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Oscillations of vector differential equations of hyperbolic type with functional arguments

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Abstract. Vector hyperbolic differential equations with functional arguments are studied, and oscillations of solutions of certain boundary value problems are investigated. The approach used is to reduce the multi-dimensional oscillation problems to the nonexistence of positive solutions of scalar functional differential inequalities by employing the concept of H-oscillation introduced by Domšlak, where H denotes some unit vector.

1. Introduction

In 1970, Domšlak [2] has introduced the concept of H-oscillation to study the oscillatory character of vector differential equations, where H is a unit vector in $\mathbb{R}^N$. Several authors have investigated H-oscillation of vector differential equations. We refer the reader to [2–4] for vector ordinary differential equations, and to [5–7] for vector partial differential equations. It seems that no work has been done on H-oscillation of vector hyperbolic differential equations with functional arguments.

The objective of this paper is to derive sufficient conditions for every solution of certain boundary value problems for vector hyperbolic differential equations with functional arguments to be H-oscillatory in a cylindrical domain.

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We are concerned with the oscillations of the vector hyperbolic differential equation with functional arguments

$$\begin{aligned} & \frac{\partial^2}{\partial t^2}\left(U(x,t)+\sum_{i=1}^{\ell} h_i(t)U(x,\rho_i(t))\right) - a(t)\Delta U(x,t) \\ & -\sum_{i=1}^{k} b_i(t)\Delta U(x,\tau_i(t)) + \sum_{i=1}^{m} c_i(x,t,U(x,\sigma_i(t)))U(x,\sigma_i(t)) \\ = \quad & F(x,t), \quad (x,t)\in\Omega\equiv G\times(0,\infty), \end{aligned} \tag{1}$$

where G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G. We assume that :

(H$_1$) $h_i(t)\in C^2([0,\infty);\mathbb{R})$ $(i=1,2,...,\ell)$,
$b_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,2,...,k)$,
$a(t)\in C([0,\infty);[0,\infty))$ and $F(x,t)\in C(\overline{\Omega};\mathbb{R}^N)$;

(H$_2$) $\rho_i(t)\in C^2([0,\infty);\mathbb{R})$, $\lim_{t\to\infty}\rho_i(t)=\infty$ $(i=1,2,...,\ell)$,
$\tau_i(t)\in C([0,\infty);\mathbb{R})$ and $\lim_{t\to\infty}\tau_i(t)=\infty$ $(i=1,2,...,k)$,
$\sigma_i(t)\in C([0,\infty);\mathbb{R})$ and $\lim_{t\to\infty}\sigma_i(t)=\infty$ $(i=1,2,...,m)$;

(H$_3$) $c_i(x,t,\Xi)\in C(\overline{\Omega}\times\mathbb{R}^N;\mathbb{R})$ $(i=1,2,...,m)$,
$c_i(x,t,\Xi)\ge p_i(t)\psi_i(|\Xi|)$ for $(x,t)\in\Omega,\ \Xi\in\mathbb{R}^N$,
$p_i(t)\in C([0,\infty);[0,\infty))$, $\psi_i(\xi)\in C([0,\infty);[0,\infty))$ and $\psi_i(\xi)$ are nondecreasing for $\xi\ge 0$.

The following two kinds of boundary conditions are considered:

(B$_1$) $U=\Psi \quad \text{on} \quad \partial G\times(0,\infty)$,

(B$_2$) $\dfrac{\partial U}{\partial\nu}+\mu U=\tilde{\Psi} \quad \text{on} \quad \partial G\times(0,\infty)$,

where $\Psi,\ \tilde{\Psi}\in C(\partial G\times(0,\infty);\mathbb{R}^N)$, $\mu\in C(\partial G\times(0,\infty);[0,\infty))$ and ν denotes the unit exterior normal vector to ∂G.

