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Picone identities for ordinary differential equations of fourth order

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Abstract. It is known that there are two kinds of Picone identities for fourth order ordinary differential equations. A new type of Picone identity is established, and Sturmian comparison theorems are derived.

1. Introduction

Picone identity is a fundamental tool in establishing Sturmian comparison theorems. We refer the reader to Cimmino [1], Kreith [6, 7] and Kuks [8] for fourth order ordinary differential equations, and to Cimmino [2], Eastham [4], Halanay and Šandor [5], Kusano and Yoshida [9] for even order ordinary differential equations. Two kind of Picone identities are known for ordinary differential equations of fourth order, see, for example, Eastham [4, p.197], Kreith [6, p.665].

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The objective of this paper is to establish a new type of Picone identity for ordinary differential equations of fourth order. We can derive Sturmian comparison theorems as applications.

2. Picone-type identities

We consider the ordinary differential operators l and L defined by

$$l[u] \equiv (a(t)u'')'' - (b(t)u')' + c(t)u, \qquad t \in (\alpha, \beta),$$

$$L[v] \equiv (A(t)v'')'' - (B(t)v')' + C(t)v, \qquad t \in (\alpha, \beta),$$

where (α, β) is a finite interval, $a(t) \in C^2[\alpha, \beta]$, $A(t) \in C^2[\alpha, \beta]$, $b(t) \in C^1[\alpha, \beta]$, $B(t) \in C^1[\alpha, \beta]$, $c(t) \in C[\alpha, \beta]$ and $C(t) \in C[\alpha, \beta]$.

The domains $\mathcal{D}_l((\alpha, \beta))$ of l is defined to be the set of all real-valued functions of class $C^4(\alpha, \beta) \cap C^2[\alpha, \beta]$. The domain $\mathcal{D}_L((\alpha, \beta))$ is defined to be the same as that of l, that is, $\mathcal{D}_l((\alpha, \beta)) = \mathcal{D}_L((\alpha, \beta))$.

The following Picone identity is known, see, for example, Kreith [7, p.270].

Theorem 1. *Let v_1 and v_2 be linearly independent solutions of $L[v] = 0$ on $[\alpha, \beta]$ such that*

$$v_1(\alpha) = v_1'(\alpha) = v_2(\alpha) = v_2'(\alpha) = 0$$

and define the functions σ and τ by

$$\sigma = v_1 v_2' - v_2 v_1',$$

$$\tau = v_1' v_2'' - v_2' v_1''.$$

If σ does not vanish in $(\alpha, \beta]$, then the following Picone identity holds :

$$\frac{d}{dt}\left[-(a(t)u'')'u + a(t)u''u' + b(t)u'u - A(t)\frac{\sigma'}{\sigma}(u')^2 + 2A(t)\frac{\tau}{\sigma}uu' - \frac{(A(t)\tau)'}{\sigma}u^2\right]$$

$$= \big(a(t) - A(t)\big)(u'')^2 + \big(b(t) - B(t)\big)(u')^2 + \big(c(t) - C(t)\big)u^2 + A(t)\left(u'' - \frac{\sigma'}{\sigma}u' + \frac{\tau}{\sigma}u\right)^2$$

in $(\alpha, \beta]$.

The next Picone identity is a special case of a result of Kusano and Yoshida [9, Theorem 1A].

Theorem 2. *If* $u \in \mathcal{D}_l((\alpha,\beta))$, $v \in \mathcal{D}_L((\alpha,\beta))$ *and if none of* v *and* v' *vanish in* (α,β), *then we have the Picone identity* :

$$\begin{aligned} &\frac{d}{dt}\left[\frac{u}{v}\left\{u(A(t)v'')' - v(a(t)u'')'\right\} + \frac{u'}{v'}\left\{v'(a(t)u'') - u'(A(t)v'')\right\}\right. \\ &\qquad\qquad\left. + \frac{u}{v}\left\{v(b(t)u') - u(B(t)v')\right\}\right] \\ &= \big(a(t) - A(t)\big)(u'')^2 + \big(b(t) - B(t)\big)(u')^2 + \big(c(t) - C(t)\big)u^2 \\ &+ A(t)\left(u'' - \frac{u'}{v'}v''\right)^2 + \big(-v'(A(t)v'')' + B(t)(v')^2\big)\left(\frac{u'}{v'} - \frac{u}{v}\right)^2 \\ &\qquad\qquad + \frac{u}{v}(uL[v] - vl[u]). \end{aligned} \tag{1}$$

Now we present new Picone identities in the following Theorems 3 and 4.

