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Convergence analysis of the interface for interfacial transport phenomena*

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Abstract. In this paper we are concerned with the convergence analysis of the interface for interfacial transport phenomena such as incompressible immiscible two-fluid flows. Some convergence results for the interface are shown by the regularized Heaviside function. In order to validate our convergence results, numerical examples are presented with an original test problem having non-trivial but explicit interface.

1. Introduction

Interfacial phenomena often appear in many engineering problems. In these problems a main concern is the interface evolution. The numerical methods for dealing the interface are classified into three categories, namely, Eulerian, Lagrangian and Arbitrary Lagrangian-Eulerian (ALE) methods. The feature of the Eulerian method is very easy to construct the algorithm due to the availability of the fixed mesh, however, we can only capture the interface implicitly. On the other hand, the Lagrangian method involves a moving mesh and can

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express the interface easily. In this method nodal points of the mesh move with fluid particles, it needs a costly resoning mesh preqocedure when the mesh becomes distorted. The ALE method has been proposed to improve the shortcomings of both Eulerian and Lagrangian methods, however, it is difficult to implement since the selection of mesh velocity is nontrivial for complex flows.

Our approach is the Eulerian. Then the interface is defined as the 0 level set of the solution of the transport equation, which is called by pseudo-density function, and the position of the interface is updated by solving the transport equation. Therefore it is important to analyze the mathematical property such as the convergence of the interface which is defined implicitly. In this paper we discuss the convergence of the interface by using the regularized Heaviside function $H_\epsilon(\cdot)$ under the assumption that the finite element scheme for the transport equation has some reasonable accuracy. In order to confirm the effectiveness of the interface convergence results, we provide some numerical experiments with a new test problem having non-trivial but explicit interface.

An overview of this paper is as follows. In Section 2, we are concerned with incompressible immiscible two-fluid flows as a model problem of interfacial transport phenomena. Section 3 is devoted to the modeling of the interface in the Eulerian approach. In Section 4 we give the fractional step projection finite element scheme for the model problem and show the stability of the scheme. In Section 5 we consider the finite element scheme for the pure advection equation and show the stability and the convergence of the scheme. In Section 6, we discuss some convergence results of the interface by the regularized Heaviside function. Finally we present some numerical experiments in Section 7.

Let us now establish the notation used throughout this paper. Let Ω be a bounded domain in $\mathbf{R}^2$ with Lipschitz boundary Γ. For a nonnegative integer m, let $H^m(\Omega)$ be the standard Sobolev space equipped with the norm $\|\cdot\|_{m,\Omega}$ and the semi-norm $|\cdot|_{m,\Omega}$. We shall denote $H^0(\Omega)$ by $L^2(\Omega)$, and the norm and the inner product of $L^2(\Omega)$ are denoted by $\|\cdot\|_{0,\Omega}$ and $(\cdot,\cdot)$, respectively. For functions defined on the cylinder $Q_T = \Omega \times (0,T)$, we shall also introduce some additional notations. Namely, for any Banach space X and $1 \le p < \infty$ let $L^p(0,T;X)$ be the space of all X-valued functions which are defined on $(0,T)$, measurable and

$$\|u\|_{L^p(0,T;X)} = (\int_0^T \|u(t)\|_X^p \, dt)^{1/p} < +\infty,$$

where $\|\cdot\|_X$ is the norm in X. Similarly we define $L^\infty(0,T;X)$. In particular, we denote $L^p(0,T;L^p(\Omega))$ by $L^p(Q_T)$, and $L^\infty(0,T;L^\infty(\Omega))$ by $L^\infty(Q_T)$, respectively.

2. Mathematical model

As a model problem for the interfacial transport phenomena, we consider incompressible immiscible two-fluid flows. Let Ω be a bounded domain of $\mathbf{R}^2$ with the boundary Γ. The domain Ω consists of two time-dependent subdomain $\Omega_i(t), i = 1,2$ such that

$$\overline{\Omega} = \overline{\Omega_1(t)} \cup \overline{\Omega_2(t)} \quad \text{and} \quad \Omega_1(t) \cap \Omega_2(t) = \emptyset \ \text{ for } 0 \le t < \infty.$$

Each of subdomains is filled by the $fluid\#1$ and $fluid\#2$, respectively. We assume that two fluids are both viscous, incompressible and immiscible. The governing equations for unsteady, viscous, incompressible, immiscible two-fluid flow system and the incompressibility condition are given by the following :

$$\begin{cases} \rho(\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\nabla p + 2\mu \,\mathrm{div} D(\boldsymbol{u}) + \boldsymbol{f} & \text{in } \Omega \times (0,T), \\ \mathrm{div}\ \boldsymbol{u} = 0 & \text{in } \Omega \times (0,T), \end{cases} \tag{2.1}$$

where $\boldsymbol{f} = {}^t[0 \quad -\rho g]$. Here $\boldsymbol{u}$ is the velocity, p the pressure, μ the viscosity, ρ the density, g the gravitational acceleration, and $D(\boldsymbol{u}) =$