Definition 1. By a *solution* of equation (1) we mean a function $U(x,t)\in C^2(\overline{G}\times[t_{-1},\infty);\mathbb{R}^N)\cap C(\overline{G}\times[\hat{t}_{-1},\infty);\mathbb{R}^N)$ which satisfies (1), where

$$\begin{aligned} t_{-1} &= \min\left\{0,\ \min_{1\le i\le \ell}\left\{\inf_{t\ge 0}\rho_i(t)\right\},\ \min_{1\le i\le k}\left\{\inf_{t\ge 0}\tau_i(t)\right\}\right\}, \\ \hat{t}_{-1} &= \min\left\{0,\ \min_{1\le i\le m}\left\{\inf_{t\ge 0}\sigma_i(t)\right\}\right\}. \end{aligned}$$

Definition 2. Let H be a unit vector in $\mathbb{R}^N$. A solution $U(x,t)$ of (1) is said to be H-*oscillatory* in Ω if the inner product $< U(x,t), H >$ has a zero in $G \times (t,\infty)$ for any $t > 0$.

We give two examples of $c_i(x,t,\Xi)$ which satisfy the hypothesis (H_3) (cf. [6]). Let $M_i(x,t) \in C^1(\Omega)$ be symmetric, positive definite matrix functions, and $\lambda_i(x,t)$ be the smallest eigenvalue of $M_i(x,t)$. Then it can be shown that

$$\begin{aligned} c_i(x,t,\Xi) &= (|\Xi|^\gamma \Xi)^T M_i(x,t)\,(|\Xi|^\gamma \Xi) \\ &\ge p_i(t)|\Xi|^{2\gamma+2} \quad \text{for} \quad (x,t)\in\Omega,\ \Xi\in\mathbb{R}^N, \end{aligned}$$

where T denotes the transpose, $\gamma \ge -1$ and $p_i(t) = \min_{x\in\overline{G}} \lambda_i(x,t)$. Another example is the following

$$c_i(x,t,\Xi) = \left(\Xi^T M_i(x,t)\Xi\right)^\delta \quad (\delta \ge 0).$$

It is easily seen that

$$c_i(x,t,\Xi) \ge \left(p_i(t)|\Xi|^2\right)^\delta = (p_i(t))^\delta\,|\Xi|^{2\delta}.$$

In Section 2 we reduce the multi-dimensional oscillation problems to the nonexistence of positive solutions of scalar functional differential inequalities. In Section 3 we treat the case where $h_i(t) \ge 0$ $(i = 1,2,...,\ell)$, $\sum_{i=1}^{\ell} h_i(t) \le 1$ and derive sufficient conditions for every solution of the boundary value problems (1), (B_i) $(i = 1,2)$ to be H-oscillatory in Ω.

2. Reduction to scalar functional differential inequalities

In this section we reduce the multi-dimensional oscillation problems to certain one-dimensional oscillation problems for scalar functional differential inequalities.

We use the following notation:

$$u_H(x,t) = < U(x,t), H >,$$
$$f_H(x,t) = < F(x,t), H >,$$

where H is a unit vector in $\mathbb{R}^N$ and $< U, V >$ denotes the inner product of $U, V \in \mathbb{R}^N$.

Theorem 1. *Assume that* (H_1)–(H_3) *hold. Let* $U(x,t)$ *be a solution of* (1). *If* $u_H(x,t)$ *is eventually positive, then* $u_H(x,t)$ *satisfies the scalar hyperbolic differential inequality*

$$\frac{\partial^2}{\partial t^2}\left(u_H(x,t) + \sum_{i=1}^{\ell} h_i(t) u_H(x, \rho_i(t))\right) - a(t)\Delta u_H(x,t) - \sum_{i=1}^{k} b_i(t)\Delta u_H(x, \tau_i(t)) + \sum_{i=1}^{m} p_i(t)\varphi_i(u_H(x, \sigma_i(t))) \le f_H(x,t), \quad (2)$$

where $\varphi_i(\xi) = \xi\psi_i(|\xi|)$. *If* $u_H(x,t)$ *is eventually negative, then* $v_H(x,t) \equiv -u_H(x,t)$ *satisfies the scalar hyperbolic differential inequality*

$$\frac{\partial^2}{\partial t^2}\left(v_H(x,t) + \sum_{i=1}^{\ell} h_i(t) v_H(x, \rho_i(t))\right) - a(t)\Delta v_H(x,t) - \sum_{i=1}^{k} b_i(t)\Delta v_H(x, \tau_i(t)) + \sum_{i=1}^{m} p_i(t)\varphi_i(v_H(x, \sigma_i(t))) \le -f_H(x,t). \quad (3)$$

