact real hypersurface M in $G_2(\mathbb{C}^{m+2})$ the Ricci curvature $\mathrm{Ric}(\xi,\xi) = 0$ implies

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)^2 = m+1. \tag{3.7}$$

Now let us denote by $\mathfrak{D}$ the orthogonal complement of $\mathfrak{D}^\perp = \mathrm{Span}\ \{\xi_1, \xi_2, \xi_3\}$ in the tangent space T_xM, $x \in M$ of M in
$G_2(\mathbb{C}^{m+2})$, which can be decomposed in such a way that

$$T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp.$$

Then we are able to consider the following cases:

Case 1: $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$.

Then (3.7) gives a contradiction such that $m+1 = 0$ for $\xi \in \mathfrak{D}$. For the case $\xi \in \mathfrak{D}^\perp$ we may put $\xi = \xi_1$. Then (3.7) implies $m = 0$, which makes a contradiction. So this case also can not be appeared.

Case 2: $\xi \in T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$.

Then in this case we know that

$$\eta_\nu(\xi) = \|\xi\| \|\xi_\nu\| \cos\theta_\nu = \cos\theta_\nu \leq 1.$$

This implies

$$m+1 = \sum_{\nu=1}^{3} \eta_\nu(\xi)^2 = \sum_{\nu=1}^{3} \cos^2\theta_\nu \leq 3,$$

which also contradicts our assumption $m \geq 3$.

Summing up all the situations mentioned above, we have the following

Theorem 3.1. *There do not exist any compact real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$, satisfying Ric $(\xi,\xi)\geq 0$ and* (3.6).

4. Killing vector fields

Let M be a compact Riemannian manifold with Riemannian metric g. Then a vector field X of M is said to be *Killing* if and only if the Riemannian metric g is invariant along the direction of X, that is, $\mathcal{L}_X g = 0$. In component wise, we can express it by $\mathcal{L}_X g_{ji} = \nabla_j X_i + \nabla_i X_j = 0$.

Now on a compact Riemannian manifold M we introduce an integral formula due to Watanabe [14] as follows:

$$\int_M \big[\mathrm{Ric}(X,X) + \frac{1}{2}\|\mathcal{L}_X g\|^2 - \|\nabla X\|^2 - (\mathrm{div}X)^2\big] * 1 = 0. \tag{4.1}$$

From this, we know that if X is Killing, then $\nabla_i X^i = 0$. So its divergence vector $\mathrm{div}X = -\sum_i \nabla_i X^i = 0$. Accordingly, the integral formula reduces to

$$\int_M (\mathrm{Ric}(X,X) - \|\nabla X\|^2) * 1 = 0. \tag{4.2}$$

Now let us apply (4.1) to a compact real hypersurface M in $G_2(\mathbb{C}^{m+2})$. Then the formula (2.1) gives the following

$$\mathrm{div}\ \xi = \sum_{i=1}^{4m-1} g(\nabla_{e_i}\xi, e_i) = \mathrm{Tr}\phi A = 0.$$

From this, if we substitute the vector ξ in (4.1), we have the following integral formula

$$\int_M (\mathrm{Ric}(\xi,\xi) - \|\nabla\xi\|^2) * 1 = -\frac{1}{2}\int_M \|\mathcal{L}_\xi g\|^2 * 1 \leq 0.$$

From this, together with the formula (3.5),we assert the following

Proposition 4.1. *Let M be a compact real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$, with the Ricci curvature Ric$(\xi,\xi)\leq 0$. If M satisfies*

$$TrA^2 \leq 4(m+1) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 + TrA\ g(A\xi,\xi), \tag{4.3}$$

then the structure vector ξ is a Killing vector field and has vanishing covariant derivative. Moreover, if the Ricci curvature is negative-definite, then a Killing vector field other than zero does not exist on M.

In the paper [3] due to Berndt and Suh we have proved that the structure vector ξ is a Killing vector field, that is $\mathcal{L}_\xi g = 0$ if and only if the structure tensor ϕ and the shape operator A commutes with each other. Moreover, in such a case we have asserted that M is congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. For such kind of tubes we introduce a Proposition given in [3] as follows:

Proposition 4.2. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three(if $r = \pi/2$) or four(otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}cot(\sqrt{8}r)\ ,\ \beta = \sqrt{2}cot(\sqrt{2}r)\ ,\ \lambda = -\sqrt{2}tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 2\ ,\ m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}\xi_1, \\ T_\beta &= Span\ \{\xi_2, \xi_3\}, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1 X\}. \end{aligned}$$

From these Propositions 4.1 and 4.2 we know that

$$A\xi_2 = \eta(A\xi_2)\xi.$$

Then this gives that

$$0 = g(A\xi_2, \xi_2) = \sqrt{2}\cot(\sqrt{2}r).$$

Then $r = \frac{\pi}{\sqrt{8}}$, which contradicts Proposition 4.2. Then summing up this situation we assert the following:

Theorem 4.3. *There does not exist any compact real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *satisfying* $Ric(\xi,\xi)\leq 0$ *and* (4.3).

Now let M be a compact real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then by the formula (2.1) its structure vector ξ satisfies the following formulas:

$$\text{div}\ \xi = \sum_{i=1}^{4m-1} g(\nabla_{e_i}\xi, e_i) = \text{Tr}\phi A = 0,$$

and

$$\|\nabla\xi\|^2 = g(\nabla\xi,\nabla\xi) = \text{Tr}\ A^2 - \sum_{i=1}^{4m-1}\eta(Ae_i)\eta(Ae_i).$$

From this, together with (3.5) and the integral formula (4.1), we know that

$$\begin{aligned} &-\frac{1}{2}\int_M \|\mathcal{L}_\xi g\|^2 * 1 \\ =&\int_M \left\{4(m+1) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 + hg(A\xi,\xi) - \text{Tr}\ A^2\right\} * 1 \leq 0. \end{aligned}$$

From this we assert the following:

Theorem 4.4. *Let* M *be a compact real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m\geq 3$. *If it satisfies*

$$\int_M \left\{4(m+1) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 + TrAg(A\xi,\xi) - Tr\ A^2\right\} * 1 \geq 0,$$

then M *is congruent to a tube of radius* r *over a totally geodesic* $G_2(\mathbb{C}^{m+1})$ *in* $G_2(\mathbb{C}^{m+2})$.

Let M be a compact real hypersurface in $G_2(\mathbb{C}^{m+2})$, which satisfies

$$\text{Tr}\ A^2 + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 \leq 4(m+1) + \text{Tr}Ag(A\xi,\xi).$$

Then we also assert the following

Corollary 4.5. *Let* M *be a compact real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m\geq 3$. *If* M *satisfies*

$$Tr\ A^2 + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 \leq 4(m+1) + Tr\ Ag(A\xi,\xi),$$

then M *is congruent to a tube of radius* r *over a totally geodesic* $G_2(\mathbb{C}^{m+1})$ *in* $G_2(\mathbb{C}^{m+2})$.

By this Corollary and Proposition 4.2 we are able to assert the following

Corollary 4.6. *Let M be a compact real hypersurface of a complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m\geq 3$. If M is a minimal hypersurface satisfying*

$$Tr\, A^2 + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 \leq 4(m+1),$$

then M is congruent to a tube of radius r, $\cot\sqrt{2}r = \sqrt{\frac{2m-1}{3}}$, over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

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