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Stable Sets of a Strategic Public Good Provision Game

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Abstract

We consider stable sets of a simple and symmetric strategic public good provision game. In particular, we consider stable sets where each one consists of strategy profiles that provide an identical level of the public good. We completely identify the public good provision levels to be supported by the stable sets for each number of players. This identification induces the following two observations. First, the efficient public good provision level is always supported by a stable set. Second, the public good provision levels at the stable sets are no lower than that at the (unique coalition-proof) Nash equilibrium. In fact, the stable sets support strictly higher public good provision levels than that at the Nash equilibrium if there are more than two players. Further, we give a welfare comparison between the stable sets and the Nash equilibrium by employing the coefficient of resource utilization.

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1 Introduction

Game theory includes two contrasting conclusions for a public good provision problem. In the cooperative framework, the players can agree upon the efficient outcomes in the sense that the core is generally nonempty. In fact, the core is large and the unique stable set. (See Champsaur, 1975.) On the other hand, in the noncooperative framework, the players generally fail to agree upon the efficient outcome in the sense that the Nash equilibrium is inefficient. Indeed, the public good is underprovided at the Nash equilibrium. The underprovision and the resulting inefficiency are main concerns in the noncooperative public good provision game. At least interpretationally, the cooperative and the noncooperative games are distinguished whether the binding agreement is allowed or not.[1] In the absence of the binding agreement, the players can unilaterally deviate from an outcome, leaving the other players' strategies unchanged in the noncooperative framework. This paper explores the possibility of more efficient outcomes to be agreed upon without the binding agreement.

To this end, we employ the coalitional contingent threat situation, henceforth CCTS, that is defined by Greenberg (1990) in his book. The CCTS is derived from a strategic form game and describes the following open negotiation among the players. Once a strategy profile is proposed, a coalition of players may openly declare to deviate from the strategy profile by changing their own strategies, provided that the other players stick to the current strategies. Other players come to know this deviation by the open declaration. Then, another coalition can counter to this deviation by declaring to revise their own strategies from the new strategy profile. The possibility of such a counter deviation is the feature of the CCTS that varies from the stan-

[1]See Aumann (1974) for this kind of arguments.

dard strategic form game analysis with the Nash equilibrium and its variants such as the strong Nash equilibrium (Aumann, 1959) and the coalition-proof Nash equilibrium (Bernheim, *et al.*, 1987).[2]

The solution concept for the CCTS is given by the stability notion *a la* von Neumann and Morgernstern (1944), and will be called the stable set in this paper.[3] Namely, the stable set in the CCTS is a set of strategy profiles satisfying the internal and external stability. It can be roughly described as follows. Consider that a stable set and a strategy profile in the stable set are proposed. According to the manner of the CCTS, a coalition of players may declare to revise the strategy profile to make themselves better off. The internal stability requires that the revised strategy profile be outside the stable set. Then, this strategy revision will be countered by another coalition by the external stability. Namely, the coalition will beneficially declare to change their strategies so that the new strategy profile is in the stable set. In this way, the players voluntarily maintain the stable set as an agreement without a binding agreement once the players have accepted the stable set.

We consider the stable sets in a CCTS derived from a strategic public good provision game. The two players case was investigated by Miyakawa (2006). This paper extends his model for an arbitrary finite number of players. Okada and Muto (1998) investigated the stable sets in the symmetric duopoly model, which is technically related to our model. They showed the existence of some complicated stable sets though there were only two players. To avoid such a difficulty, we restrict our attention to one particular class of the stable sets: we consider a class of stable sets, each of which is consisting of strategy

[2]The coalition-proof Nash equilibrium takes a subsequent deviation into account, but it is limited to the deviation by a subcoalition of the deviating coalition.

[3]Originally, the solution concepts in the the theory of social situations are defined as the optimistic stable standard of behavior and the conservative stable standard of behavior, which coincide with each other in the CCTS.

profiles that achieve an identical level of the public good provision. In other words, we consider the range of public good provision levels to be supported by stable sets.

The main purpose of this paper is the complete identification of the public good provision levels to be supported by stable sets. We derive the following observations from this identification. First, the efficient public good provision level is supported by a stable set. Second, the underprovision at the stable sets are no worse than that at the Nash equilibrium, and better than that at the Nash equilibrium when there are more than two players. Indeed, the minimum public good provision level supported by a stable set is close to two thirds of the efficient outcomes, which slightly varies according to the number of players.

The first observation shows that the players can voluntarily maintain the agreement on the efficient public good provision without a binding agreement. The efficiency of the stable set was investigated in some related models in the literature. Okada and Muto (1998) showed that the set of efficient outcomes is a stable set in a Cournot duopoly market, which is technically related to our model. Later, Miyakawa (2006) showed that the efficient provision level is supported by a two-player public good provision game. Therefore, the first observation is an extension of his result for an arbitrary finite number of players. Further, Nakanishi (2001) showed the existence of the stable set with an efficient outcome in a n-player prisoners' dilemma with continuous strategy spaces where only individual declarations of changing strategies are allowed. Hirai (2013) showed that the stable set exists if and only if the strong Nash equilibrium exists in a binary choice model of the public good provision game. Moreover, the stable set coincides with the set of strong Nash equilibria if it exists, and thus the stable set is efficient whenever it exists.

In spite of the first observation, there may exist other inefficient stable sets since multiple stable sets exist in general. The second observation indicates that the stable sets are better than the Nash equilibrium in the sense of the extent of the underprovision. However, each stable set includes a strategy profile where at least one player prefers the Nash equilibrium to it. Therefore, we cannot conclude that the stable sets are more efficient than the Nash equilibrium with respect to the Pareto criterion. Instead, we employ the concept of the coefficient of resource utilization due to Debreu (1951). Then, we show that the strategy profiles included in one stable set has the common coefficient of resource utilization, and our stable sets are at least as efficient as the Nash equilibrium with respect to the coefficient of resource utilization. Indeed, the stable sets are strictly more efficient than the Nash equilibrium when there are more than two players.

At the end of this section, we review some literature that investigated the stable set and its variants in the public good provision problem. In the standard coalitional form game of a public good provision problem, Champsaur (1975) showed that the core is the unique stable set. The analogy for the continuum of players was shown by Einy and Shitovitz (1995) by employing a different framework from the CCTS in the theory of social situation. Further, Shitovitz and Weber (1997) considered the stable set in the continuum players public good provision problem with finite types, where any feasible allocations have the equal-treatment property and a certain crowding effect presents. They showed that the set of equal-treatment Lindahl equilibrium allocations is the unique stable set as well as the core. Hirai (2008) showed the existence and characterized the stable sets in a public good economy with proportional income tax due to Guesnerie and Oddou (1981). These literature at least implicitly assumed the possibility of the binding agreements, while the present paper does not.

In the next section, we define the strategic form game of the public good provision problem. In Section 3, we define the stable set and give a characterization of it. We also state our main results that completely identify the public good provision levels being supported by stable sets. The proofs of the results in this section are relegated to the Appendix. In Section 4, we compare the extents of the underprovision and the inefficiency of the stable set with those of the Nash equilibrium. In the final section, we conclude with some remarks.

2 The public good provision game

For a finite set A, we denote $|A|$ the cardinality of A. For any pair of real numbers a and b with $a < b$, we denote $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$, $]a, b[= \{x \in \mathbb{R} | a < x < b\}$, $[a, b[= \{x \in \mathbb{R} | a \leq x < b\}$, and $]a, b] = \{x \in \mathbb{R} | a < x \leq b\}$.

We consider a simple and symmetric public good provision game in the strategic form. Let $N = \{1, ..., n\}$ be the finite set of players with $n \geq 2$. A nonempty subset of N is called a coalition. Let $\mathcal{N}$ denote the set of coalitions. Each player is endowed with an identical amount $\omega \in \mathbb{R}_{++}$ of the private good.

Each $i \in N$ chooses $x_i \in [0, \omega]$ that he contributes for the public good provision. The set of strategies for each $i \in N$ is denoted by $X_i = [0, \omega]$. For each $S \in \mathcal{N}$, we denote $X_S = \times_{i \in S} X_i$ and $x(T) = \sum_{i \in T} x_i$ for any coalition $T \subset S$ and $x_S \in X_S$. The private good contributed from the players is transformed to the public good by a linear production technology with unit cost 1. Thus, the set of the feasible (public good) provision level is $[0, n\omega]$.

For each $y \in [0, n\omega]$, we denote $X(y) = \{x \in X_N | x(N) = y\}$. Each $i \in N$ has the identical simple preferences relation on X_N that is represented by the payoff function $v_i(x) = (\omega - x_i) \sum_{i \in N} x_i$ for any $x \in X_N$.

In a public good provision game, $x \in X_N$ is said to be efficient if there exists no $x' \in X_N$ such that $v_i(x') \geq v_i(x)$ for all $i \in N$ and a strict inequality holds for at least one player. It is easy to see that if $x(N) = \frac{n\omega}{2}$, then x is efficient. Therefore, we call $\frac{n\omega}{2}$ *the efficient provision level.*[4]

3 The stable set

We introduce the stable set in a strategic form game due to Greenberg (1990). Originally, the stable set is defined as the optimistic (conservative) stable standard of behavior, henceforth OSSB (CSSB) for a CCTS, which coincides with each other in the CCTS. We omit to derive the CCTS and directly define the stable set in a strategic form game. Of course, the stable set in a strategic form game and the OSSB (CSSB) for a CCTS are essentially equivalent.

The stable set in the strategic form game inherits the spirit of von Neumann and Morgernstern (1944): The stable set is a set of strategy profiles such that its elements do not dominate each other (the internal stability) and any strategy profile outside the set is dominated by some strategy profile in the set (the external stability). To define the dominance relation in the strategic form game, we begin with the definition of the inducibility.

Definition 1 *For any $x, x' \in X_N$ and $S \in \mathcal{N}$, we say x is inducible from x' via S, denoted by $x' \rightarrow_S x$, iff $x_{N \setminus S} = x'_{N \setminus S}$.*

[4]To be precise, there generally exists an efficient $\bar{x} \in X_N$ with $\bar{x}(N) > \frac{n\omega}{2}$. For example, $\bar{x} = (0, \omega, ..., \omega)$ is efficient whereas $\bar{x}(N) > \frac{n\omega}{2}$ when $n \geq 3$. The term "efficient provision level" is used just for simplifying the arguments.