$(1/2)(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ is the deformation tensor. Furtheremore, we set as follows :

$$(\rho, \mu) = \begin{cases} (\rho_1, \mu_1) & \text{in } \Omega_1(t), \\ (\rho_2, \mu_2) & \text{in } \Omega_2(t). \end{cases} \tag{2.2}$$

We set as follows :

$$\begin{cases} \rho_{\min} = \min\{\rho_1, \rho_2\}, & \rho_{\max} = \max\{\rho_1, \rho_2\}, \\ \mu_{\min} = \min\{\mu_1, \mu_2\}, & \mu_{\max} = \max\{\mu_1, \mu_2\}. \end{cases} \tag{2.3}$$

As for the boundary condition we assume the no-slip condition :

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{2.4}$$

and we enforce the initial condition such as

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{in } \Omega, \tag{2.5}$$

where $\boldsymbol{u}_0$ is a prescribed divergence-free velocity.

3. Advection of the interface

By the assumption for the immiscibility, the motion of the interface is governed by

$$\boldsymbol{u}^{(\alpha)} \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \boldsymbol{n}, \tag{3.1}$$

where $\boldsymbol{u}^{(\alpha)}$ is the velocity of *fluid*$\#\alpha$ on the interface for $\alpha = 1, 2$, $\boldsymbol{v}$ the speed of the interface displacement and $\boldsymbol{n}$ the unit normal vector to the interface. This means that the interface moves with the fluid particles. Therefore, if we consider that the interface $I(t)$ is defined as follows :

$$I(t) = \{\boldsymbol{x} = (x_1, x_2) \mid \varphi(\boldsymbol{x}, t) = 0 \text{ for } 0 \leq t < T\}, \tag{3.2}$$

then it is easily seen that $\varphi = \varphi(\boldsymbol{x}, t)$ satisfy the following transport equation :

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi = 0 \quad \text{in } \Omega \times (0, T), \tag{3.3}$$

where $\boldsymbol{u}$ is the velocity field. As for the initial condition for (3.3), we choose an sufficiently smooth initial function so that

$$\varphi_0 = \begin{cases} < 0 & \text{in } \Omega_1(0), \\ > 0 & \text{in } \Omega_2(0). \end{cases} \tag{3.4}$$

In our approach the density ρ and the viscosity μ in the Navier-Stokes equations (2.1) depend on the sign of the solution of the transport equation (3.3) as follows :

$$(\rho, \mu) = \begin{cases} (\rho_1, \mu_1) & \text{if } \varphi < 0, \\ (\rho_2, \mu_2) & \text{if } \varphi > 0. \end{cases} \tag{3.5}$$

Thus incompressible immiscible two-fluid flow problems are reduced to the nonlinear interaction between the Navier-Stokes equations (2.1) and the transport equation (3.3).

4. Fractional step projection scheme

In this section we consider the finite element approximation for the Navier-Stokes equations (2.1). Since main concern in this paper is to discuss the convergence of the interface, we shall limit ourselves to discuss only the stability of the scheme. Here we consider the fractional step projection finite element scheme for the Navier-Stokes equations (2.1).

We define the following finite element spaces using $P1\,iso\,P2/P1$ element [5] :

$$\begin{cases} \boldsymbol{V}_h = \{\boldsymbol{v}_h \in C^0(\overline{\Omega})^2 \mid \boldsymbol{v}_h|_{\tilde{K}} \in P_1(\tilde{K})^2 \ \ \forall \tilde{K} \in \mathcal{T}_{h/2}\}, \\ Q_h = \{q_h \in C^0(\overline{\Omega}) \mid q_h|_K \in P_1(K) \ \ \forall K \in \mathcal{T}_h\} \cap L_0^2(\Omega), \end{cases} \tag{4.1}$$

where $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ and $\mathcal{T}_{h/2}$ be a new triangulation obtained by dividing each element $K \in \mathcal{T}_h$ into four equal subtriangles. Furthermore we define two subspaces of $\boldsymbol{V}_h$

$$\boldsymbol{V}_{0h} = \{\boldsymbol{v}_h \in V_h |\ \boldsymbol{v}_h = \boldsymbol{0} \text{ on } \Gamma\}, \tag{4.2}$$

and

$$\boldsymbol{V}_h^1 = \{\boldsymbol{v}_h \in V_h |\ \boldsymbol{v}_h \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}. \tag{4.3}$$

In order to define a full discretization of (2.1) we consider the uniform mesh for the time variable t and define $t_n = n\tau$ for $n = 0, 1, 2, \cdots, [T/\tau]$, where $\tau > 0$ is a time step. Then we consider the following fractional step projection finite element scheme which is slightly different from [6] and [7], where $\boldsymbol{u}_h^n$, and p_h^n, denotes the approximation of the solution $\boldsymbol{u}(\boldsymbol{x}, t_n)$, and $p(\boldsymbol{x}, t_n)$, respectively.

