Math. J. Toyama Univ.
Vol. 26(2003), 61-73

Some extensions of Grüss' inequality

Saichi Izumino*, Josip E. Pečarić** and Božidar Tepeš†

Abstract. We give some extensions of Grüss' inequalities of discrete and integral types, which refine or generalize recent results due to P. Cerone and S. S. Dragomir and those due to some other authors.

1. Introduction

Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ be n-tuples (sequences) of real numbers, and let $p = (p_1, ..., p_n)$ be an n-tuple of positive numbers. Then we define $T(a, b; p)$ by

$$T(a,b;p) := \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i a_i \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i, \tag{1.1}$$

where $P_n = \sum_{i=1}^{n} p_i$. It is (discrete) Grüss' inequality that estimates this difference under certain conditions. Čebyšev's inequality [7, p.240] is well-known; it asserts that

$$T(a,b;p) \geq 0 \quad \text{or} \quad \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i \geq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \tag{1.2}$$

under the condition that both a and b are nonincreasing (or nondecreasing), i.e.,

$$a_1 \geq \cdots \geq a_n \text{ and } b_1 \geq \cdots \geq b_n \quad (\text{or} \quad a_1 \leq \cdots \leq a_n \text{ and } b_1 \leq \cdots \leq b_n). \tag{1.3}$$

2000 *Mathematics Subject Classification.* 26D15, 26D99.
Key words and phrases. Grüss' inequality, Čebyšev's inequality.

As a complement of this inequality, Pečarić [8] proved:

Theorem A ([8, Theorem 8], [7, p. 302]). *Let a and b be nondecreasing (or nonincreasing) n-tuples of real numbers, and let p be an n-tuple of positive numbers. Then*

$$|T(a,b;p)| \le |a_n - a_1||b_n - b_1| \max_{1\le k\le n-1} \frac{P_k(P_n - P_k)}{P_n^2},$$

where $P_k = \sum_{i=1}^{k} p_i$.

Without any assumption of monotonicity on n-tuples a and b, the following extension of Theorem A was given by Andrica and Badea [1]:

Theorem B ([1, Theorem 2]). *Let a and b be n-tuples of real numbers satisfying*

$$m_1 \le a_i \le M_1 \quad \text{and} \quad m_2 \le b_i \le M_2 \quad (i = 1, ..., n), \tag{1.4}$$

and let p be an n-tuple of positive numbers. Then

$$|T(a,b;p)| \le (M_1 - m_1)(M_2 - m_2) \max_{J\subset I_n} \frac{P(J)((P_n - P(J))}{P_n^2}, \tag{1.5}$$

where $I_n = \{1, ..., n\}$ and $P(J) = \sum_{i\in J} p_i$ for $J \subset I_n$. (cf. $P_n = P(I_n)$.)

Using convexity of functions related to Grüss' inequality, Izumino and Pečarić [5] recently gave the following fact, from which Theorems A and B were induced:

Lemma C ([5, Corollary 2.4 and Lemma 2.2]). *Let a be an n-tuple of real numbers satisfying $m \le a_i \le M$ $(i = 1, ..., n)$, and let p be an n-tuple of positive numbers with $\sum_{i=1}^{n} p_i = 1$. Then*

$$\sum_{1\le i<j\le n} p_i p_j |a_i - a_j| \le (M_1 - m_1) \max_{J\subset I_n} P(J)(1 - P(J)), \tag{1.6}$$

and in particular, if a is assumed to be nonincreasing,

$$\sum_{1\le i<j\le n} p_i p_j (a_i - a_j) \le (M_1 - m_1) \max_{1\le k\le n-1} P_k(1 - P_k). \tag{1.7}$$

Concerning the integral form of Grüss' inequality, recently Cheng and Sun [3] (and Matić [6]), as an improvement of the inequality due to Grüss himself [4] gave the following result:

Theorem D ([3, Theorem 1.1], [6, Theorem 3]). *Let h and g be integrable functions on an interval $[a,b]$ and let $\phi_2 \le g(x) \le \Phi_2$ $(x \in [a,b])$ for some constants $\phi_2 < \Phi_2$. Then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b h(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b h(x)dx \int_a^b g(x)dx\right| \\
&\quad \le \frac{\Phi_2-\phi_2}{2}\cdot\frac{1}{b-a}\int_a^b \left|h(x) - \frac{1}{b-a}\int_a^b h(y)dy\right| dx \\
&\quad \left(\le \frac{\Phi_2-\phi_2}{2}\cdot\frac{1}{b-a}\left(\int_a^b \left|h(x)-\frac{1}{b-a}\int_a^b h(y)dy\right|^2 dx\right)^{1/2}\right).
\end{aligned}$$