Then, the dominance relation is defined as follows.

Definition 2 *For any $x, x' \in X_N$ and $S \in \mathcal{N}$, we say x dominates x' via S, denoted by $x \succ_S x'$, iff $x' \rightarrow_S x$ and $v_i(x) > v_i(x')$ for all $i \in S$. For any $x, x' \in X_N$, we say x dominates x', denoted by $x \succ x'$ iff x dominates x' via some $S \in \mathcal{N}$.*

A strategy profile x is dominated via a coalition when the members of the coalition can make themselves better off by changing their own strategies only. Now, we define the stable set.

Definition 3 *We say $K \subset X_N$ is a stable set iff K satisfies the following two properties.*

Internal stability: *For any $x, x' \in K$, $x \succ x'$ does not hold.*

External stability: *For any $x \in X_N \setminus K$, there exists some $x' \in K$ such that $x' \succ x$.*

In general, there are multiple stable sets, and some of those may be very complicated. To simplify the argument, we restrict our attention whether a feasible provision level is supported by a stable set. Formally, we investigate whether $X(y)$ is a stable set for each $y \in [0, n\omega]$. We call $y \in [0, n\omega]$ is *a stable provision level* iff $X(y)$ is a stable set. The simplicity of our model allows the following characterization of the stable provision level.

Proposition 1 *Let $y \in [0, n\omega]$. Then, y is a stable provision level if and only if if $y \leq \frac{n\omega}{2}$, and for any $y' \in]y, n\omega]$ and $x' \in X(y')$, there exists some $S \in \mathcal{N}$ that satisfies*

$$x'(S) > |S|\omega - y, \tag{1}$$

$$x'(S) \geq y' - y, \tag{2}$$

$$\omega y > (\omega - x'_i)\, y' \text{ for all } i \in S. \tag{3}$$

Moreover, if $|S| = 1$*, then (1) and (2) imply (3).*

The proof of Proposition 1 will be given in Appendix A.1.

Then, we state the main results of this paper. Employing the characterization of Proposition 1, we identify the feasible provision levels to be supported by the stable sets.

Theorem 1 *Assume that* $n = 2$*. Then,* $X(y)$ *is a stable set if and only if* $y \in \left[\frac{2\omega}{3}, \omega\right]$.

Theorem 2 *Assume that* $n \geq 3$.

(a) *When* $n = 3k$ *for some* $k \in \mathbb{N}$*,* $X(y)$ *is a stable set if and only if*

$$y \in \left[\frac{n\omega}{3}, \frac{n\omega}{2}\right].$$

(b) *When* $n = 3k + 1$ *for some* $k \in \mathbb{N}$*,* $X(y)$ *is a stable set if and only if*

$$y \in \left[\frac{\left(\sqrt{9n^2 - 6n - 3} - n + 1\right)\omega}{6}, \frac{n\omega}{2}\right].$$

(c) *When* $n = 3k + 2$ *for some* $k \in \mathbb{N}$*,* $X(y)$ *is a stable set if and only if*

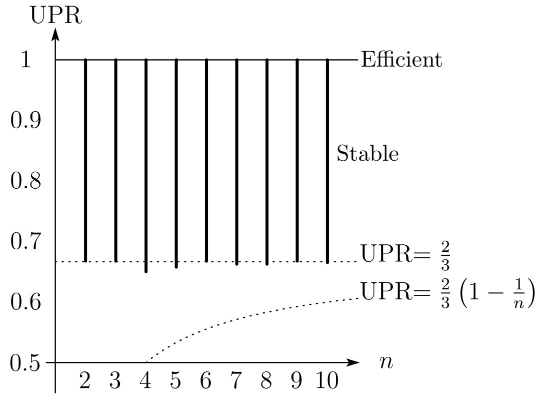
$$y \in \left[\frac{\left(\sqrt{9n^2 + 6n - 3} - n - 1\right)\omega}{6}, \frac{n\omega}{2}\right].$$

The proofs of Theorem 1 and 2 will be also given in Appendix A.2 and A.3, respectively.

We obtain two corollaries from Theorem 1 and 2. The first one extends one of the results of Miyakawa (2006) for the number of players.

Corollary 1 *The efficient provision level is stable.*

Figure 1: The UPRs of the stable provision levels.



The second corollary assures the existence of the stable set since any interval in Theorem 1 and 2 has a positive length. Note that the stable set may fail to exist in general. See Hirai (2013) for an example with no stable set, which is a binary choice version of the present model.

Corollary 2 *There exist infinitely many stable sets.*

Figure 1 summarizes the results of Therem 1 and 2 in terms of the under-provision rate, henceforth UPR. The UPR at a feasible provision level $y \leq \frac{n\omega}{2}$ is defined as $\frac{2y}{n\omega}$ that is the ratio of y to the efficient provision level.[5] Each thick vertical line represents the range of the stable provision levels in terms of the UPR at each number of players. Note that the UPRs at the minimum stable provision levels vary according to the number of players, but they are close to $\frac{2}{3}$.

[5]In our symmetric model, the UPR essentially turns out to be the index of easy riding due to Cornes and Sandler (1984). They remarked that the index of easy riding is inappropriate to measure the extent of inefficiency. See also Cornes and Sander (1996, pp.159-161).

Remark 1 In Theorem 2, the formulas of the minimum stable provision levels vary according to the remainder of n divided by 3. Indeed, as is obvious from Figure 1, the minimum stable provision level is lower than but close to $\frac{n\omega}{3}$ when n is not a multiple of 3. This fluctuation is caused by a certain integer problem.

Given a feasible provision level $y \leq \frac{n\omega}{2}$, we will later construct a strategy profile that is most difficult to be dominated by some strategy profile in $X(y)$ to check whether $X(y)$ is a stable set. To this end, imagine tentatively the model where the set of players is the continuum with measure n, while the initial endowments and the payoff functions are left unchanged. In the proof of Theorem 2, we will construct the strategy profile for each feasible provision level y as follows: $\frac{n}{3}$ players contribute all of their endowments, and the remaining players contribute sufficiently small amount so that they do not satisfy (3) in Proposition 1, which is in fact determined endogenously. Such a strategy profile is ideal "ideal" in the sense that we are constructing this in the model with continuum of players. Then, $X(y)$ includes a strategy profile that dominates this ideal one whenever y is no less than $\frac{n\omega}{3}$, which implies that $\frac{n\omega}{3}$ is the minimum stable provision level.

Returning to the original model with finite players, such an ideal strategy profile can be directly applied when n is a multiple of 3. Therefore, $\frac{n\omega}{3}$ is the minimum stable provision level in this case. On the other hand, the ideal strategy profile is not feasible when $\frac{n}{3}$ is not an integer. Then, given a feasible provision level y, we need to find an "approximately ideal" strategy profile that is most difficult to be dominated by some strategy profile in $X(y)$ among the feasible strategy profiles. This approximately ideal strategy profile is certainly departed from the ideal one constructed in the imaginary model with continuum of players. This departure makes the approximately ideal strategy profile slightly easier to be dominated by some strategy profile in $X(y)$ for a given y. Then, $X(y)$ where y is slightly lower than $\frac{n\omega}{3}$ includes a strategy profile that dominates the approximately ideal strategy profile when n is not a multiple of 3, and thus slightly lower provision levels than $\frac{n\omega}{3}$ can be supported by a stable set. See the proofs in Appendix A.3 for more detail, in particular Lemma 9.

Additionally, note that the UPR of the minimum stable provision level converges to $\frac{2}{3}$, which is the UPR at $y = \frac{n\omega}{3}$, as n goes to infinity. This comes from the fact that the approximately ideal strategy profile converges to the ideal one constructed in the imaginary model as the number of players increases. ■

Remark 2 When $n = 2$, the approximately ideal strategy profile considered in Remark 1 is so far from the ideal strategy profile that it is no longer the most difficult to be dominated among the feasible strategy profiles. Therefore, we need to consider another strategy profile not to be dominated for identifying the minimum stable provision level when $n = 2$. This makes the case with $n = 2$ distinct from the others, and thus an independent proof is necessary. ■

4 Comparison with the Nash equilibrium

The Nash equilibrium is the most popular solution in the strategic public good provision game. In our simple model, it is easy to see that

$$x^* = \left(\frac{\omega}{n+1}, ..., \frac{\omega}{n+1}\right) \in X_N$$

is the unique Nash equilibrium. We call $\frac{n\omega}{n+1}$ *the Nash provision level.* In fact, it seems consistent to our model that we regard x^* the coalition-proof

Nash equilibrium that is robust to not only the individual deviations, but also the credible coalitional deviations since we are allowing the players to form a coalition when we consider the stable set. It is also easy to check that x^* is a coalition-proof Nash equilibrium. See for example Yi (1999).

We compare the stable provision levels with the Nash provision level, and observe that any stable provision level is strictly higher than the Nash provision level when $n \geq 3$.

Proposition 2 *Any stable provision level is at least as high as the Nash provision level. Moreover, if $n \geq 3$, then any stable provision level is strictly higher than the Nash provision level.*

Proof. The case with $n = 2$ is obvious by Theorem 1. We show the case with $n \geq 3$ by employing Theorem 2.

First, assume that $n = 3k$ for some $k \in \mathbb{N}$. Then, by $n \geq 3$,

$$\frac{n\omega}{3} - \frac{n\omega}{n+1} = \frac{n\,(n-2)}{3\,(n+1)} > 0,$$

the desired inequality.

Second, assume that $n = 3k + 1$ for some $k \in \mathbb{N}$. Then, by $n \geq 4$,

$$\begin{aligned}
\frac{\left(\sqrt{9n^2 - 6n - 3} - n + 1\right)\omega}{6} & - \frac{n\omega}{n+1} \\
&= \frac{\left((n+1)\left(\sqrt{9n^2 - 6n - 3} - n + 1\right) - 6n\right)\omega}{6\,(n+1)} \\
&> \frac{\left((n+1)\left(\sqrt{9n^2 - 12n + 4} - n + 1\right) - 6n\right)\omega}{6\,(n+1)} \\
&= \frac{\left((n+1)\,(2n-1) - 6n\right)\omega}{6\,(n+1)} \\
&= \frac{\left(2n^2 - 5n - 1\right)\omega}{6\,(n+1)} \\
&> 0,
\end{aligned}$$

the desired inequality.

