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A CONSTRUCTION OF EINSTEIN METRICS BY WARPED PRODUCT

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We construct Einstein metrics by means of warped product and study their properties.

1. Warped product.

We review some properties of warped product([1],[2]). Let (B,g) and $(F,\bar{g})$ be the Riemannian manifolds and f a positive C^∞-function on B. Consider the product manifold $B\times F$ with projections $\pi; B\times F \longrightarrow B$ and $\mu; B\times F \longrightarrow F$. The warped product $B\times_f F$ is the Riemannian manifold $(B\times F, \tilde{g})$ defined by

$$\tilde{g}(X,Y) = g(\pi_* X, \pi_* Y) + f(\pi(x))^2 \bar{g}(\mu_* X, \mu_* Y) , \tag{1.1}$$

for all tangent vectors $X, Y \in T_x(B\times F)$. Let us identify $T_x(B\times F)$ with $T_{\pi(x)}(B) + T_{\mu(x)}(F)$. Note that for $p \in F$, $\mu^{-1}(p)$ is totally geodesic in $B\times_f F$ and $\pi_{|\mu^{-1}}; \mu^{-1}(p) \longrightarrow B$

is an isometry(cf. [1]). We consider the curvature tensor $\tilde{R}$ of the warped product $B\times_f F$ with $\dim B = 1$. Let F be an n-dimensional Riemannian manifold and f a positive C^∞-function on B. Let $\{d/dt, e_1, \ldots, e_n\}$ be an orthonormal frame field of $B\times_f F$, where d/dt is a unit vector field on B and e_i a vector field on F. Then we have the following two lemmas(cf. [2]), putting $f'=df/dt$ and $f'' = d^2f/dt^2$.

LEMMA 1.1.

(1) $$\tilde{g}(\tilde{R}(e_a,e_b)e_c,e_d) = (1/f)^2\bar{g}(\bar{R}(fe_a,fe_b)fe_c,fe_d) - (f'/f)^2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}),$$

(2) $$\tilde{g}(\tilde{R}(e_a,e_b)e_c,d/dt) = 0,$$

(3) $$\tilde{g}(\tilde{R}(d/dt,e_a)d/dt,e_b) = -(f''/f)\delta_{ab},$$

where $\bar{R}$ *is the curvature tensor of* F *and* $1\le a,b,c,d\le n$.

LEMMA 1.2. *Let* $\bar{S}$ *and* $\tilde{S}$ *be the scalar curvature of* F *and* $B\times_f F$ *respectively. Then we have*

$$\tilde{S} = -2n(f''/f) - n(n-1)(f'/f)^2 + (1/f)^2\bar{S}.$$

Let $\bar{R}_1$ and $\tilde{R}_1$ be the Ricci tensors of F and $B\times_f F$ respectively. Then by Lemma 1.1, we have the following

LEMMA 1.3.

(1) $$\tilde{R}_1(e_b,e_d) = (1/f)^2\bar{R}_1(fe_b,fe_d) - ((n-1)(f'/f)^2 + (f''/f))\delta_{bd},$$

(2) $\tilde{R}_1(e_a, d/dt) = 0$,

(3) $\tilde{R}_1(d/dt, d/dt) = -n(f''/f)$.

2. Results.

THEOREM 2.1. *Let* M *be a Riemannian manifold of dimension* $n>2$ *and* f *a positive* C^∞*-function on* 1*-dimensional Riemannian manifold. A necessary and sufficient condition for a warped product* $B\times_f M$ *to be Einsteinian is that* M *is Einsteinian and* f *satisfies*

$$(*) \qquad (f')^2 - ff'' = \bar{S}/n(n-1) .$$

PROOF. Suppose that $B\times_f M$ is Einsteinian. Then Lemma 1.3 implies that

$$\tilde{R}_1(e_b, e_b) = \tilde{R}_1(d/dt, d/dt) = -nf''/f ,$$

and

$$\bar{R}_1(fe_b, fe_b) = (n-1)((f')^2 - ff'') .$$

On the other hand, we have (*) by using Lemma 1.2, because

$$\tilde{S}/(n+1) = \tilde{R}_1(d/dt, d/dt) = -nf''/f .$$

Therefore we have

$$\bar{R}_1(fe_b, fe_b) = \bar{S}/n .$$

This shows that M is Einsteinian.