Let us note that the first inequality of the above theorem was shown by Sokolov [10] in 1963. Corresponding to Theorem D, the following discrete analogue has been shown by Cerone and Dragomir [2]:

Theorem E ([2, p. 376]). *Let a and b be two n-tuples of real numbers with $m_2 \le b_i \le M_2$ $(i = 1, \dots, n)$ for some constants $m_2 < M_2$, and let p be an n-tuple of positive numbers such that $\sum_{i=1}^n p_i = 1$. Then*

$$\begin{aligned}
|T(a,b;p)| &\le \frac{M_2-m_2}{2}\sum_{i=1}^n p_i|a_i - \sum_{j=1}^n p_j a_j| \\
&\le \frac{M_2-m_2}{2}\left(\sum_{i=1}^n p_i|a_i - \sum_{j=1}^n p_j a_j|^q\right)^{1/q} \quad (q>1) \qquad (1.8)\\
&\le \frac{M_2-m_2}{2}\max_{1\le i\le n}\left|a_i - \sum_{j=1}^n p_j a_j\right|.
\end{aligned}$$

Now let Ω be a measurable space with respect to a positive measure μ on the set. For a measurable function $w(x) \ge 0$ on Ω such that

$$\infty > \int_\Omega w(x)d\mu(x) > 0,$$

we write $L_w(\Omega,\mu)$ the Lebesgue space of (real-valued) μ-measurable functions f on Ω such that

$$\int_\Omega |f(x)|w(x)d\mu(x) < \infty.$$

Put

$$T(f,g;w) = \left(\int_\Omega w(x)d\mu(x)\right)^{-1} \int_\Omega w(x)f(x)g(x)d\mu(x) - \left(\int_\Omega w(x)d\mu(x)\right)^{-2} \int_\Omega w(x)f(x)d\mu(x) \int_\Omega w(x)g(x)d\mu(x).$$

Then the following result was shown by Cerone and Dragomir [2], as an extension of Theorem D and also integral analogue of Theorem E at the same time:

Theorem F ([2, Theorem 2.1]). *Let* $f,\ g \in L_w(\Omega,\mu)$ *and let* $\phi_2 \le g(x) \le \Phi_2$ $(x \in \Omega)$. *Then*

$$|T(f,g;w)| \le \frac{\Phi_2 - \phi_2}{2}\left(\int_\Omega w(x)d\mu(x)\right)^{-1} \times \int_\Omega w(x)\left| f(x) - \left(\int_\Omega w(x)d\mu(x)\right)^{-1}\int_\Omega w(y)f(y)d\mu(y)\right| d\mu(x). \tag{1.9}$$

In this paper, applying Lemma C and making use of the idea of Cerone and Dragomir [2], we give some refinements and generalizations of all theorems mentioned before.

2. Discrete Grüss' inequality

Hereafter we assume that an n-tuple $p = (p_1,\cdots,p_n)$ of positive numbers satisfies

$$\sum_{i=1}^n p_i = 1 \tag{2.1}$$

for convenience sake. Then $T(a,b;p)$ is rewritten as follows:

$$T(a,b;p) = \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \tag{2.2}$$

for n-tuples $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$. We often make use of the following expression of $T(a, b; p)$.

$$T(a, b; p) = \sum_{1 \le i < j \le n} p_i p_j (a_i - a_j)(b_i - b_j), \tag{2.3}$$

which is obtained from Binet-Cauchy identity [7, p. 85]:

$$\sum_{i=1}^{n} a_i c_i \sum_{i=1}^{n} b_i d_i - \sum_{i=1}^{n} a_i d_i \sum_{i=1}^{n} b_i c_i = \sum_{1 \le i < j \le n} (a_i b_j - a_j b_i)(c_i d_j - c_j d_i)$$

or by a direct computation. The following fact is also useful for our discussion.