Finally, assume that $n = 3k + 2$ for some $k \in \mathbb{N}$. Then, by $n \geq 5$,

$$\begin{aligned} &\frac{\left(\sqrt{9n^2 + 6n - 3} - n - 1\right)\omega}{6} - \frac{n\omega}{n+1} \\ &= \frac{\left((n+1)\left(\sqrt{9n^2 + 6n - 3} - n - 1\right) - 6n\right)\omega}{6(n+1)} \\ &> \frac{\left((n+1)\left(\sqrt{4n^2 + 4n + 1} - n - 1\right) - 6n\right)\omega}{6(n+1)} \\ &= \frac{(n(n+1) - 6n)\omega}{6(n+1)} \\ &= \frac{(n^2 - 5n)\omega}{6(n+1)} \\ &\geq 0, \end{aligned}$$

the desired inequality. ■

Though the stable provision level is almost always higher than the Nash provision level, each stable set includes a strategy profile that is not comparable with the Nash equilibrium with respect to the Pareto criterion. For example, consider the case where $n = 2m$ for some $m \in \mathbb{N}$ and a strategy profile x where $x_i = \omega$ for all $i = 1, ..., m$ and $x_i = 0$ for all $i = m+1, ..., 2m$. It is easy to see that x is in the stable set $X(m\omega)$, which supports the efficient provision level. However, players $1, ..., m$ prefer the Nash equilibrium to x, while players $m+1, ..., 2m$ prefer x to the Nash equilibrium. We can consider a similar example for the case with lower stable provision level and/or odd n. Therefore, the Pareto criterion is not appropriate to compare the efficiency at the stable sets with that at the Nash equilibrium. Instead, we apply the coefficient of resource utilization due to Debreu (1951) for the measurement of the efficiency.

Definition 4 *Let $x \in X_N$. The coefficient of resource utilization, henceforth CRU, at x is defined as $\sigma(x) = \frac{W(x)}{n\omega}$, where*

$$W(x) = \min \left\{ \sum_{i \in N} r_i + q \,\middle|\, \begin{array}{l} r_i q \geq u_i(x) \text{ for all } i \in N; \\ r_i \leq \omega \text{ for all } i \in N \end{array} \right\}$$

The CRU at x is the ratio of the minimum amount of the (private good required for attaining the payoffs $(u_1(x), ..., u_n(x))$ to the the original amount of the initial endowments $n\omega$. Note that the condition $r_i \leq \omega$ for all $i \in N$ is added for the consistency with our model in which each player is not allowed to consume the private good more than ω.

The following proposition confirms that the stable set is more efficient than the Nash equilibrium with respect to CRU.

Proposition 3 *For any $y \in \left[0, \frac{n\omega}{2}\right]$ and $x \in X(y)$, $\sigma(x) = \frac{2\sqrt{(n\omega - y)y}}{n\omega}$. Therefore, for any stable provision level y and any $x \in X(y)$, $\sigma(x) \geq \sigma(x^*)$, where $x^* \in X$ is the Nash equilibrium, and the strict inequality holds if $n \geq 3$.*

Proof. We prove the first statement. Then, the remaining proof follows from Proposition 2 since $\frac{2\sqrt{(n\omega - y)y}}{n\omega}$ is increasing in y with $0 \leq y \leq \frac{n\omega}{2}$.

It is easy to see that if $x_i = 0$ for all $i \in N$, then $x \in X(0)$ and $\sigma(x) = 0$. Fix an arbitrary $y \in \left]0, \frac{n\omega}{2}\right]$ and an arbitrary $x \in X(y)$. Define

$$r_i = \frac{(\omega - x_i)\, y}{\sqrt{(n\omega - y)\, y}} \text{ for all } i \in N \text{ and } q = \sqrt{(n\omega - y)\, y}.$$

For all $i \in N$,

$$\omega - r_i = \frac{\omega\sqrt{(n\omega - y)\, y} - (\omega - x_i)\, y}{\sqrt{(n\omega - y)\, y}} \geq \frac{\omega\left(\sqrt{(n\omega - y)\, y} - y\right)}{\sqrt{(n\omega - y)\, y}}$$

by $x_i \geq 0$. Suppose that $\sqrt{(n\omega - y)\, y} < y$. Then, $n\omega - y < y$ by $y > 0$, contradicting that $y \leq \frac{n\omega}{2}$. Thus, $\omega \geq r_i$ for all $i \in N$. It is easy to see that $r_i q = (\omega - x_i)\, y = v_i(x)$ for all $i \in N$. Hence, $W(x) \leq \sum_{i \in N} r_i + q = 2\sqrt{(n\omega - y)\, y}$.

Fix arbitrary $r'_1, ..., r'_n \leq \omega$ and q' with $r'_i q' \geq (\omega - x_i)\, y$ for all $i \in N$. Define $W' = \sum_{i \in N} r'_i + q'$. Note that $\sum_{i \in N} r'_i q' \geq (n\omega - y)\, y$. On the other hand, $(W' - p)\, p$ is maximized at $p = \frac{W'}{2}$. Thus,

$$\frac{W'^2}{4} \geq (W' - q')\, q' = \sum_{i \in N} r'_i q' \geq (n\omega - y)\, y$$

that is equivalent to $W' \geq 2\sqrt{(n\omega - y)\, y}$. Hence $W(x) = 2\sqrt{(n\omega - y)\, y}$ and $\sigma(x) = \frac{W(x)}{n\omega}$. ■

5 Concluding remarks

This paper investigated a certain class of the stable sets in a strategic public good provision game. We completely identified the range of the feasible provision levels to be supported by stable sets. Then, we observed that the efficient provision level is supported by a stable set. We also observed that the stable sets are better than the Nash equilibrium in terms of the extent of the underprovision and the coefficient of resource utilization. We conclude with a remark.

Our results heavily depend on the simplicity and the symmetry of the model. It seems difficult to characterize the stable provision levels in a general model. Perhaps, there may not exist a stable set consisting of strategy profiles achieving an identical level of the public good provision. Instead, we may work with the general model by restricting our attention to whether a set of the efficient strategy profiles is a stable set. Another direction of the extension is to consider a model with asymmetric initial endowments while the preferences remain symmetric. Such a model was employed by Shitovitz and Spiegel (1998). We may investigate whether the neutrality theorem like Bergstom, *et al.* (1986) holds, or the inequality of the initial endowments affects the range of the stable provision levels. We remain these problems for future research.

Appendix

We give the proofs for Proposition 1 and Theorem 1 and 2.

A.1 Proof of Proposition 1

We first prove four lemmas. Then, we turn to the proof of Proposition 1.

Lemma 1 *For any $y \in [0, n\omega]$, $X(y)$ is internally stable.*

Proof. Fix an arbitrary $y \in [0, n\omega]$. Fix arbitrary $x, x' \in X(y)$ and $S \in \mathcal{N}$ such that $x \to_S x'$. By $x(N) = x'(N) = y$, there exists some $j \in S$ such that $x_j \leq x'_j$. Then, $x' \succ_S x$ is impossible by $v_j(x') = (\omega - x'_j)\, y \leq (\omega - x_j)\, y = v_j(x)$. ■

Lemma 2 *For any $y \in]\frac{n\omega}{2}, n\omega]$, $X(y)$ is not externally stable.*

Proof. Fix an arbitrary $y \in \left]\frac{n\omega}{2}, n\omega\right]$. Fix an arbitrary $x^* \in X\left(\frac{n\omega}{2}\right)$. Note that x^* is Pareto efficient. Suppose that there exist some $S \in \mathcal{N}$ and $x_S \in X_S$ such that $(x_S, x^*_{N\setminus S}) \succ_S x^*$ and $(x_S, x^*_{N\setminus S}) \in X(y)$. Then, $v_i(x_S, x^*_{N\setminus S}) > v_i(x^*)$ for all $i \in S$. For all $i \in N \setminus S$, $v_i(x_S, x^*_{N\setminus S}) = (\omega_i - x^*_i)\, y \geq (\omega_i - x^*_i)\, \frac{n\omega}{2} = v_i(x^*)$ by $y > \frac{n\omega}{2}$. This contradicts the Pareto efficiency of x^*. Hence no $x \in X(y)$ dominates x^*, and $X(y)$ is not externally stable. ■

Lemma 3 *Let* $y \in \left]0, \frac{n\omega}{2}\right]$ *and* $y' \in [0, y[$. *For any* $x' \in X(y')$, *there exists some* $x \in X(y)$ *such that* $x \succ x'$.