Proof. Let $u_H(x,t)$ be eventually positive. The inner product of (1) and H yields

$$\frac{\partial^2}{\partial t^2}\left(< U(x,t), H > + \sum_{i=1}^{\ell} h_i(t) < U(x, \rho_i(t)), H >\right) - a(t) < \Delta U(x,t), H > - \sum_{i=1}^{k} b_i(t) < \Delta U(x, \tau_i(t)), H > + \sum_{i=1}^{m} c_i(x, t, U(x, \sigma_i(t))) < U(x, \sigma_i(t)), H > = < F(x,t), H > . \quad (4)$$

Proceeding as in the proof of [5, Theorem 1], we observe that

$$c_i(x, t, U(x, \sigma_i(t))) < U(x, \sigma_i(t)), H > \ge \ p_i(t)\varphi_i\left(u_H(x, \sigma_i(t))\right). \quad (5)$$

Combining (4) with (5), we conclude that $u_H(x,t)$ satisfies the inequality (2). If $u_H(x,t)$ is eventually negative, we see that

$$c_i(x,t,U(x,\sigma_i(t))) < U(x,\sigma_i(t)), H > \leq \; -p_i(t)\varphi_i\left(-u_H(x,\sigma_i(t))\right) \qquad (6)$$

(see [5, proof of Theorem 1]). We combine (4) and (6) to conclude that $v_H(x,t)$ satisfies the inequality (3).

Associated with the boundary conditions (B_i) ($i=1,2$), we consider the following scalar boundary conditions

($\tilde{\mathrm{B}}_1$) $u = \psi_H \quad \text{on} \quad \partial G \times (0,\infty),$

($\tilde{\mathrm{B}}_2$) $\dfrac{\partial u}{\partial \nu} + \mu u = \tilde{\psi}_H \quad \text{on} \quad \partial G \times (0,\infty),$

where

$$\begin{aligned} \psi_H &= \; < \Psi, H >, \\ \tilde{\psi}_H &= \; < \tilde{\Psi}, H > . \end{aligned}$$

Theorem 2. *Assume that* (H_1)–(H_3) *hold. If the scalar hyperbolic differential inequalities*

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\left(u(x,t) + \sum_{i=1}^{\ell} h_i(t)u(x,\rho_i(t))\right) - a(t)\Delta u(x,t) \\ &\quad - \sum_{i=1}^{k} b_i(t)\Delta u(x,\tau_i(t)) + \sum_{i=1}^{m} p_i(t)\varphi_i(u(x,\sigma_i(t))) \leq \pm f_H(x,t) \qquad (7) \end{aligned}$$

have no eventually positive solutions satisfying the boundary conditions ($\tilde{\mathrm{B}}_i$) ($i=1,2$)*, then every solution* $U(x,t)$ *of the boundary value problems* (1), (B_i) ($i=1,2$) *is* H*-oscillatory in* Ω*, respectively.*

Proof. Suppose to the contrary that there is a solution $U(x,t)$ of the problem (1), (B_i) which is not H-oscillatory in Ω. If $u_H(x,t)$ is eventually positive, then $u_H(x,t)$ satisfies (7) with $+f_H(x,t)$ by Theorem 1. It is easy to see that $u_H(x,t)$ satisfies the boundary conditions ($\tilde{\mathrm{B}}_i$). This contradicts the hypothesis. If $u_H(x,t)$ is eventually negative, then $v_H(x,t) = -u_H(x,t)$ is an eventually positive solution of (7) with $-f_H(x,t)$ satisfying the boundary conditions ($\tilde{\mathrm{B}}_i$). This also contradicts the hypothesis. This completes the proof.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{aligned} -\Delta w &= \lambda w \quad \text{in} \quad G, \\ w &= 0 \quad \text{on} \quad \partial G \end{aligned}$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G (see Courant and Hilbert [1]).

