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Remarks on Solutions of a Coupled Semilinear Parabolic System

Tokumori NANBU

1 Introduction

We are interested in the global solution $U(x,t) = (u(x,t), v(x,t))$ of the initial-Dirichlet problem (1.1)-(1.3) for a coupled semilinear parabolic system

$$(1.1)_1 \qquad u_t = \Delta u + f(u,v), \quad u \geq 0 \quad \text{in} \quad Q = \Omega \times R^+,$$

$$(1.1)_2 \qquad v_t = \Delta v + g(u,v), \quad v \geq 0 \quad \text{in} \quad Q = \Omega \times R^+$$

with the initial condition

$$(1.2) \qquad (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega$$

and the boundary condition

$$(1.3) \qquad (u(x,t), v(x,t)) = (0,0) \quad \text{on} \quad \partial\Omega \times R^+.$$

Here Ω is a smoothly bounded domain in $R^N\,(3 \leq N)$, $R^+ = (0,+\infty)$,the functions $u_0(x)$ and $v_0(x)$ are of class $C_0^1(\overline{\Omega})$ and nonnegative in Ω. $f(u,v)$ and $g(u,v)\,(f,g \in C^1(R^+ \times R^+))$ satisfy some conditions which will be given later.

We state some known results on the asymptotic behaviour of solution of the initial-Dirichlet problem (1.1)-(1.3) for a semilinear parabolic system.

Lemma.([PW(p.190, Th.13)]) Let u and v be a pair of functions

$$C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega \times (0,T)})$$

satisfying the inequalities

$$-u_t + a\Delta u \leq \alpha u + \beta v$$

$$-v_t + b\Delta v \leq \gamma u + \delta v$$

in $\Omega \times (0,T)$, where a and b are positive constants and α, β, γ and δ are bounded in $\Omega \times (0,T)$. Suppose further that

$$\beta \leq 0, \qquad \gamma \leq 0.$$

Then the nonnegativity of u and v on P implies the nonnegativity of u and v in $\Omega \times (0,T)$. Here P=$(\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$.

C.S.Kahane considered the existence, uniqueness and asymptotic behaviour for solutions of the initial-boundary value problem

$$(1.4)_1 \qquad u_t = a\Delta u - juv, \quad u \geq 0 \quad \text{in} \quad Q_T = \Omega \times (0,T),$$

$$(1.4)_2 \qquad v_t = b\Delta v - kuv, \quad v \geq 0 \quad \text{in} \quad Q_T = \Omega \times (0,T)$$

with the nonnegative initial condition

$$(1.5) \qquad (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega$$

and the boundary condition

$$(1.6) \qquad u(x,t) = \psi_1(x,t), \quad v(x,t) = \psi_2(x,t) \quad \text{on} \quad \partial\Omega \times (0,T)$$

where $a, b, j,$ and k are positive constants,ψ_1, ψ_2, u_0 and v_0 are nonnegative functions which satisfy $\psi_1(x,0) = u_0(x), \psi_2(x,0) = v_0(x)$ on $\partial\Omega$, and $0 < T \leq \infty$.

C.S.Kahane([KA]) proved the existence of a local solution in time by using Green's function.

Theorem 1.1.([KA]) Let u and v be a pair of functions

$$C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega \times (0,T)})$$

satisfying (1.4)-(1.6).

Then the nonnegativity of u and v on P implies

$$0 \leq u(x,t) \leq \sup_{\mathrm{P}} u, \qquad 0 \leq v(x,t) \leq \sup_{\mathrm{P}} v$$

for $(x,t) \in \Omega \times (0,T)$. Here P=$(\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$.

Theorem 1.2.([KA]) The problem (1.4)-(1.6) has a unique nonnegative solution in $\Omega \times (0,T)$ assuming given nonnegative continuous data prescribed for u and v on $(\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$.