Theorem 3. *If* $v \in \mathcal{D}_L((\alpha,\beta))$ *and* v *does not vanish in* (α,β), *then we obtain the Picone identity* :

$$\begin{aligned} &-\frac{d}{dt}\left[\frac{u}{v}\left\{u(A(t)v'')'\right\} - \frac{u'}{v}\left\{u(A(t)v'')\right\} - \frac{u}{v}\left\{u(B(t)v')\right\} - u(A(t)v'')\left(\frac{u}{v}\right)'\right] \\ &= A(t)(u'')^2 + B(t)(u')^2 + C(t)u^2 - A(t)\left(u'' - \frac{u}{v}v''\right)^2 \\ &\qquad - v\big(B(t)v - 2A(t)v''\big)\left\{\left(\frac{u}{v}\right)'\right\}^2 - \frac{u^2}{v}L[v]. \end{aligned} \tag{2}$$

Proof. The following identity holds:

$$\begin{aligned} &\frac{d}{dt}\left[-\frac{u^2}{v}(A(t)v'')' + u(A(t)v'')\left(\frac{u}{v}\right)' + \frac{u'}{v}u(A(t)v'')\right] \\ &= A(t)(u'')^2 + C(t)u^2 - A(t)\left(u'' - \frac{u}{v}v''\right)^2 \\ &\qquad + 2A(t)\frac{v''}{v}\left(u' - \frac{u}{v}v'\right)^2 - \frac{u^2}{v}L[v] \end{aligned} \tag{3}$$

which is a special case of Dunninger [3, Theorem 2.2]. We easily obtain

$$\frac{d}{dt}\left[\frac{u}{v}(uB(t)v')\right] = B(t)(u')^2 - B(t)\left(v\left(\frac{u}{v}\right)'\right)^2 + \frac{u^2}{v}(B(t)v')'. \tag{4}$$

Combining (3) with (4) yields the desired identity (2).

Theorem 4. *If $v \in \mathcal{D}_L((\alpha,\beta))$ and v does not vanish in (α,β), then we obtain the Picone identity :*

$$\begin{aligned}
&\frac{d}{dt}\Bigg[\frac{u}{v}\left\{u(A(t)v'')' - v(a(t)u'')'\right\} + \frac{u'}{v}\left\{v(a(t)u'') - u(A(t)v'')\right\} \\
&\qquad\qquad + \frac{u}{v}\left\{v(b(t)u') - u(B(t)v')\right\} - u(A(t)v'')\left(\frac{u}{v}\right)'\Bigg] \\
&= \big(a(t) - A(t)\big)(u'')^2 + \big(b(t) - B(t)\big)(u')^2 + \big(c(t) - C(t)\big)u^2 \\
&+ A(t)\left(u'' - \frac{u}{v}v''\right)^2 + v\big(-2A(t)v'' + B(t)v\big)\left\{\left(\frac{u}{v}\right)'\right\}^2 + \frac{u}{v}(uL[v] - vl[u]).
\end{aligned} \tag{5}$$

Proof. It is easy to see that

$$\begin{aligned}
ul[u] = \frac{d}{dt}\big[u(a(t)u'')'\big] - \frac{d}{dt}\big[u'(a(t)u'')\big] - \frac{d}{dt}\big[u(b(t)u')\big] \\
+ a(t)(u'')^2 + b(t)(u')^2 + c(t)u^2.
\end{aligned} \tag{6}$$

Combining (2) with (6), we arrive at (5).

Remark 1. In the case where none of v and v' does not vanish in (α,β), the Picone identity (2) reduces to (1) with $a(t) = b(t) = c(t) = 0$. It is easy to check that

$$\frac{d}{dt}\left[\frac{u'}{v}\left\{u(A(t)v'')\right\} + u(A(t)v'')\left(\frac{u}{v}\right)'\right] = \frac{d}{dt}\left[\left(\frac{u^2}{v}\right)' A(t)v''\right]. \tag{7}$$