$$\begin{cases} \rho(\dfrac{\boldsymbol{u}_h^{n+1/2} - \boldsymbol{u}_h^n}{\tau}, \boldsymbol{v}_h) = -\rho(\boldsymbol{u}_h^n \cdot \nabla \boldsymbol{u}_h^{n+1/2} + \dfrac{1}{2}(\operatorname{div} \boldsymbol{u}_h^n)\boldsymbol{u}_h^{n+1/2}, \boldsymbol{v}_h) \\ \qquad\qquad - 2\mu(D(\boldsymbol{u}_h^{n+1/2}), D(\boldsymbol{v}_h)) + (\boldsymbol{f}, \boldsymbol{v}_h) \quad \text{for } \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_{0h}, \\ \boldsymbol{u}_h^{n+1/2} = \boldsymbol{0} \quad \text{on } \Gamma. \end{cases} \tag{4.4}$$

$$\begin{cases} \rho(\dfrac{\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n+1/2}}{\tau}, \boldsymbol{v}_h) = (p_h^{n+1}, \operatorname{div} \boldsymbol{v}_h) \quad \text{for } \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^1, \\ (\operatorname{div} \boldsymbol{u}_h^{n+1}, q_h) = 0 \quad \text{for } \forall q_h \in Q_h, \\ \boldsymbol{u}_h^{n+1} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma. \end{cases} \tag{4.5}$$

As well known, projection method consists of the viscous step (4.4) and the projection step (4.5). Equation (4.4) shows that the intermediate velocity $\boldsymbol{u}_h^{n+1/2}$ is computed as solution of a discretized momentum equation without the pressure term. On the other hand, equation

(4.5) shows that the intermediate velocity $\boldsymbol{u}_h^{n+1/2}$ is decomposed into the sum of a solenoidal velocity $\boldsymbol{u}_h^{n+1}$ and the gradient of a scalar function proportional to the pressure p_h^{n+1}.

For the stability of the fractional step projection scheme (4.4)-(4.5) we have the following :

Proposition 4.1 *Assume that* $\boldsymbol{f} = \boldsymbol{0}$*. Then the fractional step projection scheme* $(4.4) - (4.5)$ *is unconditionally stable. That is, it holds that for* $n = 1, 2, \cdots, [T/\tau]$

$$\|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \frac{4\mu_{\min}\tau C}{\rho_{\min}} \sum_{k=0}^{n-1} \|\boldsymbol{u}_h^{k+1/2}\|_{1,\Omega}^2 \le \|\boldsymbol{u}_h^0\|_{0,\Omega}^2 \,, \tag{4.6}$$

where $C > 0$ *is a constant in Korn's inequality.*

Proof In (4.4) we take $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1/2}$. Considering that

$$(\boldsymbol{u}_h^n \cdot \nabla \boldsymbol{u}_h^{n+1/2} + \frac{1}{2}(\operatorname{div} \boldsymbol{u}_h^n)\boldsymbol{u}_h^{n+1/2}, \boldsymbol{u}_h^{n+1/2}) = 0 \text{ for } \boldsymbol{u}_h^{n+1/2} \in V_{0h},$$

and the Korn's inequalty :

$$\sum_{i,j=1}^{2} \|D_{ij}(\boldsymbol{u}_h^{n+1/2})\|_{0,\Omega}^2 \ge C\|\boldsymbol{u}_h^{n+1/2}\|_{1,\Omega}^2 \quad \text{for} \quad \boldsymbol{u}_h^{n+1/2} \in \boldsymbol{V}_{0h}, \tag{4.7}$$

we have

$$\frac{1}{2\tau}(\|\boldsymbol{u}_h^{n+1/2}\|_{0,\Omega}^2 - \|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^{n+1/2} - \boldsymbol{u}_h^n\|_{0,\Omega}^2) \le -\frac{2\mu_{\min}C}{\rho_{\min}}\|\boldsymbol{u}_h^{n+1/2}\|_{1,\Omega}^2.$$

Then we have

$$\|\boldsymbol{u}_h^{n+1/2}\|_{0,\Omega}^2 - \|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^{n+1/2} - \boldsymbol{u}_h^n\|_{0,\Omega}^2 + \frac{4\mu_{\min}\tau C}{\rho_{\min}}\|\boldsymbol{u}_h^{n+1/2}\|_{1,\Omega}^2 \le 0. \tag{4.8}$$

On the other hand we take $\boldsymbol{v}_h = \boldsymbol{u}_h^{n+1}$ in (4.5). Since $(\operatorname{div} \boldsymbol{u}_h^{n+1}, q_h) = 0$ for any $q_h \in Q_h$, we have

$$\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 - \|\boldsymbol{u}_h^{n+1/2}\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n+1/2}\|_{0,\Omega}^2 = 0. \tag{4.9}$$