R.Martin posed a problem on the existence and uniform bounds of solution $U = (u,v)$ for the system

$$u_t = \Delta u - uv^\beta, \quad u \geq 0 \quad \text{in} \quad Q = \Omega \times R^+,$$

$$v_t = \Delta v + uv^\beta, \quad v \geq 0 \quad \text{in} \quad Q = \Omega \times R^+$$

with the nonnegative initial condition

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega$$

and the boundary condition

$$u(x,t) = 0, \quad v(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times R^+.$$

Here $\beta \geq 1$,u_0 and v_0 are nonnegative functions in Ω.

K.Masuda([MAS]) extented the Martin's problem to the more general equations. He proved the existence and asymptotic behaviour of solutions of the following system

$$(1.7)_1 \qquad u_t = \Delta u - f(u,v) \qquad \text{in} \quad Q = \Omega \times R^+,$$

$$(1.7)_2 \qquad v_t = \Delta v + g(u,v) \qquad \text{in} \quad Q = \Omega \times R^+$$

with the initial condition

$$(1.8) \qquad (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \qquad \text{in} \quad \Omega$$

and the boundary condition

$$(1.9) \qquad u(x,t) = 0, \quad v(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times R^+.$$

Here Masuda assumed that the functions $f(u,v), g(u,v)$ are non-negative, $f(0,s) = g(s,0) = 0\,(s \in R^+)$ and $f(u,v), g(u,v)$ satisfy some additional conditions: there is a monotonically increasing function $\omega(s)$ $(s \geq 0)$ and a positive constant r with $g(u,v) \leq \omega_1(u)(v + v^r)$, and $g(u,v) \leq \omega_1(u) f(u,v)$ ($(u,v) \in R^+ \times R^+$) .

M.Escobedo and M.A.Herrero ([EH]) considered the following equation

$$u_t = \Delta u + v^p, \quad u \geq 0 \qquad \text{in} \quad Q = \Omega \times (0,T),$$

$$v_t = \Delta v + u^q, \quad v \geq 0 \qquad \text{in} \quad Q = \Omega \times (0,T).$$

Here $p(> 0)$ and $q(> 0)$ are positive constants and $0 < T \leq \infty$.

Recently , N.Bedjaoui and P.Souplet([BS]) considered the existence of the solution (u,v) of the following initial-Dirichlet problem

$$u_t = \Delta u + v^p - au^r, \quad u \geq 0 \qquad \text{in} \quad Q_T = \Omega \times (0,\infty),$$

$$v_t = \Delta v + u^q - bv^s, \quad v \geq 0 \qquad \text{in} \quad Q_T = \Omega \times (0,\infty)$$

with the nonnegative initial conditions and the zero boundary condition.

Pao([PAO]) considered the existence of solution of the following coupled parabolic system

$$(1.10)_1 \qquad u_{1_t} = \Delta u_1 + f_1(x,t,u_1,u_2) \qquad \text{in} \quad Q_T = \Omega \times (0,T),$$

$$(1.10)_2 \qquad u_{2_t} = \Delta u_2 + f_2(x,t,u_1,u_2) \qquad \text{in} \quad Q_T = \Omega \times (0,T)$$

with the initial condition

$$(1.11) \qquad (u_1(x,0), u_2(x,0)) = (u_{10}(x), u_{20}(x)) \quad \text{in} \quad \Omega$$

and the boundary condition

$$(1.12) \qquad u_1(x,t) = u_2(x,t) = 0 \quad on \quad \partial\Omega \times (0,T).$$

Here $0 < T \leq \infty$.The functions $f_1(x,t,u_1,u_2), f_2(x,t,u_1,u_2)$ satisfy some monotone conditions which will be given later.

Pao([PAO]) states the following definitions.

Definition 1 ([PAO]) Let $J_1 \times J_2$ be a bounded subset in R^2. A vector function $(f_1(u_1,u_2), f_2(u_1,u_2))$ is defined in a bounded subset in R^2.

(i) A function $f_k(u_1,u_2)$ is said to be **quasimonotone nonincreasing** if for fixed u_k, the function $f_k(u_1,u_2)$ is nonincreasing in u_j for $j \neq k$.