Proof. Fix an arbitrary $y \in \left]0, \frac{n\omega}{2}\right]$. Fix an arbitrary $y' \in [0, y[$ and an arbitrary $x' \in X(y')$. Note that $(n\omega - y')\, y' < (n\omega - y)\, y$ since $(n\omega - z)\, z$ is a quadratic concave function of z maximized at $z = \frac{n\omega}{2}$ and $y' < y \leq \frac{n\omega}{2}$. Let $S = \{i \in N | x'_i < \omega\}$ and $s = |S|$. Note that $s\omega - x'(S) = n\omega - x'(N) = n\omega - y'$ since $x'_i = \omega$ for all $i \in N \setminus S$. For each $i \in S$, there uniquely exists some $x''_i \in]x'_i, \omega]$ such that $(\omega - x'_i)\, y' = (\omega - x''_i)\, y$ by $0 \leq (\omega - x'_i)\, y' < (\omega - x'_i)\, y$. Since

$$\begin{aligned}(n\omega - (x''(S) + (n-s)\,\omega))\, y &= (s\omega - x''(S))\, y \\ &= (s\omega - x'(S))\, y' \\ &= (n\omega - y')\, y' \\ &< (n\omega - y)\, y,\end{aligned}$$

we have

$$y - (n-s)\,\omega < x''(S)\,. \tag{A.1}$$

Define $x_i = \frac{x''_i}{x''(S)}\,(y - (n-s)\,\omega)$ for each $i \in S$. Then, $x(S) = y - (n-s)\,\omega$ and $0 \leq x_i < x''_i \leq \omega$ for all $i \in S$ by (A.1). We have $(x_S, x'_{N\setminus S}) \in X(y)$ since $x'_i = \omega$ for all $i \in N \setminus S$. By $x_i < x''_i$ for all $i \in S$, $v_i(x') = (\omega - x'_i)\, y' = (\omega - x''_i)\, y < (\omega - x_i)\, y = v_i(x_S, x'_{N\setminus S})$ for all $i \in S$. Hence $(x_S, x'_{N\setminus S}) \succ_S x'$. ■

Lemma 4 *Let* $y, y' \in [0, n\omega]$ *with* $y < y'$ *and* $x' \in X(y')$. *Then, there exists some* $x \in X(y)$ *such that* $x \succ_S x'$ *if and only if there exists some* $S \in \mathcal{N}$ *that satisfies (1)-(3) in Proposition 1.*

Proof. Fix arbitrary $y, y' \in [0, n\omega]$ with $y < y'$ and an arbitrary $x' \in X(y')$. Let $S \in \mathcal{N}$ and $s = |S|$. We first rewrite (1) and (2) in the following way.

We have

$$
\begin{aligned}
(s\omega - x'(S))\,y' - (s\omega - (y - x'(N \setminus S)))\,y \\
&= (s\omega - x'(S))\,y' - (s\omega - (y - y' + x'(S)))\,y \\
&= s\omega\,(y' - y) - (y' - y)\,x'(S) - (y' - y)\,y \\
&= (y' - y)\,(s\omega - x'(S) - y)\,.
\end{aligned}
$$

By $y' - y > 0$, (1) holds if and only if

$$(s\omega - x'(S))\,y' < (s\omega - (y - x'(N \setminus S)))\,y. \tag{A.2}$$

Also, $y - x'(N \setminus S) = y - y' + x'(S)$. Thus, (2) holds if and only if

$$y - x'(N \setminus S) \geq 0. \tag{A.3}$$

Next, we show the necessity. Assume that there exists some $x \in X(y)$ such that $x \succ_S x'$. By $x \in X(y)$ and $x' \to_S x$,

$$x(S) = y - x'(N \setminus S)\,. \tag{A.4}$$

By $v_i(x) > v_i(x')$ for all $i \in S$, $(\omega - x'_i)\,y' = v_i(x') < v_i(x) = (\omega - x_i)\,y$ for all $i \in S$. Aggregating these inequalities over S,

$$(s\omega - x'(S))\,y' < (s\omega - x(S))\,y. \tag{A.5}$$

Substituting (A.4) to (A.5), we obtain (1) as well as (A.2). By (A.4) and $x(S) \geq 0$, we obtain (2) as well as (A.3). By $x_i \geq 0$ for all $i \in S$, $(\omega - x'_i)\,y' < (\omega - x_i)\,y \leq \omega y$ for all $i \in S$. Thus, we obtain (3).

Finally, we show the sufficiency. Assume that (1)-(3) hold. Note that (A.2) and (A.3) hold as well. For each $i \in S$, there exists some $g_i \in]0, \omega]$ such that $(\omega - g_i)\,y = (\omega - x'_i)\,y'$ by (3). Aggregating these equations over S, $(s\omega - g(S))\,y = (s\omega - x'(S))\,y'$. Then, by (A.2) and (A.3),

$$g(S) > y - x'(N \setminus S) \geq 0. \tag{A.6}$$

For each $i \in S$, define $x_i = \frac{g_i}{g(S)} (y - x'(N \setminus S))$. Then, $x(S) = y - x'(N \setminus S)$ and $0 \leq x_i < g_i$ for all $i \in S$ by $g_i > 0$ for all $i \in S$ and (A.6). Thus, $x' \to_S (x_S, x'_{N \setminus S})$, $(x_S, x'_{N \setminus S}) \in X(y)$, and

$$v_i \left(x_S, x'_{N \setminus S}\right) = (\omega - x_i) y > (\omega - g_i) y = (\omega - x'_i) y' = v_i (x')$$

for all $i \in S$. Hence $(x_S, x'_{N \setminus S}) \succ_S x'$. ■

Proof of Proposition 1. Let $y \in [0, n\omega]$. First, we show the sufficiency. Assume that $y \leq \frac{n\omega}{2}$ and y satisfies (1)-(3) for any $y' \in]y, n\omega]$ and $x' \in X(y')$. Then, the internal stability follows from Lemma 1, and the external stability follows from Lemma 3 and 4. Hence $X(y)$ is a stable set.

Next, we show the necessity. Assume that $X(y)$ is a stable set. Then, Lemma 2 implies $y \leq \frac{n\omega}{2}$, and Lemma 4 implies that y satisfies (1)-(3) for any $y' \in]y, n\omega]$ and $x' \in X(y')$.

Finally, we show that (1) and (2) imply (3) if $|S| = 1$. Fix an arbitrary $y' \in]y, n\omega]$ and $x' \in X(y')$. Denote $S = \{i\}$. Assume that S satisfies (1) and (2). Recall that (1) is equivalent to (A.2), and (2) is equivalent to (A.3). Then, together with $|S| = 1$,

$$(\omega - x'_i) y' < (\omega - (y - x'(N \setminus \{i\}))) y \leq \omega y.$$

Thus, we obtain (3). ■

Proposition 1 plays an important role in the subsequent proofs of Theorem 1 and 2.

A.2 Proof of Theorem 1

Proof of Theorem 1. Let $n = 2$. We first prove the necessity. Assume that $X(y)$ is a stable set. It suffices to show that $y \geq \frac{2\omega}{3}$ by Proposition 1.

Suppose that $y < \frac{2\omega}{3}$. Consider a strategy profile x' satisfying $x'_i = \frac{2\omega}{3}$ for $i = 1, 2$. Thus, $x'(\{1, 2\}) = \frac{4\omega}{3}$.

For each $i = 1, 2$, $\frac{4\omega}{3} - y > \frac{2\omega}{3} = x_i$ by $y < \frac{2\omega}{3}$. Thus, neither $\{1\}$ nor $\{2\}$ satisfies (2) in Proposition 1. Hence for each $i = 1, 2$, there exists no $x \in X(y)$ such that $x \succ_{\{i\}} x'$. Further, $\{1, 2\}$ does not satisfy (1) since

$$2\omega - y > \frac{4\omega}{3} = x'(\{1, 2\})$$

by $y < \frac{2\omega}{3}$. Thus, there neither exists $x \in X(y)$ such that $x \succ_{\{1,2\}} x'$. Hence $X(y)$ is not externally stable.

Next, we turn to the sufficiency. Fix an arbitrary $y \in \left[\frac{2\omega}{3}, \omega\right]$. Fix an arbitrary $x' \in X_N$ such that $x'_1 + x'_2 > y$. Denote $y' = x'_1 + x'_2$. By Proposition 1, it suffices to prove that some $S \in \mathcal{N}$ satisfies (1)-(3) for these y' and x'. We distinguish two cases.

Case 1. Both $x'_1 > y$ and $x'_2 > y$.

We prove that $N = \{1, 2\}$ satisfies all (1)-(3) in Proposition 1. By $x'_i > y \geq \frac{2\omega}{3}$ for each $i = 1, 2$, $x'(N) > 2y \geq 2\omega - y$. Thus, N satisfies (1). By $y > 0$, $x'(N) = y' > y' - y$. Thus, N satisfies (2). Since $x'_i > y \geq \frac{2\omega}{3}$ for each $i = 1, 2$ and $y' \leq 2\omega$, $\omega y \geq \frac{2\omega^2}{3} \geq \frac{\omega y'}{3} > (\omega - x'_i) y'$ for each $i = 1, 2$. Thus, N satisfies (3).

Case 2. Either $x'_1 \leq y$ or $x'_2 \leq y$.

We consider the case where $x'_1 \geq x'_2$. The case where $x'_1 < x'_2$ can be proved by a similar argument.

In this case, $x'_2 \leq y$ and $x'_1 \geq \frac{y'}{2}$ hold. Thus, $x'_1 > \frac{\omega}{3}$ by $y' > y \geq \frac{2\omega}{3}$. We show that $\{1\}$ satisfies (1)-(3) in Proposition 1. By $y \geq \frac{2\omega}{3}$ and $x'_1 > \frac{\omega}{3}$, $x'_1 > \frac{\omega}{3} \geq \omega - y$. Thus, $\{1\}$ satisfies (1). By $x'_2 \leq y$, $x'_1 = y' - x'_2 \geq y' - y$. Thus, $\{1\}$ satisfies (2). Since $|\{1\}| = 1$, $\{1\}$ satisfies (3) by the last statement of Proposition 1.

By Proposition 1 and Case 1 and 2, $X(y)$ is a stable set. ■

A.3 Proof of Theorem 2

We first prove five lemmas. Then, we turn to the proof of Theorem 2.

In what follows, for each $s \in \{1, ..., n\}$ and any $x \in X_N$, let $S^*(x, s)$ denote the coalition such that $|S^*(x, s)| = s$ and $x_i \geq x_j$ for any $i \in S^*(x, s)$ and $j \notin S^*(x, s)$. Of course, there may be multiple coalitions satisfying these conditions. In such a case, we can choose one coalition according to an arbitrary rule. Note that for any $y \in [0, y]$, $x \in X(y)$, and $s \in \{1, ..., n\}$,

$$x(S^*(x, s)) \geq \frac{s}{n} y. \tag{A.7}$$

Suppose that there exist some $\bar{y} \in [0, n\omega]$, $\bar{x} \in X(\bar{y})$, and $\bar{s} \in \{1, ..., n\}$ such that $\bar{x}(S^*(\bar{x}, \bar{s})) < \frac{\bar{s}}{n}\bar{y}$. By the definition of $S^*(\bar{x}, \bar{s})$, $\bar{x}_i < \frac{\bar{y}}{n}$ for all $i \notin S^*(\bar{x}, \bar{s})$. Thus, $\bar{x}(N) < \bar{y}$, contradicting that $\bar{x} \in X(\bar{y})$. Hence (A.7) holds.