Adding (4.8) and (4.9), we obtain

$$\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 - \|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}_h^{n+1/2}\|_{0,\Omega}^2 + \|\boldsymbol{u}_h^{n+1/2} - \boldsymbol{u}_h^n\|_{0,\Omega}^2$$

$$+\frac{4\mu_{\min}\tau C}{\rho_{\min}}\|\boldsymbol{u}_h^{n+1/2}\|_{1,\Omega}^2 \leq 0.$$

Therefore we have

$$\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 - \|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \frac{4\mu_{\min}\tau C}{\rho_{\min}}\|\boldsymbol{u}_h^{n+1/2}\|_{1,\Omega}^2 \leq 0. \tag{4.10}$$

Summing up (4.10) from $n = 0$ to $n = m-1$, we find that

$$\|u_h^m\|_{0,\Omega}^2 + \frac{4\mu_{\min}\tau C}{\rho_{\min}}\sum_{k=0}^{m-1}\|u_h^{k+1/2}\|_{1,\Omega}^2 \leq \|u_h^0\|_{0,\Omega}^2.$$

Therefore the fractional step projection scheme (4.4)-(4.5) is unconditionally stable.

5. Finite element scheme for transport equations

In our Eulerian approach, as shown in Section 3, the constants ρ and μ should be determined by only the sign of the pseudo-density function. Therefore, we may not regard the accurate value of the pseudo-density function all that. Furthermore, since our concern in this paper is to discuss the convergence of the interface in the interfacial transport phenomena, we consider the following *pure* advection problem [3] for $n = 0, 1, \cdots, [T/\tau] - 1$ instead of (3.3) in order to decouple the nonlinear interaction of the Navier-Stokes equations (2.1) and the transport equation (3.3).

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} + \tilde{\boldsymbol{u}} \cdot \nabla\varphi = 0 & \text{in } \Omega \times (t_n, t_{n+1}], \\ \varphi(\boldsymbol{x}, t_n) \text{ is given in } \Omega, \end{cases} \tag{5.1}$$

where $\tilde{\boldsymbol{u}}$ is taken *constant* vector in the time interval $(t_n, t_{n+1}]$. In practice, for example, we may consider that $\tilde{\boldsymbol{u}}$ is chosen as $\boldsymbol{u}_h^{n+1}$. In the sequel, we assume that

$$\begin{cases} \tilde{\boldsymbol{u}} \in L^\infty(\Omega), \\ \operatorname{div} \tilde{\boldsymbol{u}} = 0 \qquad \text{in } \Omega, \\ \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma. \end{cases} \tag{5.2}$$

Then we consider an implicit Euler scheme for the pure advection equation (5.1), which is the discretization in time and the finite element approximation is adopted in space variable.

Let V be a Sobolev space $H^1(\Omega)$ and V_h be the finite dimensional space of V such that

$$V_h = \{v_h \in C^0(\overline{\Omega}) \mid v_h|_K \in P_k(K) \text{ for } \forall K \in \mathcal{T}_h\}, \tag{5.3}$$

where $k \geq 1$. Then our finite element scheme for (5.1) is as follows :

$$\text{Find } \varphi_h^{n+1} \in V_h \text{ such that}$$

$$\begin{cases} (\dfrac{\varphi_h^{n+1} - \varphi_h^n}{\tau}, v_h) + (\tilde{\boldsymbol{u}} \cdot \nabla \varphi_h^{n+1}, v_h) = 0 \quad \text{for } \forall v_h \in V_h, \\ \varphi_h^0 = \varphi_{0h} = \Pi_h \varphi_0, \end{cases} \tag{5.4}$$

where Π_h be the interpolation operator from $H^{l+1}(\Omega)$ to V_h for $1 \leq l \leq k$.

From (3.5) the determining rules for the density ρ and the viscosity μ in (4.4)-(4.5) is expressed by the following :

$$\begin{cases} \rho = \rho_1(1 - H(\varphi_h^n)) + \rho_2 H(\varphi_h^n), \\ \mu = \mu_1(1 - H(\varphi_h^n)) + \mu_2 H(\varphi_h^n), \end{cases} \tag{5.5}$$

where $H(\cdot)$ is the Heaviside function. In this paper, considering the stable computation for two-fluid flows with high density and viscosity ratios such as the air-water system, we use the following determining rules for ρ and μ :

$$\begin{cases} \rho = \rho_1(1 - H_\epsilon(\varphi_h^n)) + \rho_2 H_\epsilon(\varphi_h^n), \\ \mu = \mu_1(1 - H_\epsilon(\varphi_h^n)) + \mu_2 H_\epsilon(\varphi_h^n), \end{cases} \tag{5.6}$$

where

$$H_\epsilon(\phi) = \begin{cases} 1 & (\text{if } \phi > \epsilon), \\ \frac{1}{2}\{1 + \frac{\phi}{\epsilon} + \frac{1}{\pi}\sin(\frac{\pi\phi}{\epsilon})\} & (\text{if } |\phi| \le \epsilon), \\ 0 & (\text{if } \phi < -\epsilon). \end{cases} \tag{5.7}$$

Therefore the computational algorithm for incompressible immiscible two-fluid flows can be written by the following :

1. $n \leftarrow 0$ and $t \leftarrow 0$.
2. Determine the parameter (ρ, μ) in $(t_n, t_{n+1}]$ by (5.6).
3. Compute $\boldsymbol{u}_h^{n+1}, p_h^{n+1}$ by (4.4)-(4.5).
4. Compute φ_h^{n+1} by (5.4).
5. $n \leftarrow n + 1, t \leftarrow t + \tau$.
6. *If* $t < T$ *then* go to 2 *else* stop.