(ii) A function $f_k(u_1,u_2)$ is said to be **quasimonotone nondecreasing** if for fixed u_k, the function $f_k(u_1,u_2)$ is nondecreasing in u_j for $j \neq k$.

Definition 2 ([PAO]) Let $J_1 \times J_2$ be a bounded subset in R^2.

(i) A function $\boldsymbol{F} = (f_1, f_2)$ is called **quasimonotone nonincreasing** in $J_1 \times J_2$ if both f_1 and f_2 are quasimonotone nonincreasing for $(u_1,u_2) \in J_1 \times J_2$.

(ii) A function $\boldsymbol{F} = (f_1, f_2)$ is called **quasimonotone nondecreasing** in $J_1 \times J_2$ if both f_1 and f_2 are quasimonotone nondecreasing for $(u_1,u_2) \in J_1 \times J_2$.

(iii) A function $\boldsymbol{F} = (f_1, f_2)$ is called **mixed quasimonotone** in $J_1 \times J_2$ if f_1 is quasimonotone nonincreasing and f_2 is quasimonotone nondecreasing for $(u_1,u_2) \in J_1 \times J_2$ (or vice versa).

Pao([PAO]) gives the definition on the ordered upper solution and lower solution for the problem (1.10)-(1.12) as follows:

Definition 3. ([PAO] p.383) A pair of function $\overline{\boldsymbol{u}} = (\overline{u}_1, \overline{u}_2)$, $\underline{\boldsymbol{u}} = (\underline{u}_1, \underline{u}_2)$ in $C(\overline{Q_T}) \cap C^{1,2}(Q_T)$ are called **ordered upper and lower solutions** of (1.10)-(1.12) if they satisfy the relation $\overline{\boldsymbol{u}} \geq \underline{\boldsymbol{u}}$ and

$$\overline{u}_k \geq \psi_k \geq \underline{u}_k \quad (k=1,2) \quad \text{on} \quad \partial\Omega \times (0,T),$$

$$\overline{u}_k(x,0) \geq u_{k0}(x) \geq \underline{u}_k(x,0) \quad (k=1,2) \quad \text{in} \quad \Omega$$

and if

(i)

$$\overline{u}_{1_t} - \Delta\overline{u}_1 - f_1(\overline{u}_1, \overline{u}_2) \geq 0 \geq \underline{u}_{1_t} - \Delta\underline{u}_1 - f_1(\underline{u}_1, \underline{u}_2)$$

$$\overline{u}_{2_t} - \Delta\overline{u}_2 - f_2(\overline{u}_1, \overline{u}_2) \geq 0 \geq \underline{u}_{2_t} - \Delta\underline{u}_2 - f_2(\underline{u}_1, \underline{u}_2)$$

when (f_1, f_2) is quasimonotone nondencreasing,

(ii)

$$\overline{u}_{1_t} - \Delta\overline{u}_1 - f_1(\overline{u}_1, \underline{u}_2) \geq 0 \geq \underline{u}_{1_t} - \Delta\underline{u}_1 - f_1(\underline{u}_1, \overline{u}_2)$$

$$\overline{u}_{2_t} - \Delta\overline{u}_2 - f_2(\underline{u}_1, \overline{u}_2) \geq 0 \geq \underline{u}_{2_t} - \Delta\underline{u}_2 - f_2(\overline{u}_1, \underline{u}_2)$$

when (f_1, f_2) is quasimonotone nonincreasing,
and
(iii)

$$\overline{u}_{1_t} - \Delta\overline{u}_1 - f_1(\overline{u}_1, \underline{u}_2) \geq 0 \geq \underline{u}_{1_t} - \Delta\underline{u}_1 - f_1(\underline{u}_1, \overline{u}_2)$$
$$\overline{u}_{2_t} - \Delta\overline{u}_2 - f_2(\overline{u}_1, \overline{u}_2) \geq 0 \geq \underline{u}_{2_t} - \Delta\underline{u}_2 - f_2(\underline{u}_1, \underline{u}_2)$$

when (f_1, f_2) is mixed quasimonotone.

