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NOTES ON KÄHLERIAN METRIC ON DOMAINS IN C^n

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0. Introduction.

The metric tensor $g_{\alpha\overline{\beta}}$ of a Kählerian space can be expressed in the form

$$g_{\alpha\overline{\beta}} = \frac{\partial^2 \phi}{\partial z_\alpha \partial \overline{z}_\beta} ,$$

with respect to local coordinates (z_α) , $\alpha=1,\ldots,n$, where $\phi(z,\overline{z})$ is a real valued function of $(z_\alpha,\overline{z}_\alpha)$([4]). Under the assumption that ϕ is a function $f(t)$ of $t = \sum_{\alpha=1}^{n} z_\alpha \overline{z}_\alpha$ on C^n or a subdomain in C^n , S.S.Eum[1] has determined ϕ for the non-flat metric of constant holomorphic curvature, S.Tachibana and R.C.Liu[3] have obtained metrics of vanishing Bochner curvature tensor, and P.F.Klembeck[2] has done a complete Kählerian metric of positive curvature. We follow these methods to study about Kählerian metrics satisfying the following curvature conditions

$$(*) \qquad R_{ij;k;l} - R_{ij;l;k} = 0 ,$$

or

$$(**) \qquad R^i{}_{jkl;m;n} - R^i{}_{jkl;n;m} = 0$$

where (;) denotes the covariant differentiation.

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1. Preliminaries.

We agree to adopt the summation convention and the following ranges of indices throughout the paper:

$$1 \leqq i,j,k,\ldots \leqq 2n,$$

$$1 \leqq \alpha,\beta,\gamma,\ldots \leqq n, \qquad \bar{\alpha} = n+\alpha$$

Consider an n complex dimensional Kählerian space with metric

$$(1.1) \qquad ds^2 = g_{jk}dz_j dz_k ,$$

where (z_α) is a local complex coordinate system and $\bar{z}_\alpha$ (= conjugate of z_α). As the metric is Kählerian, g_{jk} satisfy the following conditions

$$(1.2) \qquad \begin{aligned} &g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0, \\ &g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha} = \overline{g_{\bar{\alpha}\beta}} = \overline{g_{\beta\bar{\alpha}}} , \end{aligned}$$

and (1.1) becomes

$$ds^2 = 2g_{\alpha\overline{\beta}}dz_\alpha d\overline{z}_\beta \ .$$

g^{jk} satisfy the corresponding equations to (1.2). The Christoffel symbols $\left\{ {i \atop j\ k} \right\}$ vanish except

$$(1.3) \qquad \left\{ {\alpha \atop \beta\ \gamma} \right\} = g^{\alpha\overline{\varepsilon}} \frac{\partial g_{\beta\overline{\varepsilon}}}{\partial z_\gamma}$$

and their conjugates. As to the curvature tensor $R^i{}_{jkl}$, only the components of the form $R^\alpha{}_{\beta\gamma\overline{\delta}}$ and $R^\alpha{}_{\beta\overline{\gamma}\delta}$ and their conjugates can be different from zero, and it holds that

$$(1.4) \qquad R^\alpha{}_{\beta\gamma\overline{\delta}} = \frac{\partial \left\{ {\alpha \atop \beta\ \gamma} \right\}}{\partial \overline{z}_\delta} \ .$$

The Ricci tensor $R_{jk} = R^i{}_{jki}$ satisfy

$$R_{\beta\gamma} = R_{\overline{\beta}\overline{\gamma}} = 0 \ ,$$

$$R_{\beta\overline{\gamma}} = R^\alpha{}_{\beta\overline{\gamma}\alpha} = - R^\alpha{}_{\beta\alpha\overline{\gamma}} \ .$$

2. Kählerian metrics satisfying (*).

Let C^n be the complex number space with complex coordinate (z_α), and D^n put C^n or a subdomain in C^n. A real valued function $\phi(z,\overline{z})$ of $(z_\alpha, \overline{z}_\alpha)$ gives a Kählerian metric

$$(2.1) \qquad g_{\alpha\overline{\beta}} = \frac{\partial^2 \phi}{\partial z_\alpha \partial \overline{z}_\beta}$$

to D^n .