Lemma 5 *Let $y, y' \in [0, n\omega[$ with $y < y'$ and $s \in \{1, ..., n\}$. Then, $S^*(x', s)$ satisfies both (1) and (2) in Proposition 1 for any $x' \in X(y')$ if and only if*

$$n\left(1 - \frac{y}{y'}\right) \leq s < \frac{y}{\omega - (y'/n)}. \tag{A.8}$$

Proof. Let $y, y' \in [0, n\omega[$ with $y < y'$ and $s \in \{1, ..., n\}$. For any $x' \in X(y')$, $x'(S^*(x', s)) \geq \frac{s}{n}y'$ for any $x' \in X(y')$ and the equality holds if $x'_i = \frac{y'}{n}$ for all $i \in N$ by (A.7). Thus, $S^*(x', s)$ satisfies (1) for any $x' \in X(y')$ if and only if

$$0 < \frac{s}{n}y' - (s\omega - y) = s\left(\frac{y'}{n} - \omega\right) + y.$$

This is equivalent to the latter inequality of (A.8) since $\left(\frac{y'}{n} - \omega\right) < 0$ by $y' < n\omega$. On the other hand, $S^*(x', s)$ satisfies (2) for any $x' \in X(y')$ if and only if

$$\frac{s}{n}y' \geq y' - y.$$

This is equivalent to

$$s \geq n - \frac{y}{y'}n = n\left(1 - \frac{y}{y'}\right)$$

by $y' > 0$. Thus, we obtain the former inequality of (A.8). ■

Lemma 6 *If $\frac{(n+1)\omega}{4} \leq y \leq \frac{n\omega}{2}$, then for any $y' \in]y, n\omega[$, there exists some $s \in \{1, ..., n\}$ such that $S^*(x', s)$ satisfies (1) and (2) in Proposition 1 for any $x' \in X(y')$.*

Proof. Let $y \in \left[\frac{(n+1)\omega}{4}, \frac{n\omega}{2}\right]$.

We claim that

$$\left(\frac{y}{\omega - (y'/n)}\right) - n\left(1 - \frac{y}{y'}\right) \geq 1 \text{ for any } y' \in]y, n\omega[. \tag{A.9}$$

We can transform (A.9) as

$$ny\left(\frac{1}{n\omega - y'} - \frac{1}{y} + \frac{1}{y'}\right) \geq 1 \text{ for any } y' \in]y, n\omega[.$$

Differentiating the LHS of (A.9) by y',

$$\begin{aligned} ny\left(\frac{1}{(n\omega - y')^2} - \frac{1}{y'^2}\right) &= \frac{ny}{(n\omega - y')^2 y'^2}\left(y'^2 - (n\omega - y')^2\right) \\ &= \frac{ny}{(n\omega - y')^2 y'^2}\left(n\omega\left(2y' - n\omega\right)\right) \\ &\begin{cases} < 0 & \text{if } y' < \frac{n\omega}{2} \\ = 0 & \text{if } y' = \frac{n\omega}{2} \\ > 0 & \text{if } y' > \frac{n\omega}{2}. \end{cases} \end{aligned}$$

Thus, the LHS of (A.9) is minimized at $y' = \frac{n\omega}{2}$. Therefore, (A.9) is satisfied for any $y' \in]y, n\omega[$ since

$$ny\left(\frac{1}{n\omega - y'} - \frac{1}{y} + \frac{1}{y'}\right) \geq ny\left(\frac{2}{n\omega} - \frac{1}{y} + \frac{2}{n\omega}\right) = \frac{4y}{\omega} - n \geq 1,$$

where the last inequality follows from $y \geq \frac{(n+1)\omega}{4}$.

Then, for any $y' \in]y, n\omega[$, there exists some integer $s(y') \in \{1, ..., n\}$ such that

$$n\left(1 - \frac{y}{y'}\right) \leq s(y') < \frac{y}{\omega - (y'/n)}$$

by (A.9) and $0 < n\left(1 - \frac{y}{y'}\right) < n$. By Lemma 5, for any $y' \in]y, n\omega[$, $S^*(x', s(y'))$ satisfies (1) and (2) in Proposition 1 for any $x' \in X(y')$. ■

Hereafter, define $T^*(y, x') = \left\{i \in N \,\middle|\, x'_i > \omega\left(1 - \frac{y}{y'}\right)\right\}$ and $t^*(y, x') = |T^*(y, x')|$ for any $y \in \left[0, \frac{n\omega}{2}\right]$, $y' \in]y, n\omega]$, and $x' \in X(y')$.

Lemma 7 *Let $y \in \left[0, \frac{n\omega}{2}\right]$. Then, $T^*(y, x')$ is nonempty for any $y' \in]y, n\omega]$ and $x' \in X(y')$ if and only if $y > \frac{n\omega}{4}$.*

Proof. Let $y \in \left[0, \frac{n\omega}{2}\right]$. We claim that $T^*(y, x') \neq \emptyset$ for any $y' \in]y, n\omega]$ and any $x' \in X(y')$ if and only if

$$n\omega\left(1 - \frac{y}{y'}\right) < y' \text{ for any } y' \in]y, n\omega]. \quad \text{(A.10)}$$

We first show the contraposition of the necessity. Assume that there exists some $\tilde{y} \in]y, n\omega]$ such that $n\omega\left(1 - \frac{y}{\tilde{y}}\right) \geq \tilde{y}$. Then, $\left(\frac{\tilde{y}}{n}, ..., \frac{\tilde{y}}{n}\right) \in X(\tilde{y})$ and $T^*\left(y, \left(\frac{\tilde{y}}{n}, ..., \frac{\tilde{y}}{n}\right)\right) = \emptyset$ by $\omega\left(1 - \frac{y}{\tilde{y}}\right) \geq \frac{\tilde{y}}{n}$.

Next, we show the contraposition of the sufficiency. Assume that there exist some $\bar{y} \in]y, n\omega]$ and $\bar{x} \in X(\bar{y})$ such that $T^*(y, \bar{x}) = \emptyset$. Then, $\bar{x}_i \leq \omega\left(1 - \frac{y}{\bar{y}}\right)$ for all $i \in N$. Thus, $\bar{y} = \bar{x}(N) \leq n\omega\left(1 - \frac{y}{\bar{y}}\right)$.

It remains to show that (A.10) holds if and only if $y > \frac{n\omega}{4}$. By $y' > 0$, (A.10) is equivalent to

$$y'^2 - n\omega y' + n\omega y > 0 \text{ for any } y' \in]y, n\omega]. \quad \text{(A.11)}$$

The LHS of (A.11) is a quadratic convex function of y' that is minimized at $y' = \frac{n\omega}{2}$. Thus, (A.11) holds if and only if $n\omega y - \frac{n^2\omega^2}{4} > 0$. This is equivalent to $y > \frac{n\omega}{4}$. ■

Proposition 1 and Lemma 7 show that any $X(y)$ with $y \leq \frac{n\omega}{4}$ cannot be a stable set because it is not externally stable.

Lemma 8 *Let* $y \in \left[0, \frac{n\omega}{2}\right]$. *If* $y > \frac{n\omega}{4}$, *then* $T^*(y, x')$ *satisfies (1) in Proposition 1 for any* $y' \in]y, n\omega]$ *and* $x' \in X(y')$ *with* $t^*(y, x') < n\left(1 - \frac{y}{y'}\right)$.

Proof. Fix arbitrary $y \in \left]\frac{n\omega}{4}, \frac{n\omega}{2}\right]$ and $y' \in]y, n\omega]$. Fix an arbitrary $x' \in X(y')$ such that $t^*(y, x') < n\left(1 - \frac{y}{y'}\right)$.

By the definition of $T^*(y, x')$, $y' = x'(N) = x'(T^*(y, x')) + x'(N \setminus (T^*(y, x'))) \leq x'(T^*(y, x')) + (n - t)\,\omega\left(1 - \frac{y}{y'}\right)$ that is equivalent to $x'(T^*(y, x')) \geq y' - (n - t^*(y, x'))\,\omega\left(1 - \frac{y}{y'}\right)$. Thus, it suffices to show that

$$\left(y' - (n - t^*(y, x'))\,\omega\left(1 - \frac{y}{y'}\right)\right) - (t\omega - y) > 0. \tag{A.12}$$

By $t^*(y, x') < n\left(1 - \frac{y}{y'}\right)$ and $y' > 0$,

$$\begin{aligned}
\text{LHS of (A.12)} &= y + y' - n\omega + (n - t^*(y, x'))\,\frac{\omega y}{y'} \\
&> y + y' - n\omega + \left(n - n\left(1 - \frac{y}{y'}\right)\right)\frac{\omega y}{y'} \\
&= y + y' - n\omega + \frac{n\omega y^2}{y'^2} \\
&= y + y' - \frac{n\omega}{y'^2}\left(y'^2 - y^2\right) \\
&= (y + y')\left(1 - \frac{n\omega}{y'^2}(y' - y)\right) \\
&= \frac{n\omega\,(y + y')}{y'^2}\left(y - y' + \frac{y'^2}{n\omega}\right).
\end{aligned}$$

By $y' > y > \frac{n\omega}{4}$, (A.12) holds if $y > y' - \frac{y'^2}{n\omega}$. Consider a function $f(z) = z - \frac{z^2}{n\omega}$ that is a quadratic concave function maximized at $z = \frac{n\omega}{2}$. Thus,

$$y' - \frac{y'^2}{n\omega} \leq f\left(\frac{n\omega}{2}\right) = \frac{n\omega}{2} - \frac{n^2\omega^2}{4n\omega} = \frac{n\omega}{4}.$$

By $y > \frac{n\omega}{4}$, $y > y' - \frac{y'^2}{n\omega}$. ■

Lemma 9 *Let $y \in \left]\frac{n\omega}{4}, \frac{n\omega}{2}\right]$. Then, for any $y' \in]y, n\omega]$ and $x' \in X(y')$ with $t^*(y, x') < n\left(1 - \frac{y}{y'}\right)$, $T^*(y, x')$ satisfies (2) in Proposition 1 if and only if $y \geq h(n)$ where*

$$h(n) = \begin{cases} \frac{n\omega}{3} & \text{if } n = 3k \text{ for some } k \in \mathbb{N}, \\ \frac{\left(\sqrt{9n^2-6n-3}-n+1\right)\omega}{6} & \text{if } n = 3k+1 \text{ for some } k \in \mathbb{N}, \\ \frac{\left(\sqrt{9n^2+6n-3}-n-1\right)\omega}{6} & \text{if } n = 3k+2 \text{ for some } k \in \mathbb{N}. \end{cases}$$

Moreover, for any $y \leq h(n)$, there exist some $y' \in]y, n\omega]$ and $x' \in X(y')$ such that $t^*(y, x') < n\left(1 - \frac{y}{y'}\right)$.