Let $\overline{\boldsymbol{u}}$, $\underline{\boldsymbol{u}}$ be the lower solution and upper solution.
Define the sector

$$< \overline{\boldsymbol{u}} \, , \underline{\boldsymbol{u}} > \equiv \left\{ (u_1, u_2) \in C(\overline{D}_T) \times C(\overline{D}_T); (\underline{u}_1, \underline{u}_2) \leq (u_1, u_2) \leq (\overline{u}_1, \overline{u}_2) \right\}.$$

Pao([PAO]) assumed that there exist bounded continuous function $K_i \equiv K_i(t, x)$ such that (f_1, f_2) satisfies the Lipshitz condition

$$\text{(HF)} \qquad |f_i(t, x, u_1, u_2) - f_i(t, x, v_1, v_2)| \leq K_i(|u_1 - v_1| + |u_2 - v_2|)$$

for $(u_1, u_2), (v_1, v_2) \in < \overline{\boldsymbol{u}} \, , \, \underline{\boldsymbol{u}} >$ and $(t, x) \in \overline{D_T}\,(i = 1, 2)$.
Pao([PAO]) proved the existence of solution for the problem (1.10)-(1.12).

Theorem 1.3. (Theorem 3.1 of Chapter 8 in [PAO]) Let $(\overline{u}_1, \overline{u}_2)$, $(\underline{u}_1, \underline{u}_2)$ be ordered upper and lower solutions of (1.10)-(1.12) , and let (f_1, f_2) be quasimonotone nondecreasing in $< \overline{\boldsymbol{u}} \, , \underline{\boldsymbol{u}} >$ and satisfy the condition (HF).Then the problem has a unique solution $u = (u_1, u_2)$ in $< \overline{\boldsymbol{u}} \, , \, \underline{\boldsymbol{u}} >$ such that

$$(\underline{u}_1, \underline{u}_2) \leq (u_1, u_2) \leq (\overline{u}_1, \overline{u}_2) \quad \text{in} \quad \overline{D}_T.$$

Theorem 1.4. (Theorem 3.2 of Chapter 8 in [PAO]) Let $(\overline{u}_1, \overline{u}_2)$, $(\underline{u}_1, \underline{u}_2)$ be ordered upper and lower solutions of (1.10)-(1.12) , and let (f_1, f_2) be quasimonotone nonincreasing in $< \overline{\boldsymbol{u}} \, , \, \underline{\boldsymbol{u}} >$ and satisfy the condition (HF).Then the problem has a unique solution $u = (u_1, u_2)$ in $< \overline{\boldsymbol{u}} \, , \underline{\boldsymbol{u}} >$ such that

$$(\underline{u}_1, \underline{u}_2) \leq (u_1, u_2) \leq (\overline{u}_1, \overline{u}_2) \quad \text{in} \quad \overline{D}_T.$$

Theorem 1.5. (Theorem 3.3 of Chapter 8 in [PAO]) Let $(\overline{u}_1, \overline{u}_2)$, $(\underline{u}_1, \underline{u}_2)$ be ordered upper and lower solutions of (1.10)-(1.12) , and let (f_1, f_2) be mixed quasimonotone in $< \overline{\boldsymbol{u}} \, , \underline{\boldsymbol{u}} >$ and satisfy the condition (HF).Then the problem has a unique solution $u = (u_1, u_2)$ in $< \overline{\boldsymbol{u}} \, , \underline{\boldsymbol{u}} >$ such that

$$(\underline{u}_1, \underline{u}_2) \leq (u_1, u_2) \leq (\overline{u}_1, \overline{u}_2) \quad \text{in} \quad \overline{D}_T.$$

2 Semilinear parabolic system of the quasimonotone nonincreasing type

We consider the existence and the asymptotic behaviour of solution $U(x, t) = (u(x, t), v(x, t))$ of the initial-Dirichlet problem for a semilinear parabolic system

$$(2.1)_1 \qquad u_t = \Delta\, u - j\, u^p v^q, \quad u \geq 0 \quad \text{in} \quad Q = \Omega \times R^+,$$

