

Title: Chaos Resulting from Nonlinear Relations between Different Variables

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Abstract: In this study, we further develop the perturbation method of Marotto (1979) and investigate the general mechanisms responsible for nonlinear dynamics, which are typical of multidimensional systems. We focus on the composites of interdependent relations between different variables. First, we prove a general result on chaos, which shows that the cyclic composites of nonlinear interdependent relations are sources of chaotic dynamics in multidimensional systems. By considering several examples, we conclude that the cyclic composites play an important role in detecting chaotic dynamics.

Keywords: Chaos; Cyclic Composite; Perturbation Method.

0. Introduction

In this study, we have tried to answer the question, what types of nonlinearities yield chaotic oscillations? In particular, we are concerned with the general mechanism responsible for chaotic dynamics, which is typical of high-dimensional systems in R^n , $n \geq 2$. So far, little attention has been paid to such a general mechanism. In the present paper, we define the notion of cyclic composite that consists of interdependent relations between different variables. By using the cyclic composite, we can often reduce high-dimensional systems to one-dimensional systems. By reducing the number of dimensions and employing the perturbation method introduced by Marotto (1979), we prove a general result on the existence of chaos in high-dimensional systems. By demonstrating the application of the result to some examples, we clarify that the nonlinearity of cyclic composites is an important source of chaotic oscillations in high-dimensional systems. In particular, we demonstrate the application of the result to an interdependent consumer model and detect chaos.

This paper is organized as follows. In Section 1, some notions and a mathematical result on chaos are given. Moreover, we state a general sufficient condition for the existence of a snap-back repeller. In Section 2, we demonstrate the application of the results obtained by us to the interdependent consumer model. In Section 3, conclusions are given. In Appendix, the proofs of several results obtained in this paper are given.

1. Preliminaries and the Main Result

In this section, we assume that all the functions dealt with in this study are of class C^1 . Marotto (1978, 2005) was the first to prove that topological chaos is associated with the existence of a snap-back repeller. We explain it briefly. A fixed point c for $F: R^m \rightarrow R^m$ is called a snap-back repeller if all the eigenvalues of $JF(c)$ have absolute values larger than 1, and there exists a point $z_0 \neq c$ in $W_{loc}^U(c)$ (the local unstable set of c) and some positive integer N such that $F^N(z_0) = c$ and $\det JF(z_k) \neq 0$ for $k \in \{1, \dots, N\}$, where $z_k = f^k(z_0)$ and $Jf(u)$ denotes the Jacobian matrix of a map f evaluated at u . Marotto (1978) proved that if a fixed point c for $F: R^m \rightarrow R^m$ is a snap-back repeller, then $x_{n+1} = F(x_n)$ shows topological chaos such that the following conditions are true.

- (C.1) There exists a positive integer p_0 such that for each $p > p_0$, F has a point of period p .
- (C.2) There exists an uncountable set $S \subset R^m$ containing no periodic points of F such that
- (C.2.1) $F(S) \subset S$;
- (C.2.2) for every $y, z \in S$ with $y \neq z$,

$$\limsup_{k \rightarrow \infty} |F^k(y) - F^k(z)| > 0$$
; and
- (C.2.3) for every $y \in S$ and any periodic point z of F ,

$$\limsup_{k \rightarrow \infty} |F^k(y) - F^k(z)| > 0$$
.
- (C.3) There exists an uncountable subset $S_0 \subset S$ such that for every $y, z \in S_0$,

$$\liminf_{k \rightarrow \infty} |F^k(y) - F^k(z)| = 0$$
.

The perturbation method of Marotto (1979) is often useful for the detection of snap-back repellers. Marotto (1979) has listed some applications of the perturbation method to biological models. Dohtani et al. (1996) have discussed the application of the method to an economic model. By using the perturbation method, we herein derive a general result on the existence of a snap-back repeller, which states that the composites of nonlinear relations between different variables are sources of chaotic dynamics.