Lemma 2.1 (cf. [7, p. 296]).

$$|T(a, b; p)| \le T(a, a; p)^{1/2} T(b, b; p)^{1/2} \text{ (Cauchy's inequality)}. \tag{2.4}$$

Now we give a refinement of Theorems B, E (for $q = 2$) (and also a result due to Pečarić and Tepeš [9, Theorem 2.4]):

Theorem 2.2. *Let a and b be n-tuples of real numbers satisfying* (1.4), *and let p be an n-tuple of positive numbers satisfying* (2.1). *Write $\bar{a} = \sum_{i=1}^{n} p_i a_i$ and $\bar{b} = \sum_{i=1}^{n} p_i b_i$. Then*

$$\begin{aligned} &|T(a, b; p)| \\ &\quad \le \frac{M_2 - m_2}{2} \sum_{i=1}^{n} p_i |a_i - \bar{a}| \\ &\quad \le (M_2 - m_2) \sum_{1 \le i < j \le n} p_i p_j |a_i - a_j| \\ &\quad \le (M_1 - m_1)(M_2 - m_2) \max_{J \subset I_n} P(J)(1 - P(J)) \\ &\qquad \left(\le \frac{1}{4} (M_1 - m_1)(M_2 - m_2) \right). \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&|T(a,b;p)| \\
&\leq \frac{1}{2}(M_1-m_1)^{1/2}(M_2-m_2)^{1/2}\left(\sum_{i=1}^{n} p_i|a_i-\bar{a}|\right)^{1/2}\left(\sum_{i=1}^{n} p_i|b_i-\bar{b}|\right)^{1/2} \\
&\leq (M_1-m_1)^{1/2}(M_2-m_2)^{1/2}\left(\sum_{1\leq i<j\leq n} p_ip_j|a_i-a_j|\right)^{1/2} \\
&\qquad\qquad\qquad\qquad\qquad \times\left(\sum_{1\leq i<j\leq n} p_ip_j|a_i-a_j|\right)^{1/2} \\
&\leq (M_1-m_1)(M_2-m_2)\max_{J\subset I_n} P(J)(1-P(J)).
\end{aligned} \tag{2.6}$$

Proof. For (2.5), the first inequality is nothing but the one in Theorem E. For the second inequality, note that by the triangular inequality we have

$$\begin{aligned}
\sum_{i=1}^{n} p_i|a_i-\bar{a}| &= \sum_{i=1}^{n} p_i\left|\sum_{j=1}^{n} p_j(a_i-a_j)\right| \\
&\leq 2\sum_{1\leq i<j\leq n} p_ip_j|a_i-a_j|.
\end{aligned} \tag{2.7}$$

Hence we have the desired inequality. For the third inequality, we obtain it from (1.6) in Lemma C. (The last inequality is obtained easily.)

Next for (2.6), we have, by Lemma 2.1 and the first inequality of (2.5) (or Theorem E),

$$\begin{aligned}
&|T(a,b;p)| \leq T(a,a;p)^{1/2}T(b,b;p)^{1/2} \\
&\leq \left\{\frac{1}{2}(M_1-m_1)\left(\sum_{i=1}^{n} p_i|a_i-\bar{a}|\right)\right\}^{1/2}\left\{\frac{1}{2}(M_2-m_2)\left(\sum_{i=1}^{n} p_i|b_i-\bar{b}|\right)\right\}^{1/2}.
\end{aligned} \tag{2.8}$$

This shows the first inequality. For the second inequality, we obtain, in (2.7), that (for a)

$$\sum_{i=1}^{n} p_i|a_i-\bar{a}| \leq 2\sum_{1\leq i<j\leq n} p_ip_j|a_i-a_j|.$$

Similarly we have, for b,

$$\sum_{i=1}^{n} p_i|b_i - \bar{b}| \leq 2 \sum_{1\leq i<j\leq n} p_i p_j |b_i - b_j|.$$

Hence we have the desired inequality. For the third inequality, we obtain, from (1.6) in Lemma C,

$$\sum_{1\leq i<j\leq n} p_i p_j |a_i - a_j| \leq (M_1 - m_1) \max_{J\subset I_n} P(J)(1 - P(J)),$$

and similarly for b

$$\sum_{1\leq i<j\leq n} p_i p_j |b_i - b_j| \leq (M_2 - m_2) \max_{J\subset I_n} P(J)(1 - P(J)).$$

