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NON-INVARIANT TWO DIMENSIONAL AFFINE DOMAINS

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Introduction

A commutative ring A with identity is said to be invariant if whenever B is a ring such that two polynomial rings $A[X_1,\dots,X_n]$ and $B[Y_1,\dots,Y_n]$ in n variables are isomorphic, then A and B are isomorphic. In their work [1], Abhyankar, Eakin and Heinzer proved that an integral domain of transcendence degree one over a field is invariant, and asked whether two dimensional affine domains over a field are invariant. The purpose of the present paper is to study the invariance on some two dimensional affine domains over a field.

In section 1 we study the invariance on R[Z], where R is a one dimensional affine domain over a field k and Z is an indeterminate. In particular, when k is a perfect field, we give a complete description on a commutative ring B which satisfies $R[Z,X_1,\dots,X_n] \cong B[Y_1,\dots,Y_n]$.

In section 2 we construct a non-invariant two dimensional affine domain over a field of characteristic zero.

1. Invariance on R[Z]

We first consider the invariance on two dimensional affine domains of the form R[Z] where Z is an indeterminate.

1.1. We recall the defintion of F-rings from [2]. Let $R \subset S$ be a reduced ring extension. We say that R is F-closed in S if any element $t \in S$ such that $t^2, t^3, nt \in R$ for some positive integer n (n may depend on t) is always contained in R. If a reduced ring R is F-closed in any reduced ring extension $R \subset S$, then R is called an F-ring. It should be noticed that, in the case where a reduced ring R contains only a finite number of minimal prime ideals, R is an F-ring if and only if R is F-closed in its total quotient ring Q(R). In this case there is the smallest F-ring in Q(R) containing R, which is denoted by F(R).

1.2. If a reduced ring R contains a field k of characteristic $p \geqq 0$, then the following two conditions are equivalent to each other:

(i) R is an F-ring.

(ii) R is seminormal or $p = 0$.

For the proof see [9].

1.3. An integral domain R is said to be *strongly invariant* if the equality

$$R[X_1,\dots,X_n] = B[Y_1,\dots,Y_n]$$

of polynomial rings always implies the equality $R = B$ of coefficient rings R and B.

1.4. *Theorem. Let R be a strongly invariant integral domain. Then a polynomial ring $R[Z]$ in one variable is invariant if and only if R is an F-ring.*

Proof. Suppose $R[Z,X_1,\dots,X_n] = B[Y_1,\dots,Y_n]$. Then B contains R as a subring by [5]. First we claim that B is isomorphic to $R[Z]$ if and only if B is R-isomorphic to $R[Z]$. It is sufficient to show the "only if" part of the claim. Let assume B is isomorphic to $R[Z]$. Then B is a polynomial ring $S[T]$ in one variable T over a coefficient ring S which is isomorphic to R. Replacing B by $S[T]$, we get

$$R[Z,X_1,\dots,X_n] = S[T,Y_1,\dots,Y_n],$$

and hence we have $R = S$, which means that B is R-isomorphic to $R[Z]$. With this claim, we see that our assertion is an immediate consequence of [2].

1.5. *Remark*. We can not extend Theorem 1.4 to the case of a polynomial ring in two variables as follows: Let R be a strongly invariant integral domain. Suppose R does not contain a field of characteristic zero. Then a polynomial ring $R[Z_1, Z_2]$ in two variables is not invariant by [4].

1.6. Corollary. *Let R be a one dimensional affine domain over a field k of characteristic $p > 0$. Suppose k is algebraically closed in R. If R is not a polynomial ring over k, then the following conditions are equivalent:*

(i) $R[Z, X_1, \ldots, X_n] \cong B[Y_1, \ldots, Y_n]$ *for some* $n>1$,

(ii) $R[Z, X_1] \cong B[Y_1]$,

(iii) $B \cong R[Z^q, Z+a_1 Z^p + a_2 Z^{2p} + \ldots + a_s Z^{sp}]$,
where $q = p^e$ *and* $a_i \in F(R)$ $(i = 1, \ldots, s)$ *for some non-negative integers* e *and* s. *In particular* $B \cong R[Z]$ *if and only if* $a_i \in R$.

Proof. According to [1] R is strongly invariant. Thus the corollary is an immediate consequence of Theorem 1.4 and [3].

1.7. Remark. If k is a perfect field, then a polynomial ring $k[X, Y]$ in two variables is invariant [6,8]. Therefore, in this case, a two dimensional k-affine domain of the form $R[Z]$ is invariant if and only if R is an F-ring.

2. Non-invariant affine domains

This section is devoted to give an example of non-invariant two dimensional affine domains over a field k of characteristic zero.

2.1. Theorem. Let k be a field of characteristic zero. We set

$$A = k[X, Y+Y^3, (X-1)(X-2)Y, (X-1)(X-2)Y^2],$$

$$B = k[X, XY+X^3Y^3, (X-1)(X-2)Y, (X-1)(X-2)Y^2],$$

where X, Y are indeterminates. Then we have

(i) $A[Z] \cong B[Z]$,

(ii) $A \not\cong B$.