The stability of the scheme (5.4) is easily shown.

Proposition 5.1 For each $n = 0, 1, 2, \cdots, [T/\tau]$ it holds

$$\|\varphi_h^n\|_{0,\Omega} \le \|\varphi_{0h}\|_{0,\Omega}. \tag{5.8}$$

Noticing that for any $v \in H^{l+1}(\Omega)$

$$\|v - \Pi_h v\|_{0,\Omega} + h|v - \Pi_h v|_{1,\Omega} \le Ch^{l+1}|v|_{l+1,\Omega}\,, \tag{5.9}$$

we have the convergence of the finite element scheme (5.4).

Theorem 5.2([4], [5]) *Assume that* $\varphi_0 \in H^k(\Omega)$ *and the solution* φ *to* (3.3) *satisfies*

$$\varphi \in L^2(0,T;H^{k+1}(\Omega)),\ \frac{\partial\varphi}{\partial t} \in L^2(0,T;H^k(\Omega)) \text{ and } \frac{\partial^2\varphi}{\partial t^2} \in L^\infty(Q_T). \tag{5.10}$$

Then φ_h^n *defined by* (5.4) *satisfies for each* $n = 0, 1, 2, \cdots, [T/\tau]$

$$\|\varphi_h^n - \varphi(\cdot,t_n)\|_{0,\Omega} \le C(h^k + \tau), \tag{5.11}$$

where $C > 0$ *is a constant.*

6. Convergence of the interface

Since the interface is defined as the 0 level set of the pseudo-density function in Eulerian approach, it is difficult to discuss the convergence of the interface directly. As for this, we have shown [4] the $L^2(\Omega)$-error estimate of $\varphi_h^n - \varphi(\cdot,t_n)$ with respect to the Heaviside function $H(\cdot)$, which means the convergence for the interface in some sense.

Theorem 6.1([4]) *Assume the same condition of Theorem* 5.2*. Furthermore, we assume that for sufficiently small* $\delta > 0$ *and* $h > 0$ *there exist constants* $C_1 > 0$ *and* $C_2 > 0$*, independent of* δ *and* h*, such that*

$$\begin{cases} \sup\limits_{h>0} m[|\varphi_h^n| < \delta] \le C_1\delta, \\ m[|\varphi(\cdot,t_n)| < \delta] \le C_2\delta. \end{cases} \tag{6.1}$$

If there exists a constant $C_3 > 0$ *such that*

$$\frac{\tau}{h^k} \le C_3, \tag{6.2}$$

then we have for some constant $C = C(p) > 0$

$$\|H(\varphi_h^n) - H(\varphi(\cdot,t_n))\|_{L^p(\Omega)} \le Ch^{2k/3p}. \tag{6.3}$$

Therefore, for example in the cases of P1-element or P1 iso P2-element we have the following convergence result :

$$\|H(\varphi_h^n) - H(\varphi(\cdot, t_n))\|_{0,\Omega} \leq Ch^{1/3}, \tag{6.4}$$

however, we think that it is a little lower from our numerical experiments of view and the assumption (6.1) does not have the physical meaning.

In this section we propose a new convergence result by using the *regularized* Heaviside function $H_\epsilon(\cdot)$. We have already used it in the determining rules (5.6) in order to compute the scheme stably even if the case of high ratios of densities and viscosities. Therefore it is very natural to use the regularized Heaviside function to discuss the convergence of the interface in the finite element approximation for the immiscible two-fluid flows.

Let $H_\epsilon(\cdot)$ be a regularized Heaviside function with some constant $\epsilon > 0$ defined by

$$H_\epsilon(\phi) = \begin{cases} 1 & (\text{if } \phi > \epsilon), \\ f(\dfrac{\phi}{\epsilon}) & (\text{if } |\phi| \leq \epsilon), \\ 0 & (\text{if } \phi < -\epsilon), \end{cases} \tag{6.5}$$

where $f(\xi)$ is a smooth transition function such that $f(-1) = 0$ and $f(1) = 1$. This means that the interface is considered as a transition region $\mathcal{R}$ with some finite thickness :

$$\mathcal{R} = \{(x_1, x_2) \in \Omega \mid |\varphi(x_1, x_2)| \leq \epsilon\}.$$

In this transition region $\mathcal{R}$ we consider that the density and the viscosity vary continuously from ρ_1(resp. μ_1) to ρ_2(resp. μ_2). Then we can show the following property of the regularized Heaviside function $H_\epsilon(\cdot)$.