The following notation will be used:

$$\begin{aligned} F_H(t) &= \left(\int_G \Phi(x)dx\right)^{-1} \int_G f_H(x,t)\Phi(x)dx, \\ \Psi_H(t) &= \left(\int_G \Phi(x)dx\right)^{-1} \int_{\partial G} \psi_H \frac{\partial \Phi}{\partial \nu}(x)dS, \\ \tilde{F}_H(t) &= \frac{1}{|G|}\int_G f_H(x,t)dx, \\ \tilde{\Psi}_H(t) &= \frac{1}{|G|}\int_{\partial G} \tilde{\psi}_H \, dS, \end{aligned}$$

where $|G| = \int_G dx$.

Theorem 3. *Assume that* (H_1)–(H_3) *hold, and the following hypothesis* (H_4) *holds* :

(H_4) $\varphi_i(\xi) = \xi\psi_i(|\xi|)$ $(i = 1, 2, ..., m)$ *are convex in* $(0, \infty)$.

If the functional differential inequalities

$$\begin{aligned} &\frac{d^2}{dt^2}\left(y(t) + \sum_{i=1}^{\ell} h_i(t)y(\rho_i(t))\right) + \lambda_1 a(t)y(t) \\ &+\lambda_1 \sum_{i=1}^{k} b_i(t)y(\tau_i(t)) + \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \le \pm G_H(t) \qquad (8) \end{aligned}$$

have no eventually positive solutions, then every solution $U(x,t)$ *of the boundary value problem* (1), (B_1) *is* H*-oscillatory in* Ω*, where*

$$G_H(t) = F_H(t) - a(t)\Psi_H(t) - \sum_{i=1}^{k} b_i(t)\Psi_H(\tau_i(t)).$$

Proof. Suppose to the contrary that there is a solution $U(x,t)$ of the problem (1), (B_1) which is not H-oscillatory in Ω. First we consider the case where $u_H(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. We observe that $\varphi_i(\xi) \in C(\mathbb{R};\mathbb{R})$ $(i = 1,2,...,m)$, $\varphi_i(-\xi) = -\varphi_i(\xi)$, $\varphi_i(\xi) > 0$ for $\xi > 0$, and $\varphi_i(\xi)$ are nondecreasing in $(0,\infty)$. We easily see that

$$U_H(t) \equiv \left(\int_G \Phi(x)dx \right)^{-1} \int_G u_H(x,t)\Phi(x)dx$$

is an eventually positive solution of the inequality (8) with $+G_H(t)$ (cf. [8, Theorem 3.1], [9, Theorem 1]). Hence, we are led to a contradiction. The case where $u_H(x,t) < 0$ in $G \times [t_0,\infty)$ can be treated similarly, and we are also led to a contradiction. This completes the proof.

Theorem 4. *Assume that* (H_1)–(H_4) *hold. If the functional differential inequalities*

$$\frac{d^2}{dt^2}\left(y(t) + \sum_{i=1}^{\ell} h_i(t) y(\rho_i(t)) \right) + \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \le \pm \tilde{G}_H(t) \qquad (9)$$

have no eventually positive solutions, then every solution $U(x,t)$ *of the boundary value problem* (1), (B_2) *is* H*-oscillatory in* Ω*, where*

$$\tilde{G}_H(t) = \tilde{F}_H(t) + a(t)\tilde{\Psi}_H(t) + \sum_{i=1}^{k} b_i(t)\tilde{\Psi}_H(\tau_i(t)).$$

Proof. The proof is quite similar to that of Theorem 3, and hence will be omitted.

3. Sufficient conditions for H-oscillation

In this section we give sufficient conditions for every solution of the boundary value problems (1), (B_i) $(i = 1,2)$ to be H-oscillatory in Ω.