Proof. We begin with five claims, where Claim 1,3-5 are equivalent transformations and Claim 2 is an auxiliary claim. Throughout this proof let $y \in \left]\frac{n\omega}{4}, \frac{n\omega}{2}\right]$.

Claim 1 *y satisfies $x'(T^*(y, x')) \geq y' - y$ for any $y' \in]y, n\omega]$ and any $x' \in X(y')$ with $t^*(y, x') < n\left(1 - \frac{y}{y'}\right)$ if and only if y satisfies*

$$y - (n - t^*(y, x'))\,\omega\left(1 - \frac{y}{y'}\right) \geq 0 \tag{A.13}$$

for any $y' \in]y, n\omega]$ and any $x' \in X(y')$ with $t^*(y, x') < n\left(1 - \frac{y}{y'}\right)$.

Proof of Claim 1. Let $y \in \left]\frac{n\omega}{4}, \frac{n\omega}{2}\right]$. We begin with the sufficiency. Assume that (A.13) holds for any $y' \in]y, n\omega]$ and any $x' \in X(y')$ with

$t^*(y,x') < n\left(1-\frac{y}{y'}\right)$. For any $y' \in]y, n\omega]$ and any $x' \in X(y')$, $y' = x'(N) = x'(T^*(y,x')) + x'(N \setminus T^*(y,x')) \leq x'(T^*(y,x')) + (n - t^*(y,x'))\,\omega\left(1-\frac{y}{y'}\right)$ that is equivalent to

$$x'(T^*(y,x')) - y' \geq -(n - t^*(y,x'))\,\omega\left(1-\frac{y}{y'}\right).$$

By substituting this inequality to (A.13), we obtain $x'(T^*(y,x')) \geq y' - y$ for any $y' \in]y, n\omega]$ and any $x' \in X(y')$ with $t^*(y,x') < n\left(1-\frac{y}{y'}\right)$.

Then, we turn to the necessity. Assume that $x'(T^*(y,x')) \geq y' - y$ for any $y' \in]y, n\omega]$ and any $x' \in X(y')$ with $t^*(y,x') < n\left(1-\frac{y}{y'}\right)$. Suppose that there exist some $y'' \in]y, n\omega]$ and some $x'' \in X(y'')$ with $t^*(y,x'') < n\left(1-\frac{y}{y''}\right)$ such that

$$y - (n - t^*(y,x''))\,\omega\left(1-\frac{y}{y''}\right) < 0. \tag{A.14}$$

We claim that there exists some $\tilde{x} \in X(y'')$ such that $t^*(y,\tilde{x}) \leq t^*(y,x'')$ and $\tilde{x}_i = \omega\left(1-\frac{y}{y''}\right)$ for all $i \in N \setminus T^*(y,\tilde{x})$. Let $\tilde{t} = \min_{x \in X(y'')} t^*(y,x)$. Note that $\tilde{t} > 0$ by Lemma 7 and $y > \frac{n\omega}{4}$. Denote $z = \omega\left(1-\frac{y}{y''}\right)$. Define $\tilde{x}_i = \frac{y''-\left(n-\tilde{t}\right)z}{\tilde{t}}$ for all $i = 1,...,\tilde{t}$ and $\tilde{x}_i = z$ for all $i = \tilde{t}+1,...,n$.

We show that $\tilde{x} \in X(y'')$. By its construction, $\tilde{x}(N) = y''$ and $0 < \tilde{x}_i < \omega$ for all $i = \tilde{t}+1,...,n$ are obvious. Moreover, $\tilde{x}_i > z$, otherwise $T(\tilde{x}) = \emptyset$, contradicting that $\tilde{t} > 0$. Thus, we show $\tilde{x}_i \leq \omega$ for all $i = 1,...,\tilde{t}$. Suppose that $\tilde{x}_i > \omega$ for all $i = 1,...,\tilde{t}$. Then, $\tilde{t}\omega + (n-\tilde{t})z < \tilde{x}(N) = y''$. This contradicts the definition of $\tilde{t}$ because more than $\tilde{t}$ players are at least necessary to contribute more than z to provide y'' of the public good. Hence $\tilde{x}_i \leq \omega$ for all $i = 1,...,\tilde{t}$ and $\tilde{x} \in X(y'')$.

By the definition of z, $i \notin T(\tilde{x})$ for all $i = \tilde{t}+1,...,n$. Then, by the definition of $\tilde{t}$, $T^*(y,\tilde{x}) = \{1,...,\tilde{t}\}$. Thus, $t^*(y,\tilde{x}) = \tilde{t} \leq t^*(y,x'')$ and $\tilde{x}_i = \omega\left(1-\frac{y}{y''}\right)$ for all $i \in N \setminus T^*(y,\tilde{x})$.

By $\tilde{t} \leq t^*(y, x'')$, $\tilde{x}\left(\{1, ..., \tilde{t}\}\right) = y'' - \left(n - \tilde{t}\right)\omega\left(1 - \frac{y}{y''}\right)$, and (A.14),

$$\begin{aligned} 0 &> y - (n - t^*(y, x''))\,\omega\left(1 - \frac{y}{y''}\right) \\ &\geq y - \left(n - \tilde{t}\right)\omega\left(1 - \frac{y}{y''}\right) \\ &= y + \tilde{x}(\{1, ..., \tilde{t}\}) - y''. \end{aligned}$$

Thus, $\tilde{x}\left(\{1, ..., \tilde{t}\}\right) < y'' - y$, contradicting the presumption since $\tilde{t} \leq t^*(y, x'') < n\left(1 - \frac{y}{y''}\right)$. Hence the necessity is proved. □

Define

$$y'(t, y) = \frac{n\omega + \sqrt{n^2\omega^2 - 4(n - t)\,\omega y}}{2}$$

for any $t \in \mathbb{N}$. Note that $y'(t, y)$ is well defined if and only if $n^2\omega^2 - 4(n - t)\,\omega y \geq 0$. Note also that $y'(t, y)$ is the maximum solution to the inequality $y'(t, y) \leq t\omega + (n - t)\,\omega\left(1 - \frac{y}{y'(t,y)}\right)$ that is equivalent to $y'(t, y)^2 - n\omega y'(t, y) + (n - t)\,\omega y \leq 0$.

Claim 2 *For any $t = 1, ..., n$, $y'(t, y)$ is well-defined if and only if there exist some $y' \in]y, n\omega]$ and $x' \in X(y')$ such that $t^*(y, x') = t$.*

Proof of Claim 2. We can easily confirm the necessity by letting $y' = y'(t, y)$, $x'_i = \omega$ for all $i = 1, ..., t$, and $x'_i = \omega\left(1 - \frac{y}{y'(t,y)}\right)$ for all $i = t+1, ..., n$. We turn to the sufficiency.

Let $t = 1, ..., n$. Assume that there exist some $y'' \in]y, n\omega]$ and $x'' \in X(y'')$ such that $t^*(y, x'') = t$. By $x''_i \leq \omega$ for all $i \in T^*(y, x'')$ and $x''_i \leq \omega\left(1 - \frac{y}{y''}\right)$ for all $i \in N \setminus T(y, x'')$, $y'' = x''(N) \leq t\omega + (n - t)\,\omega\left(1 - \frac{y}{y''}\right)$, equivalently $y''^2 - t\omega y' + (n - t)\omega y \leq 0$. Therefore, the inequality $y'^2 - t\omega y' + (n - t)\omega(1 - y) \leq 0$ has a solution $y' = y''$. Thus, the maximum solution is well-defined, and it is $y'(t, y)$. □

Claim 3 *y satisfies (A.13) for any $y' \in]y, n\omega]$ and any $x' \in X(y')$ with $t^*(y,x') < n\left(1 - \frac{y}{y'}\right)$ if and only if y satisfies*

$$y - (n-t)\,\omega\left(1 - \frac{y}{y'(t,y)}\right) \geq 0 \qquad \text{(A.15)}$$

for any $t \in \mathbb{N}$ such that $y'(t,y)$ is well-defined and $t < n\left(1 - \frac{y}{y'(t,y)}\right)$.

Proof of Claim 3. We first show the sufficiency. Assume that (A.15) holds for any $t \in \mathbb{N}$ such that $y'(t,y)$ is well-defined and $t < n\left(1 - \frac{y}{y'(t,y)}\right)$. Fix arbitrary $\bar{y} \in]y, n\omega]$ and $\bar{x} \in X(\bar{y})$ such that $t^*(y,\bar{x}) < n\left(1 - \frac{y}{\bar{y}}\right)$. Denote $\bar{t} = t^*(y,\bar{x})$. Note that $y'(\bar{t},y)$ is well-defined by Claim 2. Then, $\bar{t}$ satisfies (A.15) by the presumption. By $\bar{y} = \bar{x}(N) \leq \bar{t}\omega + (n-\bar{t})\,\omega\left(1 - \frac{y}{\bar{y}}\right)$, $\bar{y}^2 - n\omega\bar{y} + (n-t)\,\omega y \leq 0$. Since $y' = y'(\bar{t},y)$ is the maximum solution for the inequality $y'^2 - n\omega y' + (n-t)\,\omega y \leq 0$, $\bar{y} \leq y'(\bar{t},y)$. Together with (A.15) and $\bar{t} = t^*(y,\bar{x})$,

$$y - (n - t^*(y,\bar{x}))\,\omega\left(1 - \frac{y}{\bar{y}}\right) \geq y - (n-\bar{t})\,\omega\left(1 - \frac{y}{y'(\bar{t},y)}\right) \geq 0,$$

the desired inequality.