Proof. We begin with some remarks on A and B. Put $F = (X-1)(X-2)$. Since F and X are coprime to each other in $k[X]$, we can choose $f, g \in k[X]$ such that $X^3f+F^3g = 1$. Thus

$$\begin{aligned} XYf+Y^3 &= XYf+(X^3f+F^3g)Y^3 \\ &= (XY+X^3Y^3)f+(FY)^3g \in B, \end{aligned}$$

which shows that Y is integral over B. Also, Y is integral over A because $Y+Y^3 \in A$. Therefore if we denote by A' and B' the integral closures of A and B respectively in the quotient field $Q(A) = Q(B) = k(x,y)$, then we have $A' = B' = k[X,Y]$. Notice that the principal ideal $(F) = Fk[X,Y]$ of $k[X,Y]$ is also an ideal of both A and B. Put

$$k[x,y,z] = k[X,Y,Z]/Fk[X,Y,Z],$$

where x, y and z are the residue classes modulo $Fk[X,Y,Z]$ of X, Y and Z respectively. Then $k[x,y,z]$ is a polynomial ring in two variables y and z over a reduced Artin ring $k[x]$ defined by $(x-1)(x-2) = 0$. By the canonical map we regard $A/(F)$ and $A'/(F)$ as subrings of $k[x,y,z]$. So if we write $c(R'/R)$ for the conductor of an integral ring extension $R \subset R'$, then

$$c((A'/(F))/(A/(F))) = 0,$$

and hence we have $c(A'/A)=(F)$. A similar argument can be applied to B to get $c(B'/B) =(F)$. Furthermore it is easy to see that A and B have the forms

$$A = k[X, Y+Y^3]+(F) \text{ and } B = k[X, XY+X^3Y^3]+(F).$$

Now we shall prove (i). Put $G = (-X+3)/2$, then

$$GX \equiv 1 \pmod{(F)}.$$

Let us denote by H^* the image of any element $H \in k[X,Y,Z]$ under the natural homomorphism $k[X,Y,Z] \to k[x,y,z]$. Thus $G^* = x^{-1}$. If we define a 2×2 matrix (G_{ij}) by

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -G & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then (G_{ij}) is invertible in $k[X]$ and

$$\begin{pmatrix} G_{11}^* & G_{12}^* \\ G_{21}^* & G_{22}^* \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}.$$

So the $k[X]$-automorphism

$$\Phi : \begin{pmatrix} Y \\ Z \end{pmatrix} \to \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix}$$

of $k[X,Y,Z]$ is well defined. Notice that $\Phi(Y)^* = xy$ and $\Phi(Z)^* = x^{-1}z$. The image of $A[Z]$ by Φ is

$$\Phi(A[Z]) = \Phi(k[X,Y+Y^3,Z]) + \Phi(Fk[X,Y,Z])$$

$$= k[X, \Phi(Y)+\Phi(Y)^3, \Phi(Z)] + Fk[X,Y,Z].$$

Therefore

$$\Phi(A[Z])^* = k[x,xy+x^3y^3,x^{-1}z] = k[x,xy+x^3y^3,z].$$

On the other hand we have

$$B[Z]^* = k[x,xy+x^3y^3,z].$$

Thus we have $\Phi(A[Z])^* = B[Z]^*$, and so $A[Z] \cong B[Z]$. This completes the proof of (i). Next we shall prove (ii). Assume that there exists an isomorphism $\Psi : A \cong B$. As is easily seen, Ψ can be extended to an isomorphism $A' \cong B'$ of integral closures of A and B, i.e., Ψ may be considered as a k-automorphism of $k[X,Y]$ satisfying the condition $\Psi(A) = B$. Now (F) is a conductor of both integral extensions A'/A and B'/B. Since (F) is stable under the automorphism Ψ, it follows that $\Psi(F) = aF$ for some $a \in k - (0)$. This shows that $\Psi(X) \in k[X]$, and hence

$$k[X,Y] = k[\Psi(X), \Psi(Y)] = k[X,\Psi(Y)].$$

So $\Psi(Y)$ is written in the form $\Psi(Y) = bY+H$ where $b \in k-(0)$ and $H \in k[X]$. Recall that $\Psi(A)^* = B^*$ is well defined as a subring of $k[x,y]$ $(\subset K[x,y,z])$. Thus we have

$$k[\Psi(X)^*, \Psi(Y)^* + \Psi(Y)^{*3}] = k[x, xy + x^3y^3].$$

The element $xy+x^3y^3$ is algebraically independent over a reduced ring $k[x] = k[\Psi(X)^*]$, so that $\Psi(Y)^*+\Psi(Y)^{*3}$ is of the form

$$\Psi(Y)^* + \Psi(Y)^{*3} = (by+H^*)+(by+H^*)^3 = P(xy+x^3y^3)+Q,$$

where P is a unit of k[x] and Q is an elememt of k[x]. It follows from easy calculation that

$$by+b^3y^3 = Pxy+Px^3y^3,$$

and so $b = Px$, $b^3 = Px^3$, which imply that $X \equiv b^2 \pmod{(F)}$ in $k[X]$, a contradiction.

2.2. Remark. In [7], Hochster noticed that the polynomial ring in two variables over the coordinate ring of a real two sphere is not invariant. The ring of his example is a four dimensional regular affine domain over the real field. On the other hand, by Remark 1.5 if k is a field of positive characteristic, then $k[X,X^{-1},Y,Z]$ is not invariant. This example is a regular three dimensional affine domain. We do not know whether there exist non-invariant regular affine k-domains of dimension two.

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