Lemma 6.1 *Assume that there exists a constant* $M > 0$ *such that*

$$|H_\epsilon(x) - H_\epsilon(y)| \leq \frac{M}{\epsilon}|x - y| \text{ for } x, y \in \mathbf{R}. \tag{6.6}$$

Then we have the following estimates for $p \geq q \geq 1$ *:*

$$\|H_\epsilon(\phi_1) - H_\epsilon(\phi_2)\|_{L^q(\Omega)} \leq \frac{M}{\epsilon}|\Omega|^{\frac{1}{q}-\frac{1}{p}}\|\phi_1 - \phi_2\|_{L^p(\Omega)} \text{ for } \phi_1, \phi_2 \in L^p(\Omega), \tag{6.7}$$

where $|\Omega|$ *denotes the Lebesgue measure of* Ω*.*

Proof It is easy to see from (6.6) the following estimate :

$$\|H_\epsilon(\phi_1) - H_\epsilon(\phi_2)\|_{L^p(\Omega)} \leq \frac{M}{\epsilon}\|\phi_1 - \phi_2\|_{L^p(\Omega)} \text{ for } \phi_1, \phi_2 \in L^p(\Omega). \tag{6.8}$$

If $p > q \geq 1$, by the Hölder's inequality we can show for a bounded domain Ω

$$\|f\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p}}\|f\|_{L^p(\Omega)} \text{ for } f \in L^p(\Omega). \tag{6.9}$$

Combining (6.8) and (6.9), we have (6.7). Since inequality (6.7) is still valid for $p = q$, the lemma is proved.

Remark 6.1 In practice, we take $f(\xi)$ as follows

$$f(\xi) = \frac{1}{2}\{1 + \xi + \frac{1}{\pi}\sin(\pi\xi)\} \text{ for } |\xi| \leq 1. \tag{6.10}$$

Then it is easy to see the function $H_\epsilon(\phi)$ with (6.10) satisfies the condition (6.6) with $M = 1$.

From Lemma 6.1 and Theorem 5.2 we have the main result :

Theorem 6.2 *Assume that the assumption in Lemma* 1 *and the same condition of Theorem* 5.2*. If there exists a constant* $C > 0$ *such that*

$$\frac{\tau}{h^k} \leq C, \tag{6.11}$$

then we have for some constant $C > 0$

$$\|H_\epsilon(\varphi_h^n) - H_\epsilon(\varphi(\cdot, t_n))\|_{0,\Omega} \le \frac{CMh^k}{\epsilon}. \tag{6.12}$$

Remark 6.2 From the result of Theorem 6.2 we can take ϵ as $\sqrt{h}$ in the finite element approximation with the $P2$ iso $P1$-element. Since (5.11) holds with $k = 1$ in this case, then the convergence rate of the approximate interface is $O(\sqrt{h})$. In the following section, we shall validate this theoretical result by some numerical experiments.

7. Numerical experiments

7.1. Test problem having non-trivial interface

Let Ω be $(-1, 1) \times (-1, 1)$. We consider a counter-clockwise flow field in Ω. Its stream function is

$$\Phi = (1 - x^2)(1 - y^2), \tag{7.1}$$

and the corresponding velocity field is

$$(u, v) = (-2y(1 - x^2), 2x(1 - y^2)). \tag{7.2}$$

Let $m = 1 - \Phi$. It is presented in the polar coordinates (r, θ) as

$$m = r^2 - r^4 \frac{\sin^2 2\theta}{4}.$$

Solving this quadratic equation with respect to r^2, we obtain

$$r^2 = \frac{2(1 \pm \sqrt{1 - m \sin^2 2\theta})}{\sin^2 2\theta}. \tag{7.3}$$

The r^2 corresponding to Ω is '$-$' one.

We then consider motion of a particle that is transported by the flow field (7.2) on a stream line m $(0 \le m \le 1)$. Let position of the particle be $(r(t), \theta(t))$ or $(x(t), y(t))$. The transportation $d\mathbf{x}/dt = \mathbf{u}$ gives

$$\begin{aligned}\theta_t &= \theta_x u + \theta_y v \\ &= -\frac{\sin\theta}{r}\{-2r\sin\theta(1 - r^2\cos^2\theta)\} + \frac{\cos\theta}{r}\{2r\cos\theta(1 - r^2\sin^2\theta)\} \\ &= 2 - r^2\sin^2 2\theta. \qquad (7.4)\end{aligned}$$

Substituting (7.3) into (7.4), we obtain the equation of motion of the particle as

$$\theta_t = 2\sqrt{1 - m\sin^2 2\theta}. \qquad (7.5)$$

Integrating (7.5) from $t = t_0$ to t_1, we have

$$\int_{\theta(t_0)}^{\theta(t_1)} \frac{d\theta}{\sqrt{1 - m\sin^2 2\theta}} = 2(t_1 - t_0), \qquad (7.6)$$

hence

$$F(2\theta(t_1)|m) - F(2\theta(t_0)|m) = 4(t_1 - t_0), \qquad (7.7)$$

where

$$F(\phi|m) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} \qquad (7.8)$$

is the elliptic integral of the first kind. (It means that the motion of a particle relates to that of a simple pendulum.)