Hence we can obtain the desired inequality. □

Applying Cauchy's inequality, we have a variant of (2.5) in the above theorem:

Theorem 2.3. *With the same assumptions as in Theorem* 2.2*,*

$$\begin{aligned}
&|T(a,b;p)| \\
&\quad \leq \frac{M_2 - m_2}{2} \sum_{i=1}^{n} p_i |a_i - \bar{a}| \\
&\quad \leq \frac{M_2 - m_2}{2} \left(\sum_{i=1}^{n} p_i (a_i - \bar{a})^2 \right)^{1/2} \\
&\quad \leq \frac{1}{\sqrt{2}} (M_2 - m_2) \left(\sum_{1\leq i<j\leq n} p_i p_j (a_i - a_j)^2 \right)^{1/2} \\
&\quad \leq \frac{1}{\sqrt{2}} (M_1 - m_1)(M_2 - m_2) \left(\max_{J\subset I_n} P(J)(1 - P(J)) \right)^{1/2} \\
&\qquad\qquad \left(\leq \frac{1}{2\sqrt{2}} (M_1 - m_1)(M_2 - m_2) \right).
\end{aligned} \tag{2.9}$$

Proof. The second inequality is obtained from Cauchy's inequality. For the third and the fourth inequalities, using Cauchy's inequality again and the fact $|a_i - a_j| \le M_1 - m_1$, we have

$$\begin{aligned}\sum_{i=1}^{n} p_i(a_i - \bar{a})^2 &\le 2 \sum_{1\le i<j\le n} p_i p_j (a_i - a_j)^2 \\ &\le 2(M_1 - m_1) \sum_{1\le i<j\le n} p_i p_j |a_i - a_j|.\end{aligned} \tag{2.10}$$

Hence, we obtain the desired inequalities (by applying (1.6) in Lemma C to the last term of (2.10)). □

Now we have a refinement of Theorem A from the above theorems and (1.7).

Corollary 2.4. *If we, in Theorems* 2.2 *and* 2.3*, add the assumption that* a *is nonincreasing, then we can replace* $\max\limits_{J\subset I_n} P(J)(1-P(J)$ *by* $\max\limits_{1\le k\le n-1} P_k(1-P_k)$ *in the theorems.*

Applying Čebyšev's inequality, we have:

Corollary 2.5. *If we assume, in Corollary* 2.4*, that* b *is also nonincreasing, then we can further replace* $|T(a,b;p)|$ *by* $T(a,b;p)(\ge 0)$ *in the theorems.*

3. Integral-type Grüss' inequalities

To show an integral analogue of Grüss' inequality considered in the preceding section, we define the Lebesgue space $L_\mu(\Omega)$ for a finite positive measure μ on Ω by

$$L_\mu(\Omega) = \left\{ f; f \text{ is } \mu - \text{measurable and } \int_\Omega |f(x)| d\mu(x) < \infty \right\}$$

(in a little more general setting than $L_w(\Omega,\mu)$ defined before). For convenience sake, we always assume that

$$\int_\Omega d\mu(x) = \mu(\Omega) = 1.$$

Now we define $T(f,g;\mu)$ for $f,\ g,\ fg \in L_\mu(\Omega)$, by

$$T(f,g;\mu) = \int_\Omega f(x)g(x)d\mu(x) - \int_\Omega f(x)d\mu(x)\int_\Omega g(x)d\mu(x). \tag{3.1}$$

Corresponding to (2.3), we can obtain the representation of $T(f,g;\mu)$ as follows:

$$T(f,g;\mu) = \frac{1}{2}\int_\Omega\int_\Omega (f(x)-f(y))(g(x)-g(y))d\mu(x)d\mu(y). \tag{3.2}$$

Integral-type Čebyšev's inequality [7, p. 273]:

$$T(f,g;\mu) \ge 0 \ \text{ or } \ \int_\Omega f(x)g(x)d\mu(x) \ge \int_\Omega f(x)d\mu(x)\int_\Omega g(x)d\mu(x) \tag{3.3}$$

is then induced from the condition that

$$(f(x)-f(y))(g(x)-g(y)) \ge 0 \ \text{ for } \ x,\ y \in \Omega. \tag{3.4}$$

(This is the case, say, if Ω is an interval of the real line R and both $f,\ g$ are nonincreasing (or nondecreasing.)