Our problem of this section is to study a function $\phi = f(t)$ of $t = \sum_{\alpha=1}^{n} z_\alpha \bar{z}_\alpha$ such that the corresponding Kählerian metric satisfies the curvature condition ($*$). ($*$) implys that

(2.2) $$- R_{sj}R^{s}{}_{ikl} - R_{is}R^{s}{}_{jkl} = 0$$

equivalently,

(2.2)' $$R_{\alpha\bar{\lambda}}R^{\alpha}{}_{\beta\gamma\bar{\delta}} + R_{\beta\bar{\alpha}}R^{\bar{\alpha}}{}_{\bar{\lambda}\gamma\bar{\delta}} = 0 \quad \text{(conj.)} .$$

From (2.1)', we have

(2.3) $$g_{\alpha\bar{\beta}} = f'\delta_{\alpha\beta} + f''\bar{z}_\alpha z_\beta ,$$

where dashes mean differentiation with respect to t . As the metric g_{ji} is positive definite, it holds that

(2.4) $$f' > 0, \qquad f' + tf'' > 0 .$$

Putting $u=f'$, it follows from (1.3) and (2.4) that

(2.5) $$\left\{ {\alpha \atop \beta\ \gamma} \right\} = \frac{u'}{u}(\bar{z}_\beta \delta_{\alpha\gamma} + \bar{z}_\gamma \delta_{\alpha\beta}) + \sigma z_\alpha \bar{z}_\beta \bar{z}_\gamma ,$$

where

(2.6) $$\sigma(t) = \frac{uu'' - 2u'^2}{u(u+tu')} .$$

Some computations and (1.4) show the following equations [1],[3]:

(2.7) $$R^{\alpha}{}_{\beta\gamma\bar{\delta}} = \frac{uu'-u'^2}{u^2} z_\delta(\bar{z}_\beta \delta_{\alpha\gamma} + \bar{z}_\gamma \delta_{\alpha\beta}) + \sigma' z_\alpha \bar{z}_\beta \bar{z}_\gamma z_\delta$$

$$+ \frac{u'}{u}(\delta_{\beta\delta}\delta_{\alpha\gamma} + \delta_{\gamma\delta}\delta_{\alpha\beta}) + \sigma z_\alpha(\bar{z}_\beta\delta_{\gamma\delta} + \bar{z}_\gamma\delta_{\beta\delta}),$$

and

(2.8) $$R_{\beta\bar{\delta}} = \lambda\bar{z}_\beta z_\delta + \mu\delta_{\beta\delta} ,$$

where λ and μ are functions defined by

(2.9) $$\lambda = -\frac{(n+1)(uu''-u'^2)}{u^2} - \sigma' t - \sigma ,$$
$$\mu = -\frac{(n+1)u'}{u} - \sigma t .$$

We can show by using (2.7) and (2.8) that (2.2)' reduces to

$$\frac{\lambda u'}{u} + t\lambda\sigma + \mu\sigma - \frac{(uu''-u'^2)\mu}{u^2}(z_\lambda\bar{z}_\gamma\delta_{\beta\delta} - \bar{z}_\beta z_\delta\delta_{\lambda\gamma}) = 0.$$

Now we assume that $n \geqq 2$ and the above equation holds identically on D^n. Then we know that

(2.10) $$\frac{\lambda u'}{u} + t\lambda\sigma + \mu\sigma - \frac{(uu''-u'^2)\mu}{u^2} = 0 ,$$

because of the similar reasons in [1] and [3]. By (2.6), we can write (2.10) in the form,

(2.11) $$(\lambda u - \mu u')(u' + tu\sigma) = 0 .$$

First we note that if $f(t)$ satisfies $\lambda u - \mu u' = 0$, g_{ij} is Einsteinian. Next it follows from (2.6) that $u' + tu\sigma = 0$ is equivalent to

(2.12) $$tuu'' + uu' - tu'^2 = 0 .$$

We can see that the general solution of (2.12) is given by

$$u = bt^{a} , \tag{2.13}$$

where a and b are any constant. Hence we get

$$u + tu' = b(1+a)t^{a} . \tag{2.14}$$

Thus, we get the following results if $n \geqq 2$.

I. If $f(t)$ gives a Kählerian metric satisfying (2.2) on a subdomain containing the origin 0 in C^{n} , it is Einsteinian.

II. If $f(t)$ gives a Kählerian metric with parallel Ricci tensor on a subdomain containing the origin 0 in C^{n} , it is Einsteinian.

III. If $f(t)$ gives a Kählerian metric satisfying (2.2) on a subdomain in C^{n} not to contain the origin 0 , it is either Einsteinian, or the form

$$g_{\alpha\bar{\beta}} = bt^{a-1}(t\delta_{\alpha\beta} + a\bar{z}_{\alpha}z_{\beta}),$$

where a is a constant satisfying $a > -1$ and b positive constant.