Since

$$\frac{d}{dt}\left[\left(\frac{u^2}{v}\right)' A(t)v'' + \left(u' - \frac{u}{v}v'\right)^2 \frac{A(t)v''}{v'}\right] = \frac{d}{dt}\left[\frac{(u')^2}{v'}A(t)v''\right] \tag{8}$$

and

$$-\frac{d}{dt}\left[\left(u'-\frac{u}{v}v'\right)^2\frac{A(t)v''}{v'}\right]$$
$$=-A(t)\left(u''-\frac{u}{v}v''\right)^2+2\frac{A(t)v''}{v}\left(u'-\frac{u}{v}v'\right)^2$$
$$-v'(A(t)v'')'\left(\frac{u'}{v'}-\frac{u}{v}\right)^2+A(t)\left(u''-\frac{u'}{v'}v''\right)^2, \qquad (9)$$

combining (7)–(9) yields

$$\frac{d}{dt}\left[\frac{u'}{v}\left\{u(A(t)v'')\right\}+u(A(t)v'')\left(\frac{u}{v}\right)'\right]$$
$$=\frac{d}{dt}\left[\frac{u'}{v'}\left\{u'(A(t)v'')\right\}\right]-A(t)\left(u''-\frac{u}{v}v''\right)^2+2\frac{A(t)v''}{v}\left(u'-\frac{u}{v}v'\right)^2$$
$$-v'(A(t)v'')'\left(\frac{u'}{v'}-\frac{u}{v}\right)^2+A(t)\left(u''-\frac{u'}{v'}v''\right)^2. \qquad (10)$$

Substituting (10) into the left hand side of (2), we observe that (2) reduces to (1) with $a(t)=b(t)=c(t)=0$.

3. Sturmian comparison theorems

By using the Picone identity established in Section 2, we derive Sturmian comparison theorems.

Theorem 5. *Assume that $A(t)\geq 0$ in (α,β). If there exists a nontrivial solution $u\in\mathcal{D}_l((\alpha,\beta))$ of $l[u]=0$ in (α,β) such that*

$$u(\alpha)=u'(\alpha)=u(\beta)=u'(\beta)=0$$

and

$$V[u]\equiv\int_\alpha^\beta\left[(a(t)-A(t))(u'')^2+(b(t)-B(t))(u')^2+(c(t)-C(t))u^2\right]dt$$
$$\geq 0,$$

then every solution $v\in\mathcal{D}_L((\alpha,\beta))$ of $L[v]=0$ in (α,β) satisfying

$$v\big(B(t)v-2A(t)v''\big)\geq 0 \quad \text{in } (\alpha,\beta), \qquad (11)$$

$$B(t)v-2A(t)v''\neq 0 \quad \text{in } (\alpha,\beta) \qquad (12)$$

has a zero on $[\alpha,\beta]$.

Proof. Suppose to the contrary that there exists a solution $v \in \mathcal{D}_L((\alpha,\beta))$ of $L[v]=0$ in (α,β) which satisfies (11), (12) and the property that $v \neq 0$ on $[\alpha,\beta]$. Integrating (5) over $[\alpha,\beta]$, we find that

$$0 \geq V[u] + \int_\alpha^\beta v\big(B(t)v - 2A(t)v''\big)\left\{\left(\frac{u}{v}\right)'\right\}^2 dt$$
$$\geq 0$$

and therefore we obtain

$$\int_\alpha^\beta v\big(B(t)v - 2A(t)v''\big)\left\{\left(\frac{u}{v}\right)'\right\}^2 dt = 0.$$

The assumptions (11) and (12) imply that $\left(\frac{u}{v}\right)' \equiv 0$ in (α,β), that is, $u = kv$ for some nonzero constant k. Since $u(\alpha) = u(\beta) = 0$ and $v \neq 0$ on $[\alpha,\beta]$, we are led to a contradiction. The proof is complete.

Theorem 6. *Assume that* $A(t) \geq 0$ *in* (α,β). *If there exists a nontrivial function* $u \in C^2[\alpha,\beta]$ *such that*

$$u(\alpha) = u'(\alpha) = u(\beta) = u'(\beta) = 0, \tag{13}$$

$$M[u] \equiv \int_\alpha^\beta \big[A(t)(u'')^2 + B(t)(u')^2 + C(t)u^2\big]\, dt \leq 0, \tag{14}$$

then every solution $v \in \mathcal{D}_L((\alpha,\beta))$ *of* $L[v]=0$ *in* (α,β) *satisfying* (11) *and* (12) *has a zero in* (α,β) *unless* u *is a constant multiple of* v.