From (3.2), we can obtain the following inequality corresponding to Lemma 2.1.

Lemma 3.1 ([7, p. 296]). *For* $f,\ g \in L^2_\mu(\Omega)\ \left(= \{f; f^2 \in L_\mu(\Omega)\}\right)$,

$$|T(f,g;\mu)| \le T(f,f;\mu)^{1/2}T(g,g;\mu)^{1/2}. \tag{3.5}$$

The following result is an integral version of Theorem 2.2 and it extends Theorem F and also [6, Theorem 3.1]:

Theorem 3.2. *Let* $f,g \in L_\mu(\Omega)$ *(or* $L^2_\mu(\Omega)$*), and let*

$$m_1 \le f(x) \le M_1 \ \text{ and } \ m_2 \le g(x) \le M_2 \ \ (x \in \Omega) \tag{3.6}$$

for some constants $m_i < M_i$ $(i = 1,2)$. *Then*

$$\begin{aligned} |T(f,g;\mu)| &\le \frac{M_2-m_2}{2}\int_\Omega |f(x)-\bar f|d\mu(x) \quad \left(\bar f = \int_\Omega f(x)d\mu(x)\right) \\ &\le \frac{M_2-m_2}{2}\int_\Omega\int_\Omega |f(x)-f(y)|d\mu(x)d\mu(y) \\ &\le \frac{M_2-m_2}{2}\left\{\int_\Omega\int_\Omega |f(x)-f(y)|^2 d\mu(x)d\mu(y)\right\}^{1/2} \\ &\qquad \left(= \frac{M_2-m_2}{2}\left\{\int_\Omega |f(x)-\bar f|^2 d\mu(x)\right\}^{1/2}\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&|T(f,g;\mu)|\\
&\le \frac{1}{2}(M_1-m_1)^{1/2}(M_2-m_2)^{1/2}\\
&\qquad\quad \times\left(\int_\Omega |f(x)-\bar f|d\mu(x)\right)^{1/2}\left(\int_\Omega |g(x)-\bar g|d\mu(x)\right)^{1/2}\\
&\le \frac{1}{2}(M_1-m_1)^{1/2}(M_2-m_2)^{1/2}\left(\int_\Omega\int_\Omega |f(x)-f(y)|d\mu(x)d\mu(y)\right)^{1/2}\\
&\qquad\qquad\qquad\qquad \times\left(\int_\Omega\int_\Omega |g(x)-g(y)|d\mu(x)d\mu(y)\right)^{1/2}\\
&\left(\le (M_1-m_1)(M_2-m_2)\sup_{E\subset\Omega}\mu(E)(1-\mu(E)),\quad \textit{if } \Omega \textit{ is locally compact}\right).
\end{aligned}\tag{3.8}$$

Proof. It follows from (3.1) that

$$T(f,g;\mu)=\int_\Omega (f(x)-\bar f)g(x)d\mu(x). \tag{3.9}$$

First for (3.7), we see, from (3.9), that

$$\begin{aligned}
|T(f,g;\mu)| \le &\left|\int_\Omega (f(x)-\bar f)\left(g(x)-\frac{M_2+m_2}{2}\right)d\mu(x)\right|\\
&+\left|\int_\Omega (f(x)-\bar f)\left(\frac{M_2+m_2}{2}\right)d\mu(x)\right|.
\end{aligned}$$

Since

$$\left|g(x)-\frac{M_2+m_2}{2}\right|\le \frac{M_2-m_2}{2} \quad \text{and} \quad \int_\Omega (f(x)-\bar f)d\mu(x)=0,$$

we have

$$|T(f,g;\mu)|\le \frac{M_2-m_2}{2}\int_\Omega |f(x)-\bar f|d\mu(x),$$

which is the first inequality. To see the second and the third inequalities, we have only to notice that

$$\begin{aligned}
\int_\Omega |f(x)-\bar f|d\mu(x) &= \int_\Omega \left|\int_\Omega (f(x)-f(y))d\mu(y)\right|d\mu(x)\\
&\le \int_\Omega\int_\Omega |f(x)-f(y)|d\mu(y)d\mu(x) \qquad (3.10)\\
&\le \left\{\int_\Omega\int_\Omega (f(x)-f(y))^2 d\mu(y)d\mu(x)\right\}^{1/2}.
\end{aligned}$$

For the identity after the third inequality, we can obtain it by an elementary computation.