Next, we show the necessity. Assume that (A.13) holds for any $y' \in]y, n\omega]$ and any $x' \in X(y')$ with $t^*(y,x') < n\left(1 - \frac{y}{y'}\right)$. Fix an arbitrary $\hat{t} \in \mathbb{N}$ such that $y'\left(\hat{t},y\right)$ is well-defined and $\hat{t} < n\left(1 - \frac{y}{y'\left(\hat{t},y\right)}\right)$. Let $\hat{x} \in X$ such that $\hat{x}_i = \omega$ for all $i = 1,...,\hat{t}$, and $\hat{x}_i = \omega\left(1 - \frac{y}{y'\left(\hat{t},y\right)}\right)$ for all $i = \hat{t}+1,...,n$. By the definition of $y'\left(\hat{t},y\right)$, $\hat{x} \in X\left(y'\left(\hat{t},y\right)\right)$. Note that $\hat{t} = t^*(y,\hat{x})$. By (A.13), $\hat{x} \in X(y'(\hat{t},y))$, and $\hat{t} < n\left(1 - \frac{y}{y'\left(\hat{t},y\right)}\right)$,

$$y - (n - t^*(y,\hat{x}))\omega\left(1 - \frac{y}{y'(\hat{t},y)}\right) \geq 0.$$

Substituting $\hat{t} = t^*(y,\hat{x})$, we obtain (A.15). □

Claim 4 *y satisfies (A.15) for any $t \in \{1, ..., n\}$ such that $y'(t, y)$ is well defined and $t < n\left(1 - \frac{y}{y'(t,y)}\right)$ if and only if y satisfies*

$$y \geq \frac{\left(\sqrt{t\left(4n - 3t\right)} - t\right)\omega}{2} \tag{A.16}$$

for any $t \in \{1, ..., n\}$ such that $y'(t, y)$ is well defined and $t < n\left(1 - \frac{y}{y'(t,y)}\right)$.

Proof of Claim 4. Let $t \in \mathbb{N}$ such that $y'(t, y)$ is well-defined and $t < n\left(1 - \frac{y}{y'(t,y)}\right)$. By the definition of $y'(t, y)$, (A.15) is equivalently transformed as

$$\left(y - (n-t)\omega\right)\sqrt{n^2\omega^2 - 4(n-t)\omega y} \geq \omega\left(n(n-t)\omega - (3n - 2t)y\right). \tag{A.17}$$

By $t < n\left(1 - \frac{y}{y'(t,y)}\right)$ and $y'(t, y) \leq n\omega$,

$$\begin{aligned} y - (n-t)\omega &< y - \left(n - n\left(1 - \frac{y}{y'(t,y)}\right)\right)\omega \\ &= y - \frac{n\omega y}{y'(t,y)} \\ &\leq y - y \\ &= 0. \end{aligned}$$

Thus, the both sides of (A.17) are negative. Then, by squaring both sides of (A.17),

$$\left(y - (n-t)\omega\right)^2\left(n^2\omega^2 - 4(n-t)\omega y\right) \leq \omega^2\left(n(n-t)\omega - (3n-2t)y\right)^2.$$

Simplifying this inequality,

$$-4(n-t)\omega y^3 - 4t(n-t)\omega^2 y^2 + 4t(n-t)^2\omega^3 y \leq 0.$$

By $y > 0$, $\omega > 0$, and $n - t > 0$, we finally have $y^2 + t\omega y - t(n-t)\omega^2 \geq 0$. Solving this inequality for y regarding with $y > 0$, we obtain (A.16). $\square$

Let $t^* \in \{1, ..., n\}$ maximize the RHS of (A.16), and define

$$y^* = \frac{\left(\sqrt{t^*(4n - 3t^*)} - t^*\right)\omega}{2}.$$

Claim 5 *Assume that $y'(t^*, y^*)$ is well-defined and $t^* < n\left(1 - \frac{y^*}{y'(t^*,y^*)}\right)$. Then, y satisfies (A.16) for any $t \in \{1, ..., n\}$ such that $y'(t, y)$ is well-defined and $t < n\left(1 - \frac{y}{y'(t,y)}\right)$ if and only if $y \geq y^*$.*

Proof of Claim 5. Assume that $y'(t^*, y^*)$ is well-defined and $t^* < n\left(1 - \frac{y^*}{y'(t^*,y^*)}\right)$.

First, assume that $y \geq y^*$. For any $t \in \{1, ..., n\}$, if $y'(t, y)$ is well-defined, then $y'(t, y^*)$ is well-defined and $y'(t, y) \leq y'(t, y^*)$ by the definition of $y'(t, y)$. Thus, for any $t \in \{1, ..., n\}$, if $y'(t, y)$ is well-defined and $t < n\left(1 - \frac{y}{y'(t,y)}\right)$, then $y'(t, y^*)$ is well-defined and $t < n\left(1 - \frac{y^*}{y'(t,y^*)}\right)$. Then, for any $t \in \{1, ..., n\}$ such that $y'(t, y)$ is well-defined and $t < n\left(1 - \frac{y}{y'(t,y)}\right)$,

$$y \geq y^* = \frac{\sqrt{t^*(4n - 3t^*)} - t^*}{2} \geq \frac{\sqrt{t(4n - 3t)} - t}{2}$$

by the definitions of t^* and y^*.

Next, assume that $y < y^*$. By the definition of $y'(t, y)$, $y'(t^*, y)$ is well-defined and $y'(t^*, y) > y'(t^*, y^*)$ since $y'(t^*, y^*)$ is well-defined and $y < y^*$. Then, $t^* < n\left(1 - \frac{y^*}{y'(t^*,y^*)}\right) < n\left(1 - \frac{y}{y'(t^*,y)}\right)$ by $y < y^*$. By the definition of t^* and y^*,

$$y < y^* = \frac{\sqrt{t^*(4n - 3t^*)} - t^*}{2}.$$

Hence y does not satisfy (A.16) for t^* while $y'(t^*, y)$ is well-defined and t^* satisfies $t^* < n\left(1 - \frac{y}{y'(t^*,y)}\right)$. $\square$

For proving the former part of Lemma 9, it remains to prove that $y^* =$

$h(n)$, $y'(t^*, y^*)$ is well-defined, and $t^* < n\left(1 - \frac{y^*}{y'(t^*,y^*)}\right)$ by Claim 1-5. Note that this is also sufficient for the last statement of the Lemma 9 as follows. The necessity part of the proof of Claim 5 proves that $y'(t^*, y)$ is well-defined and t^* satisfies $t^* < n\left(1 - \frac{y}{y'(t^*,y)}\right)$ for any $y < y^*$, *provided that* $y'(t^*, y^*)$ is well-defined and t^* satisfies $t^* < n\left(1 - \frac{y^*}{y'(t^*,y^*)}\right)$. Then, for any $y < y^*$, by defining $x'_i = \omega$ for all $i = 1, ..., t^*$ and $x'_i = \omega\left(1 - \frac{y}{y'(t^*,y)}\right)$ for all $i = t^* + 1, ..., n$, we have $x' \in X(y'(t^*, y))$ and $t^*(y, x') = t^* < n\left(1 - \frac{y}{y'(t^*,y)}\right)$.

To find t^*, it suffices to find t that maximizes $\sqrt{t(4n - 3t)} - t$. To this end, we consider a function $f(z) = \sqrt{z(4n - 3z)} - z$ defined on $z \in]0, n[$. Differentiating $f(z)$ twice,

$$f'(z) = \frac{2n - 3z}{\sqrt{z(4n - 3z)}} - 1;$$
$$f''(z) = \frac{-z(4n - 3) - (2n - 3z)^2}{(z(4n - 3z))^{\frac{3}{2}}}.$$

By $f''(z) < 0$ for any $z \in]0, n[$, $f(z)$ is a concave function that is maximized when $f'(z) = 0$. Solving $2n - 3z = \sqrt{z(4n - 3z)}$ regarding with $z < n$, we obtain $z = \frac{n}{3}$.

Recall that t in (A.16) must be a natural number as it is the cardinality of a coalition. Thus, the RHS of (A.16) is maximized at some natural number next to $\frac{n}{3}$. In what follows, we complete the proof by distinguishing three cases according to the statement of Lemma 9.

(a) Let $k \in \mathbb{N}$ and $n = 3k$. Then, $\frac{n}{3} = k$ is the natural number. Thus, $t^* = k$ and $y^* = \frac{n\omega}{3}$.

We need to show that $y'(t^*, y^*)$ is well-defined and $t^* < n\left(1 - \frac{y^*}{y'(t^*,y^*)}\right)$. We have

$$y'(t^*, y^*) = \frac{n\omega + \sqrt{n^2\omega^2 - 4\left(n - \frac{n}{3}\right)\omega\left(\frac{n\omega}{3}\right)}}{2} = \frac{2n\omega}{3}.$$

Therefore, $y'(t^*, y^*)$ is well-defined. Moreover, $n\left(1 - \frac{y^*}{y'(t^*,y^*)}\right) = \frac{n}{2} > \frac{n}{3} = k = t^*$, the desired inequality. Hence (a) is proved.

(b) Let $k \in \mathbb{N}$ and $n = 3k+1$. Thus, $\frac{n}{3} = k + \frac{1}{3}$. Then, $t^* = k$ or $k+1$ by the concavity of $f(z)$. Thus, it suffices to check the sign of $g(k) = f(k+1) - f(k)$. Note that $g(k) = \sqrt{9k^2+10k+1} - \sqrt{9k^2+4k} - 1$ by $n = 3k+1$. We claim that $g(k) < 0$. This is equivalent to

$$\sqrt{9k^2+10k+1} < \sqrt{9k^2+4k} + 1.$$

Since the both sides are positive, we can equivalently transform by squaring both sides as

$$9k^2 + 10k + 1 < 9k^2 + 4k + 1 + 2\sqrt{9k^2+4k}$$

that can be simplified as $3k < \sqrt{9k^2+4k}$, the desired inequality. Thus, $t^* = k = \frac{n-1}{3}$ and

$$y^* = \frac{\sqrt{(3n^2-2n-1)/3} - (n-1)/3}{2} = \frac{\sqrt{9n^2-6n-3} - n + 1}{6}.$$

Note that $t^* \geq \frac{n}{4}$ by $n \geq 4$ in this case.