Proof. Let $v \in \mathcal{D}_L((\alpha,\beta))$ be any solution of $L[v]=0$ in (α,β) which satisfies (11), (12) and the condition $v \neq 0$ in (α,β). In view of the boundary condition (13) and the fact $u \in C^2[\alpha,\beta]$, we see that u belongs to the Sobolev space $\overset{\circ}{\mathrm{H}}_2\,((\alpha,\beta))$ which is the closure in the norm

$$\|u\| = \|u\|_2 = \left(\int_\alpha^\beta \sum_{j=0}^{2} |u^{(j)}(t)|^2 dt\right)^{1/2} \tag{15}$$

of the class $C_0^\infty((\alpha,\beta))$ of infinitely differentiable functions with compact support in (α,β). Let $\{u_m(t)\}$ be a sequence of functions in $C_0^\infty((\alpha,\beta))$

converging to u in norm (15). Then, the Picone identity (2) with $u = u_m$ holds. Integrating (2) with $u = u_m$ over (α, β), we find that

$$M[u_m] = \int_\alpha^\beta \left[A(t) \left(u_m'' - \frac{u_m}{v} v'' \right)^2 + v\big(B(t)v - 2A(t)v''\big) \left\{ \left(\frac{u_m}{v} \right)' \right\}^2 \right] dt$$
$$\geq 0.$$

Since $A(t)$, $B(t)$ and $C(t)$ are uniformly bounded on $[\alpha, \beta]$, there is a constant $K > 0$ such that

$$|M[u_m] - M[u]|$$
$$= \left| \int_\alpha^\beta \left[A(t)((u_m'')^2 - (u'')^2) + B(t)((u_m')^2 - (u')^2) + C(t)(u_m^2 - u^2) \right] dt \right|$$
$$\leq K \int_\alpha^\beta \left| u_m''(u_m - u)'' + u''(u_m - u)'' \right| dt$$
$$+ K \int_\alpha^\beta |u_m'(u_m - u)' + u'(u_m - u)'| dt$$
$$+ K \int_\alpha^\beta |u_m(u_m - u) + u(u_m - u)| dt.$$

Application of Schwarz inequality yields

$$|M[u_m] - M[u]| \leq 3K(\|u_m\| + \|u\|)\|u_m - u\|.$$

Since $\lim\limits_{m\to\infty} \|u_m - u\| = 0$, we observe that $\lim\limits_{m\to\infty} M[u_m] = M[u] \geq 0$, and hence $M[u] = 0$ in view of (14). Let J denote an arbitrary interval with $\bar{J} \subset (\alpha, \beta)$ and define

$$H_J[u] \equiv \int_J \left[A(t) \left(u'' - \frac{u}{v} v'' \right)^2 + v\big(B(t)v - 2A(t)v''\big) \left\{ \left(\frac{u}{v} \right)' \right\}^2 \right] dt$$

for $u \in C^2[\alpha, \beta]$. We easily see that

$$0 \leq H_J[u_m] \leq M[u_m]$$

and that the inequality

$$|H_J[u_m] - H_J[u]| \leq K_1(\|w_m\|_J + \|w\|_J)\|w_m - w\|_J$$

holds, where K_1 is a positive constant, $w_m = u_m/v$, $w = u/v$ and the subscript J indicates the integrals involved in the norm (15) are taken over J. As $v \neq 0$ on $\bar{J}$, we see that $\lim_{m\to\infty} \|w_m - w\| = 0$ when $\lim_{m\to\infty} \|u_m - u\| = 0$, and therefore $\lim_{m\to\infty} H_J[u_m] = H_J[u]$. Since $\lim_{m\to\infty} M[u_m] = M[u] = 0$, we obtain $\lim_{m\to\infty} H_J[u_m] = H_J[u] = 0$. Hence, $(\frac{u}{v})' \equiv 0$ in J, that is, $u = kv$ in J for some nonzero constant k. We conclude that $u = kv$ in (α, β) by continuity, or u is a constant multiple of v. This completes the proof.

Theorem 7. *Assume that $A(t) \geq 0$ in (α, β). If there exists a nontrivial solution $u \in \mathcal{D}_l((\alpha, \beta))$ of $l[u] = 0$ in (α, β) such that*

$$u(\alpha) = u'(\alpha) = u(\beta) = u'(\beta) = 0,$$

$$V[u] \geq 0,$$

then every solution $v \in \mathcal{D}_L((\alpha, \beta))$ of $L[v] = 0$ in (α, β) satisfying (11) *and* (12) *has a zero in (α, β) unless u is a constant multiple of v.*

Proof. Using (6), we find that

$$V[u] = \int_\alpha^\beta u l[u]\, dt - M[u]$$

for any $u \in \mathcal{D}_l((\alpha, \beta))$ satisfying (13). Hence, we conclude that $V[u] = -M[u]$ for the solution u of $l[u] = 0$ satisfying (13). The conclusion follows from Theorem 6.

Remark 2. The condition (11) holds true if $B(t) \geq 0$ and $vv'' \leq 0$ in (α, β).

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