Next for (3.8), by Lemma 3.1 and the first inequality of (3.7), we have

$$\begin{aligned}|T(f,g;\mu)| &\le T(f,f;\mu)^{1/2}T(g,g;\mu)^{1/2}\\ &\le \frac{1}{2}(M_1-m_1)^{1/2}(M_2-m_2)^{1/2}\left(\int_\Omega |f(x)-\bar{f}|d\mu(x)\right)^{1/2}\\ &\qquad\times\left(\int_\Omega |g(x)-\bar{g}|d\mu(x)\right)^{1/2},\end{aligned} \tag{3.11}$$

which is the first inequality. The second inequality is obvious from the first one of (3.10). Now if Ω is locally compact, then we can assume that f and g are continuous. Put, for f,

$$I_f = \int_\Omega\int_\Omega |f(x)-f(y)|d\mu(y)d\mu(x).$$

Then I_f is approximated by the sum

$$\sum_{i,j=1}^n |f(x_i)-f(x_j)|\mu(E_i)\mu(E_j) = 2\sum_{1\le i<j\le n}|f(x_i)-f(x_j)|\mu(E_i)\mu(E_j)$$

with respect to a decomposition of measurable sets $\{E_i\}$, $x_i \in E_i$ $(i = 1,...,n)$ in Ω. By (1.6) in Lemma C, we can see that

$$\sum_{1\le i<j\le n}|f(x_i)-f(x_j)|\mu(E_i)\mu(E_j) \le (M_1-m_1)\sup_{E\subset\Omega}\ \mu(E)(1-\mu(E)),$$

so that

$$I_f \le (M_1-m_1)\sup_{E\subset\Omega}\ \mu(E)(1-\mu(E)).$$

Similarly we have

$$I_g := \int_\Omega\int_\Omega |f(x)-f(y)|d\mu(y)d\mu(x) \le (M_2-m_2)\sup_{E\subset\Omega}\ \mu(E)(1-\mu(E)).$$

Hence we can deduce the third inequality from the second one. □

Corollary 3.3. *With the same assumptions as in Theorem* 3.2 *and the additional condition* (3.4), *we can replace* $|T(f,g;\mu)|$ *by* $T(f,g;\mu)$ (≥ 0).

References

[1] D. Andrica and C. Badea, Grüss' inequality for positive linear functionals, *Period. Math. Hungar.*, **19 (2)** (1988), 155-167.

[2] P. Cerone and S. S. Dragomir, A refinement of Grüss inequality and applications, *RGMIA Research Report Collection*, **5 (2)** (2000), Article 14.

[3] X. L. Cheng and J. Sun, A note on the perturbed trapezoid inequality, *J. Inequal. in Pure and Applied Math.*, **3 (2)** (2002), Article 29.

[4] G. Grüss, Über des Maximum des absoluten Betragen von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.*, **39** (1935), 215-226.

[5] S. Izumino and J. E. Pečarić, Some extensions of Grüss' inequality and its applications, *Nihonkai Math. J.*, **13** (2002), 159-166.

[6] M. Matić, Improvement of some estimation related to the remainder on generalized Taylor's formula, *Math. Inequal. Appl.*, **5** (2002), 637-648.

[7] D. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Boston, London, 1993.

[8] J. E. Pečarić, On an inequality of T. Popovicui, *Bull. Şti. Tehn. Inst. Politehn. Temişoara* **2, 24 (38)** (1979), 9-15.

[9] J. E. Pečarić and B. Tepeš, Improvements of some inequalities for moments of guessing functions (to appear).

[10] I. G. Sokolov, Improvement of an inequality of P. L. Čebyšev, *Teoretična i prikladna Matematika, vip.* **II** (1963), 19-21 (in Russian).

* Faculty of Education
Toyama University
Gofuku, Toyama 930-8555
JAPAN
email: s-izumino@h5.dion.ne.jp

** Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb
Croatia
email: pecaric@mahazu.hazu.hr

† Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb
Croatia
email: btepes@ffzg.hr

(Received August 22, 2003)