$(2.1)_2$ $$v_t = \Delta v - k\, u^r v^s, \quad v \geq 0 \qquad \text{in} \quad Q = \Omega \times R^+$$

with the initial condition

(2.2) $$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \qquad \text{in} \quad \Omega$$

and the boundary condition

(2.3) $$u(x,t) = 0, \quad v(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times R^+.$$

Here u_0 and v_0 are nonnegative functions which belong to $C_0^1(\overline{\Omega})$, and

$$j > 0, k > 0, p \geq 1, q \geq 1, r \geq 1, s \geq 1$$

are constants.

The case $f(u,v) = -ju^p v^q, g(u,v) = -ku^r v^s$ is quasimonotone nonincreasing.

Theorem 2.1. The problem (2.1)-(2.3) has a unique nonnegative solution in $\Omega \times R^+$, and the solution (u,v) satisfies

$$|u(\cdot,t)|_\infty + |v(\cdot,t)|_\infty \leq C_1 \exp(-\lambda_0 t) \quad (0 < t)$$

where C_1 depends only on u_0, v_0.
Here λ_0 is the smallest eivenvalue of the problem

$$-\Delta\phi_0 = \lambda_0\phi_0, \phi_0(x) > 0 \quad \text{in} \quad \Omega$$

with

$$\phi_0(x) = 0 \quad (x \in \partial\Omega).$$

proof. For suitable constants C_1, C_2 the functions $\overline{u} = (C_1 \exp(-\lambda_0 t\phi_0(x)$, $C_2 \exp(-\lambda_0 t)\phi_0(x))$, $\underline{u} = (0,0)$ are ordered upper and lower solutions for the problem (2.1)-(2.3). The existence of solution of the problem (2.1)-(2.3) is proved by the theorem 1.4. The decay estimate is obtained by the uppersolution method.

3 Semilinear parabolic system of the mixed quasimonotone typ

We consider the existence and the asymptotic behaviour of solution $U(x,t) = (u(x,t), v(x,t))$ of the initial-Dirichlet problem for a quasilinear parabolic system

$(3.1)_1$ $$u_t = \Delta u - j\, u^p v^q - v^\alpha, \quad u \geq 0 \qquad \text{in} \quad Q = \Omega \times R^+,$$

$(3.1)_2$ $$v_t = \Delta v + u^\beta + k\, u^r v^s, \quad v \geq 0 \qquad \text{in} \quad Q = \Omega \times R^+$$

with the initial condition

$$(3.2) \qquad (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{in} \quad \Omega,$$

and the boundary condition

$$(3.3) \qquad u(x,t) = 0, \quad v(x,t) = 0 \quad \text{on} \quad \partial\Omega \times R^+.$$

Here u_0 and v_0 are nonnegative functions which belong to $C_0^1(\overline{\Omega})$,and

$$(\text{H.3.1}) \qquad j > 0, k > 0, \ p \geq 1, q \geq 1, \alpha \geq 1, \beta \geq 1, r \geq 1, \ 1 \leq s < \frac{N+2}{N-2}.$$

We remark that the case $(f,g) = (-j\ u^p v^q - v^\alpha, k\,u^r v^s + u^\beta)$ is mixed quasimonotone .