Using an initial condition of the transported function, we can exactly compute distribution at $t = t_1$ of the function by (7.7). We are able to know deformation of the interface exactly. The velocity field satisfies assumptions in the theorems. Moreover, the domain is a square so that it is covered by triangulation without curved triangles. We therefore remark that this problem is suitable as a test problem for transported interface problems. Figures 1 and 2 show deformation of interfaces in the cases that the initial interfaces are the x-axis and a circle ($x^2 + (y - 1/3)^2 = 1/9$), respectively.

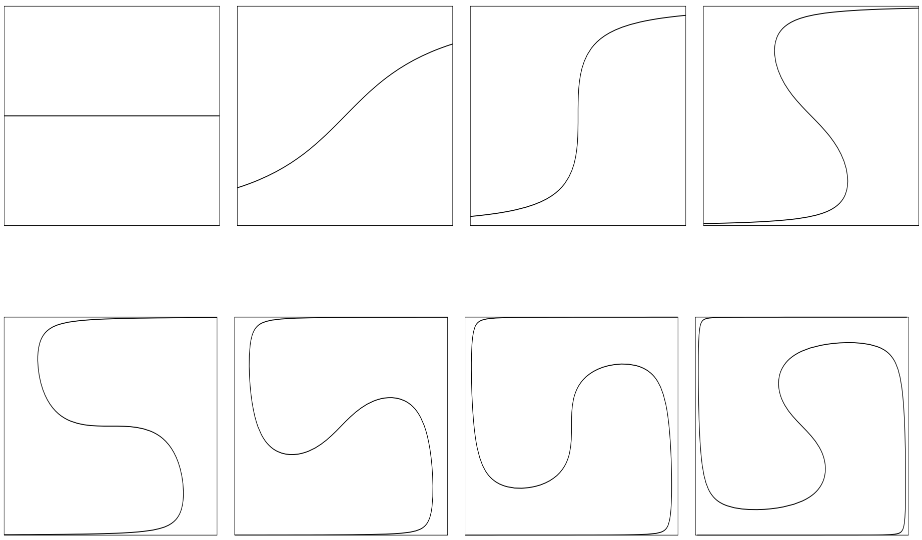

Figure 1: Deformation of a line interface: exact shapes for $t = 0, \pi/8, 2\pi/8, 3\pi/8, \pi/2, 5\pi/8, 6\pi/8, 7\pi/8$

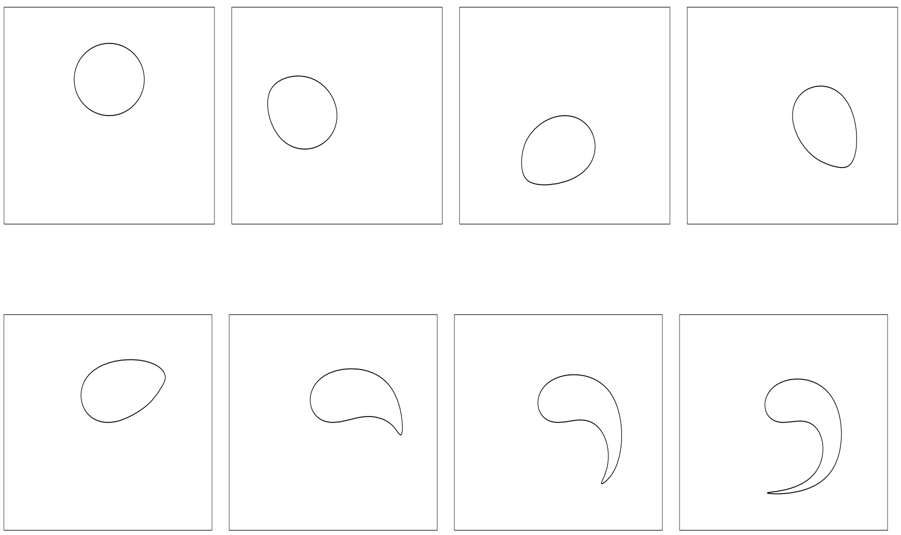

Figure 2: Deformation of a liquid drop: exact shapes for $t = 0, \pi/4, \pi/2, 3\pi/4, \pi, 2\pi, 3\pi, 4\pi$

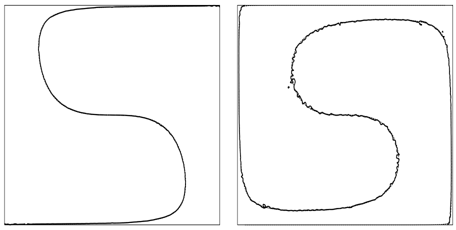

Figure 3: Approximate and exact interface shape

7.2. Numerical convergence of interfaces

We here investigate numerical convergence of interfaces when the transport equation is solved by the finite element scheme (5.4) with $P1$-element. Four edges of Ω are divided into n ($n = 10, 20, 40$ or 80) segments. In order to avoid effect of mesh uniformity, non-uniform triangulations are generated using FreeFEM+ [1]. We select time steps as $\Delta t = 5\pi/4n$ while $T = \pi$.