Conversely suppose that M is an Einstein manifold satisfying (*) . Then since

$$\bar{R}_1(fe_b, fe_d) = (\bar{S}/n)\delta_{bd}$$
$$= (n-1)((f')^2 - ff'')\delta_{bd} ,$$

we have

$$\tilde{R}_1(e_b, e_b) = \tilde{R}_1(d/dt, d/dt) ,$$

taking account of (3) of Lemma 1.3. This concludes that $B\times_f M$ is Einsteinian.

Let E be the 1-dimensional Euclidean space with standard Riemannian metric and (M,g) an n-dimensional Einstein manifold with non positive scalar curvature S. Let f be a positive C^∞-function on E and consider the warped product $E\times_f M$. Now putting $k=S/n(n-1)$, we consider only the case $S\leq 0$. Because if $S>0$, then we cannot have the solution of differential equation (*) such that $f>0$ on E . Then by an elementary computation, we have

$$(**) \qquad f = (1/2)(\beta\exp(\alpha t) - (k/\alpha^2\beta)\exp(-\alpha t)) ,$$

where α is an arbitrary constant and β an arbitrary positive constant.

If M is complete, then by a result of R. L. Bishop and B. O'Neill(see [1]), we have the following

THEOREM 2.2. *If* M *is a complete Einstein manifold with non positive scalar curvature, then a warped product* $E\times_f M$ *with* (**) *is a complete Einstein manifold with negative scalar curvature.*

Making use of Theorem 2.2 inductively, we have

COROLLARY 2.3. *If* M *is a complete Einstein manifold with non positive scalar curvature, then the product manifold* $E^m \times M$ *admits complete Einstein metrics with negative scalar curvature.*

Moreover suppose that M is an Einstein manifold with negative curvature. Then from Lemma 1.1 and (**), we can see that $E \times_f M$ is of negative curvature. Thus we have

THEOREM 2.4. *If* M *is a complete Einstein manifold with negative curvature, then the warped product* $E \times_f M$ *satisfying* (**) *is a complete Einstein manifold with negative curvature.*

Making use of Theorem 2.4 inductively, we have

COROLLARY 2.5. *If* M *is a complete Einstein manifold with negative curvature, then the product manifold* $E^m \times M$ *admits complete Einstein metrics with negative curvature.*

EXAMPLES. Let M be an irreducible Riemannian globally symmetric space of non compact type. Then, since M is an Einstein manifold of negative curvature, we find from Theorem 2.4 that the product manifold $E \times M$ admits complete Einstein metrics with negative curvature, and furthermore from Corollary 2.5 that the product manifold $E^m \times M$ admits complete Einstein metrics with negative curvature.

Now we can compute directly the length of the curvature tensors of the warped product $E \times_f CH^n$ and $E \times_f QH^n$ such that f satisfies (**) , where CH^n is the complex n-dimensional hyperbolic space and HH^n is the quaternion hyperbolic space, because we have already known their explicit expression of the curvature tensors. Then we find various complete non homogeneous Einstein metrics of negative curvature, because we can choose positive constant α and β such that the scalar functions(of the length of curvature tensor) are constant.

REFERENCES

[1] R. L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Amer. Math. Soc., 145(1969), 1-43.

[2] N. Ejiri, A negative answer to a conjecture of conformal transformations of Riemannian manifolds, J. Math. Soc. Japan, 33(1981), 261-266.

[3] S. Ishihara, Notes on quaternion Kählerian manifolds, J. Differential Geometry, 9(1974), 483-500.

[4] Y. Watanabe, On the characteristic functions of quaternion Kählerian spaces of constant Q-sectional curvature, Kōdai Math. Sem. Rep., 28(1977), 284-299.

[5] K. Yano, Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press, (1965).

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