For $u \in H_0^1(\Omega)$, we define

$$J_0(u) = \frac{1}{2}\|\nabla u\|_2^2 - k\frac{1}{1+s}\|u\|_{1+s}^{1+s}$$

and

$$J_1(u) = \|u\|_2^2 - \|u\|_{1+s}^{1+s}.$$

By Sobolev's Lemma, we can define

$$d = \inf_{u \in H_0^1(\Omega), u \neq 0} \sup_{\lambda \geq 0} J_0(\lambda u) (> 0).$$

Since $s \left(\in [1, \frac{N+2}{N-2})\right)$, we set the potential well set as

$$W = \left\{ u | u \in H_0^1(\Omega), 0 < J_1(u), \ 0 \leq J_0(\lambda u) < d, \text{ for } \lambda \in [0,1] \right\}.$$

Theorem 3.1. Assume that $v_0(x)$ is small enough in the sense of potential well. Under (H.3.1) the problem (3.1)-(3.3) has a unique nonnegative solution in $\Omega \times R^+$, and the solution(u,v) satisfies

$$(3.4) \qquad |u(t)|_\infty \leq C_1 \exp(-\lambda_0 t) \ (0 < t)$$

and

$$(3.5) \qquad \|\nabla v(t)\|_2 \leq C \exp(-\lambda^* t) \ (0 < t).$$

Here λ_0 is the smallest eivenvalue of the problem

$$-\Delta\phi_0 = \lambda_0 \phi_0, \phi_0(x) > 0 \quad \text{in} \quad \Omega$$

with

$$\phi_0(x) = 0 \quad (x \in \partial\Omega).$$

λ^* is a constant which is determined by the given data.

proof. By the standard way (Cf.[LSU]) we can show the existence of a local solution in time for the problem (3.1)-(3.3). Then from $(3.1)_1$ we can obtain

$$(\text{i}) \qquad |u(t)|_\infty \leq C_1 \exp(-\lambda_0 t) \ (0 < t).$$

From $(3.1)_2$, using the estimate (i) and the L^p method (Cf.[NAN1] ,[NAN2] and [NAN3]), we can prove that

$$\|\nabla v(t)\|_2 \leq C \exp(-\lambda^* t) \ (0 < t).$$

4 Semilinear parabolic system of the quasimonotone nondecreasing type

We consider the existence and the asymptotic behaviour of solution $U(x,t) = (u(x,t), v(x,t))$ of the initial- Dirichlet problem for a quasilinear parabolic system

$$(4.1)_1 \qquad u_t = \Delta u + av^p - bu^r, \quad u \geq 0 \qquad \text{in} \quad Q_T = \Omega \times R^+,$$

$$(4.1)_2 \qquad v_t = \Delta v + cu^q - dv^s, \quad v \geq 0 \qquad \text{in} \quad Q_T = \Omega \times R^+$$

with the initial condition

$$(4.2) \qquad (u(x,0), v(x,0)) = (u_0(x), v_0(x)) \qquad \text{in} \quad \Omega$$

and the boundary condition

$$(4.3) \qquad u(x,t) = 0, \quad v(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times R^+.$$

Here u_0 and v_0 are nonnegative functions which belong to $C_0^1(\overline{\Omega})$, and

$$(\text{H.4.1.}) \qquad p \geq 1, q \geq 1, r \geq 1, s \geq 1, a > 0, b > 0, c > 0, d > 0$$

are constants. We assume that

$$(\text{H.4.2}) \qquad 1 \leq pq < rs$$

and

$$(\text{H.4.3}) \qquad 1 \leq pq = rs, \quad b^q d^r \geq b^q d^r.$$

We remark that the case $(f(u,v), g(u,v)) = (av^p - bu^r, cu^q - dv^s)$ is quasimonotone nondecreasing.

Theorem 4.1. Under (H.4.1) ,(H.4.2), the problem (4.1)-(4.3) has a unique nonnegative solution in $\Omega \times (0, \infty)$.

Proof. We can easily construct ordered upper and lower solutions for the problem (4.1)-(4.3). Then by Theorem 1.3 we can prove Theorem4.1.

Theorem 4.2. Under (H.4.1),(H.4.3), the problem (4,1)-(4,3) has a unique nonnegative solution and we have

$$|u(\cdot,t)|_\infty + |v(\cdot,t)|_\infty \leq C\exp(-\lambda_0 t)\,(0 < t).$$

Proof. We can easily construct ordered upper and lower solutions for the problem (4.1)-(4.3). The decay estimate can be proved by the uppersolution method.

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