Deformation of a line interface

The initial condition of transported function is $\phi = y$. Figure 3 shows numerical and exact interfaces at $t = \pi/2$ and $t = \pi$. Figure 4 shows graphs of h versus errors measured by the regularized Heaviside function H_ϵ ($\epsilon = \sqrt{h}$). Four curves show the error values when $t = \pi/4, \pi/2, 3\pi/4, \pi$, respectively.We can observe that they decrease as $O(h)$ in this case.

Deformation of a drop

The initial condition of transported function,

$$\phi = \frac{e^{-9(x^2+(y-1/3)^2)} - e^{-1}}{e^{-1}},$$

satisfies $\Delta\phi = 0$ on $I(t = 0)$.

Figure 5 shows numerical and exact interfaces at $t = \pi/8, \pi/4, 3\pi/8$

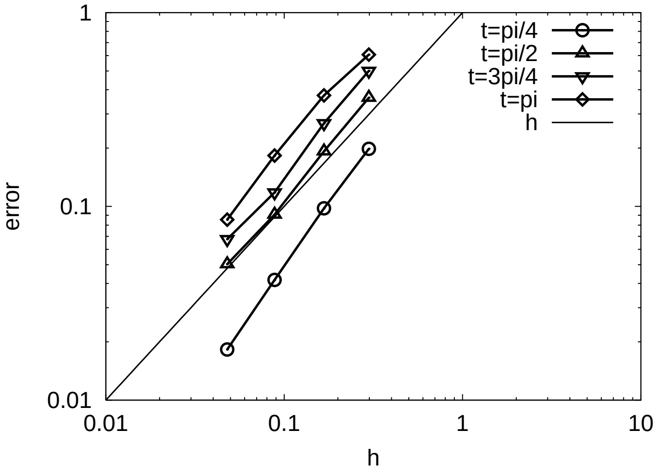

Figure 4: mesh size h vs. numerical error

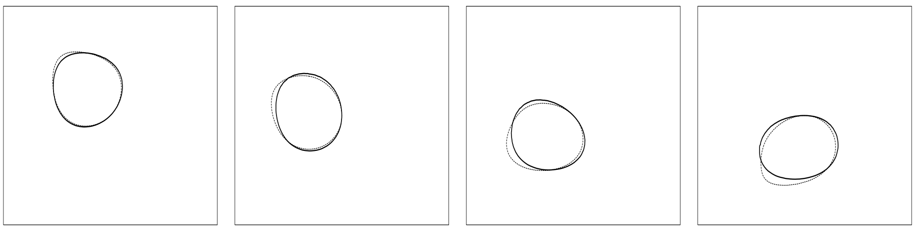

Figure 5: Approximate and exact interface shape

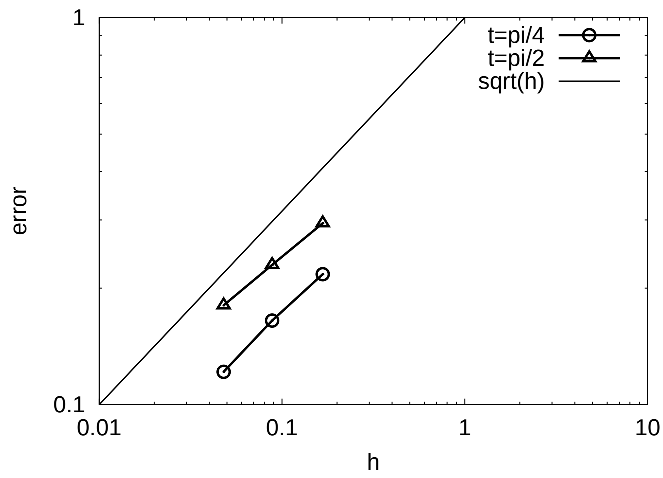

Figure 6: mesh size h vs. numerical error

and $\pi/2$. Figure 6 shows graphs of h versus errors, which is measured by using the regularized Heaviside function $H_\epsilon(\cdot)$ with $\epsilon = \sqrt{h}$.

Two curves show the error values when $t = \pi/4, \pi/2$, respectively. Convergence ratio $O(\sqrt{h})$ is observed, which agrees well with the result in Remark 6.2. Throughout the experiments, the effect of the interface convergence theorem is confirmed.

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