3. Kählerian metrics satisfying ($**$).

Our problem of this section is to determine a function $\phi = f(t)$ of t on D^{n} such that the corresponding Käherian metric satisfies the curvature condition ($**$) . ($**$) implys that

$$R^{s}{}_{jkl}R^{i}{}_{smn} - R^{i}{}_{skl}R^{s}{}_{jmn}$$

(3.1)

$$- R^{i}{}_{jsl}R^{s}{}_{kmn} - R^{i}{}_{jks}R^{s}{}_{lmn} = 0 ,$$

equivalently,

$$R^{\varepsilon}{}_{\beta\gamma\bar{\delta}}R^{\alpha}{}_{\varepsilon\lambda\bar{\mu}} - R^{\alpha}{}_{\varepsilon\gamma\bar{\delta}}R^{\varepsilon}{}_{\beta\lambda\bar{\mu}}$$

(3.1)'

$$- R^{\alpha}{}_{\beta\varepsilon\bar{\delta}}R^{\varepsilon}{}_{\gamma\lambda\bar{\mu}} - R^{\alpha}{}_{\beta\gamma\bar{\varepsilon}}R^{\bar{\varepsilon}}{}_{\bar{\delta}\lambda\bar{\mu}} = 0 \quad \text{(conj.)} .$$

For convenience sake, we shall put

(3.2) $$\xi(t) = uu' + tuu'' - tu'^2 .$$

Substituting (2.7) into (3.1)', we get

$$0 = \frac{u'\sigma'\xi}{u^2(u+tu')} z_\alpha \bar{z}_\beta \bar{z}_\gamma z_\delta \bar{z}_\lambda z_\mu + \frac{\sigma'\xi}{u} \delta_{\alpha\lambda} \bar{z}_\beta \bar{z}_\gamma z_\delta z_\mu$$

$$+ \frac{(-\sigma'u+\sigma u')\xi}{u^2(u+tu')} \delta_{\beta\mu} z_\alpha \bar{z}_\gamma z_\delta \bar{z}_\lambda - \frac{u'\sigma\xi}{u^2(u+tu')} \delta_{\gamma\delta} z_\alpha \bar{z}_\beta \bar{z}_\lambda z_\mu$$

$$+ \frac{(-\sigma'u+\sigma u')\xi}{u^2(u+tu')} \delta_{\gamma\mu} z_\alpha \bar{z}_\beta z_\delta \bar{z}_\lambda + \frac{(\sigma'u-2u\sigma)\xi}{u^2(u+tu')} \delta_{\delta\lambda} z_\alpha \bar{z}_\beta \bar{z}_\gamma z_\mu$$

$$- \frac{\sigma\xi}{u(u+tu')}(\delta_{\beta\delta}\delta_{\gamma\mu} z_\alpha \bar{z}_\lambda + \delta_{\beta\mu}\delta_{\gamma\delta} z_\alpha \bar{z}_\lambda)$$

$$+ \frac{\sigma\xi}{u^2}(\delta_{\alpha\gamma}\delta_{\delta\lambda}\bar{z}_\beta z_\mu + \delta_{\alpha\lambda}\delta_{\gamma\delta}\bar{z}_\beta z_\mu + \delta_{\alpha\beta}\delta_{\delta\lambda}\bar{z}_\gamma z_\mu$$

$$+ \delta_{\alpha\lambda}\delta_{\beta\delta}\bar{z}_\gamma z_\mu - \delta_{\alpha\beta}\delta_{\gamma\mu} z_\delta \bar{z}_\lambda - \delta_{\alpha\beta}\delta_{\gamma\mu} z_\delta \bar{z}_\lambda) .$$

Now we assume that $n \geqq 6$ and the above equation holds identically on D^n. Then we see that

$$\sigma = 0 , \quad \text{or} \quad \xi = 0 ,$$

because of the similar reason in [1] and [4]. The case $\sigma = 0$ reduces to one of S.S.Eum[1], and the case $\xi = 0$ to (2.12). Thus, we get the following results.

I. If $f(t)$ gives a Kählerian metric satisfying (3.1) on a subdomain containing the origin 0 in C^n, it is Fubinian.

II. If $f(t)$ gives a Kählerian metric on a subdomain containing the origin 0 in C^n to be locally symmetric, it is Fubinian.

III. If $f(t)$ gives a Kählerian metric satisfying (3.1) on a subdomain in C^n not to contain the origin, it is either Fubinian, or the form

$$g_{\alpha\overline{\beta}} = bt^{a-1}(t\delta_{\alpha\beta} + a\overline{z}_\alpha z_\beta),$$

where a is a constant satisfying $a > -1$ and b a positive constant.

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