Theorem 5. *Assume that* (H_1)–(H_4) *hold, and assume, moreover, that :*

(H_5) $h_i(t) \ge 0$ $(i = 1,2,...,\ell)$, $\sum_{i=1}^{\ell} h_i(t) \le 1$;

(H_6) $\rho_i(t) \le t \ (i = 1, 2, ..., \ell)$;

(H_7) *there exists a function* $\theta(t) \in C^2([t_0, \infty); \mathbb{R})$ *such that* $\theta(t)$ *is oscillatory and* $\theta''(t) = G_H(t)$ [*resp.* $\theta''(t) = \tilde{G}_H(t)$], *where* $t_0 > 0$ *is some number* ;

(H_8)

$$\int_{t_0}^{\infty} p_j(s)\varphi_j \left(\left[\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s)) \right) c \pm \Theta(\sigma_j(s)) \right]_+ \right) ds = \infty,$$

for some $j \in \{1, 2, ..., m\}$ *and every* $c > 0$, *where*

$$[\varphi(s)]_+ = \max\{\varphi(s), 0\},$$
$$\Theta(t) = \theta(t) - \sum_{i=1}^{\ell} h_i(t)\theta(\rho_i(t)).$$

Then every solution $U(x,t)$ *of the boundary value problem* (1), (B_1) [*resp.* (1), (B_2)] *is* H*-oscillatory in* Ω.

Proof. The conclusion follows by combining a result of Tanaka [8, Theorem 2.1] with Theorems 3 and 4.

Example. We consider the boundary value problem

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\left(U(x,t) + \frac{1}{2}U(x,t-\pi) \right) - \frac{\partial^2 U}{\partial x^2}(x,t) \\ &\quad -\frac{\partial^2 U}{\partial x^2}(x,t-\pi) + 2\,U(x,t-2\pi) \\ &\quad = \begin{pmatrix} \frac{3}{2}\cos x \cdot \sin t \\ \left(1 + \frac{1}{2}e^{-\pi} + 2e^{-2\pi}\right) e^t \end{pmatrix}, \ (x,t) \in (0,\pi) \times (0,\infty), \end{aligned} \tag{10}$$

$$-\frac{\partial U}{\partial x}(0,t) = \frac{\partial U}{\partial x}(\pi,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t > 0. \tag{11}$$

Here $n = 1$, $G = (0,\pi)$, $\Omega = (0,\pi) \times (0,\infty)$, $\ell = k = m = 1$, $N = 2$, $h_1(t) = 1/2$, $\rho_1(t) = t - \pi$, $a(t) = b_1(t) = 1$, $\tau_1(t) = t - \pi$, $\sigma_1(t) = t - 2\pi$, $c_1(x,t,\Xi) = 1$, $p_1(t) = 2$, $\psi_1(\xi) = 1$, $\varphi_1(\xi) = \xi$, $\mu = 0$, $\tilde{\Psi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and

$$F(x,t) = \begin{pmatrix} \frac{3}{2}\cos x \cdot \sin t \\ \left(1 + \frac{1}{2}e^{-\pi} + 2e^{-2\pi}\right) e^t \end{pmatrix}.$$

Letting $H = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we see that $\tilde{F}_{e_1}(t) = \tilde{\Psi}_{e_1}(t) = 0$, and hence $\tilde{G}_{e_1}(t) = 0$. We can choose $\theta(t) = 0$, and observe that $\Theta(t) = 0$ and

$$\int^{\infty} 2 \cdot \frac{1}{2} c \, ds = \infty$$

for every $c > 0$. Hence, Theorem 5 implies that every solution $U(x,t)$ of the problem (10), (11) is e_1-oscillatory in $(0,\pi) \times (0,\infty)$. In fact

$$U(x,t) = \begin{pmatrix} \cos x \cdot \sin t \\ e^t \end{pmatrix}$$

is such a solution. We note that the above solution $U(x,t)$ is not e_2-oscillatory in $(0,\pi) \times (0,\infty)$, where $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

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