We turn to showing that $y'(t^*, y^*)$ is well-defined and $t^* < n\left(1 - \frac{y^*}{y'(t^*,y^*)}\right)$. We have

$$\begin{aligned}\frac{n\omega}{3} - \frac{\left(\sqrt{9n^2-6n-3} - n + 1\right)\omega}{6} &= \frac{\omega}{6}\left(3n - 1 - \sqrt{9n^2-6n-3}\right)\\ &> \frac{\omega}{6}\left(3n - 1 - \sqrt{9n^2-6n+1}\right)\\ &= 0.\end{aligned}$$

By $\frac{n\omega}{3} > \frac{\left(\sqrt{9n^2-6n-3}-n+1\right)\omega}{6} = y^*$ and $t^* = k = \frac{n-1}{3} \geq \frac{n}{4}$,

$$n^2\omega^2 - 4(n - t^*)\omega y^* > n^2\omega^2 - 4\left(n - \frac{n}{4}\right)\omega\left(\frac{n\omega}{3}\right) = 0.$$

Thus, $y'(t^*, y^*)$ is well-defined, and $y'(t^*, y^*) > \frac{n\omega}{2}$ by this inequality and the definition of $y'(t^*, y^*)$. Moreover,

$$n\left(1 - \frac{y^*}{y'(t^*, y^*)}\right) > n\left(1 - \frac{n\omega/3}{n\omega/2}\right) = \frac{n}{3} > \frac{n-1}{3} = k = t^*,$$

the desired inequality. Hence (b) is proved.

(c) Let $k \in \mathbb{N}$ and $n = 3k + 2$. Thus, $\frac{n}{3} = k + \frac{2}{3}$. Then, $t^* = k$ or $k + 1$ by the concavity of $f(z)$. Thus, it suffices to check the sign of $g(k) = f(k+1) - f(k)$. Note that $g(k) = \sqrt{9k^2 + 14k + 5} - \sqrt{9k^2 + 8k} - 1$ by $n = 3k + 2$. We claim that $g(k) > 0$. This is equivalent to

$$\sqrt{9k^2 + 14k + 5} > \sqrt{9k^2 + 8k} + 1.$$

Since the both sides are positive, we can equivalently transform as

$$9k^2 + 14k + 5 > 9k^2 + 8k + 1 + 2\sqrt{9k^2 + 8k}$$

that can be simplified as $3k + 2 > \sqrt{9k^2 + 4k}$. This is the desired inequality since $3k + 2 = \sqrt{9k^2 + 12k + 4} > \sqrt{9k^2 + 4k}$. Thus, $t^* = k + 1 = \frac{n+1}{3}$ and

$$y^* = \frac{\sqrt{(3n^2 + 2n - 1)/3} - (n+1)/3}{2} = \frac{\sqrt{9n^2 + 6n - 3} - n - 1}{6}.$$

We turn to showing that $y'(t^*, y^*)$ is well-defined and $t^* < n\left(1 - \frac{y^*}{y'(t^*, y^*)}\right)$. We have

$$\begin{aligned}
\frac{n\omega}{3} - \frac{\left(\sqrt{9n^2 + 6n - 3} - n - 1\right)\omega}{6} &= \frac{\omega}{6}\left(3n + 1 - \sqrt{9n^2 + 6n - 3}\right) \\
&> \frac{\omega}{6}\left(3n + 1 - \sqrt{9n^2 + 6n + 1}\right) \\
&= 0.
\end{aligned}$$

By $\frac{n\omega}{3} > \frac{\left(\sqrt{9n^2+6n-3} - n - 1\right)\omega}{6} = y^*$ and $t^* = k + 1 = \frac{n+1}{3} > \frac{n}{3}$,

$$n^2\omega^2 - 4(n - t^*)\omega y^* > n^2\omega^2 - 4\left(n - \frac{n}{3}\right)\omega\left(\frac{n\omega}{3}\right) = \frac{n^2\omega^2}{9} > 0.$$

Thus, $y'(t^*, y^*)$ is well-defined, and $y'(t^*, y^*) > \frac{2n\omega}{3}$ by this inequality and the definition of $y'(t^*, y^*)$. Moreover,

$$n\left(1-\frac{y^*}{y'\left(t^*, y^*\right)}\right) > n\left(1-\frac{n\omega/3}{2n\omega/3}\right) = \frac{n}{2} > \frac{n+1}{3} = k+1 = t^*$$

by $n \geq 5$, the desired inequality. Hence (c) is proved. ■

Proof of Theorem 2. The necessity follows from Proposition 1 and Lemma 7, 9. Thus, we turn to the proof of the sufficiency. For each $n \geq 3$, let

$$h(n) = \begin{cases} \frac{n\omega}{3} & \text{if } n = 3k \text{ for some } k \in \mathbb{N}; \\ \frac{\left(\sqrt{9n^2-6n-3}-n+1\right)\omega}{6} & \text{if } n = 3k+1 \text{ for some } k \in \mathbb{N}; \\ \frac{\left(\sqrt{9n^2+6n-3}-n-1\right)\omega}{6} & \text{if } n = 3k+2 \text{ for some } k \in \mathbb{N}. \end{cases}$$

Fix an arbitrary y such that $\max\left\{h(n), \frac{(n+1)\omega}{4}\right\} \leq y \leq \frac{n\omega}{2}$. We first prove that $X(y)$ is a stable set. Then, we turn to showing that $h(n) \geq \frac{(n+1)\omega}{4}$ for all $n \geq 3$.

Fix an arbitrary $y' \in]y, n\omega]$ and an arbitrary $x' \in X(y')$. We show that some $S \in \mathcal{N}$ satisfies (1)-(3) for these y' and x'. Note that $T(y, x') \neq \emptyset$ by Lemma 7 and $y > \frac{n\omega}{4}$. We distinguish three cases.

Case 1. $y' = n\omega$.

In this case, it is obvious that $X(n\omega)$ is a singleton and $x' = (\omega, ..., \omega)$. We show that N satisfies (1)-(3) in Proposition 1 for $y' = n\omega$ and $x' = (\omega, ..., \omega)$. Since $y > 0$, (1) follows from $x'(N) = n\omega > n\omega - y$, and (2) follows from $x'(N) = n\omega > y' - y$. Since $y > 0$ and $x'_i = \omega$ for all $i \in N$, (3) follows from $\omega y > 0 = (\omega - x'_i)y'$ for all $i \in N$.

Case 2. $y' < n\omega$ and $t^*(y, x') \geq n\left(1 - \frac{y}{y'}\right)$.

By $y \geq \frac{(n+1)\omega}{4}$ and Lemma 6, there exists some $s \in \{1, ..., n\}$ such that $S^*(x', s)$ satisfies both (1) and (2) in Proposition 1. Let $\bar{s}$ be the minimum integer such that $S^*(x', \bar{s})$ satisfies both (1) and (2). By Lemma 5, $\bar{s}$ is the minimum integer satisfying $\bar{s} \geq n\left(1-\frac{y}{y'}\right)$. Then, $\bar{s} \leq t^*(y,x')$ and $S^*(x',\bar{s}) \subset T^*(y,x')$ by the definitions of these two sets. Hence $S^*(x',\bar{s})$ satisfies (1)-(3) in Proposition 1.

Case 3. $y' < n\omega$ and $t^*(y,x') < n\left(1-\frac{y}{y'}\right)$.

Since $y \geq \max\left\{h(n), \frac{(n+1)\omega}{4}\right\} > \frac{n\omega}{4}$, $T^*(y,x')$ satisfies (1) by Lemma 8, and (2) by Lemma 9. Together with the definition of $T(y,x')$, $T(y,x')$ satisfies (1)-(3).

By Case 1-3, there exists some coalition that satisfies (1)-(3) in Proposition 1. Hence $X(y)$ is a stable set.

We turn to proving that $h(n) \geq \frac{(n+1)\omega}{4}$ for any $n \geq 3$. It is easy to see that $\frac{n\omega}{3} \geq \frac{(n+1)\omega}{4}$ by $n \geq 3$. Thus, we obtain the sufficiency of (a).

Let $n = 3k+1$ for some $k \in \mathbb{N}$. It suffices to show that

$$2\sqrt{9n^2-6n-3} \geq 5n+1 \tag{A.18}$$

since

$$\frac{\left(\sqrt{9n^2-6n-3}-n+1\right)\omega}{6} - \frac{(n+1)\omega}{4} = \frac{\left(2\sqrt{9n^2-6n-3}-5n-1\right)\omega}{12}.$$

By the positivity of both sides of (A.18), it suffices to show that

$$36n^2-24n-12 = \left(2\sqrt{9n^2-6n-3}\right)^2 \geq (5n+1)^2 = 25n^2+10n+1$$

that can be simplified as $11n^2-34n-13 \geq 0$. It is easy to check that this inequality holds for any $n \geq 4$. Thus, we obtain the sufficiency of (b).

Let $n = 3k+2$ for some $k \in \mathbb{N}$. It suffices to show that

$$2\sqrt{9n^2+6n-3} \geq 5n+5 \tag{A.19}$$

since

$$\frac{\left(\sqrt{9n^2+6n-3}-n-1\right)\omega}{6} - \frac{(n+1)\omega}{4} = \frac{\left(2\sqrt{9n^2+6n-3}-5n-5\right)\omega}{12}.$$

By the positivity of both sides of (A.19), it suffices to show that

$$36n^2 + 24n - 12 = \left(2\sqrt{9n^2 + 6n - 3}\right)^2 \geq (5n+5)^2 = 25n^2 + 50n + 25$$

that can be simplified as $11n^2 - 26n - 37 > 0$. It is easy to check that this inequality holds for any $n \geq 5$. Thus, we obtain the sufficiency of (c). Hence the sufficiency for each (a)-(c) is proved. ■

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