First, we define the notion of cyclic composites. Consider a parameterized s -dimensional discrete-time system:

$$(1.1) \quad \begin{cases} x_{1,n+1} = f_1(x_{1,n}, x_{2,n}, \dots, x_{s,n}, \varepsilon), \\ x_{2,n+1} = f_2(x_{1,n}, x_{2,n}, \dots, x_{s,n}, \varepsilon), \\ \dots\dots\dots \\ x_{s,n+1} = f_s(x_{1,n}, x_{2,n}, \dots, x_{s,n}, \varepsilon), \end{cases}$$

where $\varepsilon \in \Omega \subseteq R^k$. We here suppose that the set Ω contains a parameter $\varepsilon^\#$ that satisfies

$$\begin{aligned} \varphi_1(x_2) &= f_1(x_1, x_2, \dots, x_s, \varepsilon^\#), \quad \varphi_2(x_3) = f_2(x_1, x_2, \dots, x_s, \varepsilon^\#), \quad \dots\dots, \\ \varphi_{s-1}(x_s) &= f_{s-1}(x_1, x_2, \dots, x_s, \varepsilon^\#), \quad \varphi_s(x_1) = f_s(x_1, x_2, \dots, x_s, \varepsilon^\#). \end{aligned}$$

Then, we define a cyclic auxiliary system as follows:

$$(1.2) \quad \begin{cases} x_{1,n+1} = \varphi_1(x_{2,n}), \\ x_{2,n+1} = \varphi_2(x_{3,n}), \\ \dots\dots\dots \\ x_{s-1,n+1} = \varphi_{s-1}(x_{s,n}), \\ x_{s,n+1} = \varphi_s(x_{1,n}). \end{cases}$$

We define $\varphi \equiv \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_s$. For simplicity, we introduce the following notion.

Definition 1. The map φ is said to be a cyclic composite. ■

For the original system given by (1.1), we have the following result that shows that the nonlinearities of cyclic composites may be a source of chaotic dynamics.

Theorem 1. Suppose that the cyclic composite φ possesses a snap-back repeller r . Then $p = (p_1, \dots, p_s) \equiv (r, \varphi_2 \circ \dots \circ \varphi_s(r), \varphi_3 \circ \dots \circ \varphi_s(r), \dots, \varphi_{s-1} \circ \varphi_s(r), \varphi_s(r))$ is a snap-back repeller of the cyclic auxiliary system given by (1.2). Hence, there exists an open neighborhood $U(\varepsilon^\#)$ of $\varepsilon^\#$ and a continuous function $\tau: U(\varepsilon^\#) \cap \Omega \rightarrow R^s$ such that $\tau(\varepsilon)$ is a snap-back repeller of the original system given by (1.1) for any $\varepsilon \in U(\varepsilon^\#) \cap \Omega$ and $\tau(\varepsilon^\#) = p$ holds. ■

Proof. See Appendix. ■

We have used the perturbation method in Theorem 1. The theorem states that if the cyclic auxiliary system has a snap-back repeller, any system that is sufficiently close to the cyclic auxiliary system (i.e., any system slightly perturbed with respect to the cyclic auxiliary system) has a snap-back repeller, too. The reason for this is that snap-back repellers are robust under small perturbations. We provide two numerical examples to illustrate the application of Theorem 1.

Example 1: First, we provide a simple numerical example. Consider the following two-dimensional systems:

$$(1.3.1) \quad \begin{cases} x_{n+1} = f(x_n) - ay_n, \\ y_{n+1} = x_n, \end{cases}$$

$$(1.3.2) \quad \begin{cases} x_{n+1} = bx_n + g(y_n), \\ y_{n+1} = x_n. \end{cases}$$

The well-known Hénon map is of the type shown in (1.3.1). In fact, the f -function of the Hénon map is given by $f(x) = x^2 + 2$. Thus, the nonlinearity of the “diagonal” element is responsible for the chaotic dynamics of the Hénon map. The chaos detection method that involves focusing on the diagonal elements has often been utilized in economics. Dohtani et al. (1996) have demonstrated the use of this method. On the other hand, an example where Theorem 1 holds is given by (1.3.2). Theorem 1 shows that for suitable forms of the g -function, (1.3.2) also gives rise to chaos. Unlike the Hénon map, chaotic dynamics such as those of systems given by (1.3.2) are a result of the nonlinearity of the “off-diagonal” element. ■

In Example 1, we considered a case in which all the variables are separable. Theorem 1 is applicable to non-separable cases, too. It is useful to briefly illustrate the application of Theorem 1 to such a case.

Example 2: Consider the following two-dimensional system:

$$(1.4) \quad \begin{cases} x_{n+1} = ax_n - bx_n y_n + cy_n, \\ y_{n+1} = dx_n - ex_n^2 + hy_n^3, \end{cases}$$

where $\varepsilon \equiv (a, b, c, d, e, h) \in R_{++}^6 \equiv \{a > 0, b > 0, c > 0, d > 0, e > 0, h > 0\}$.

Before considering this system, we consider the following cyclic auxiliary system:

$$\begin{cases} x_{n+1} = cy_n \equiv \varphi_1(y_n), \\ y_{n+1} = dx_n - ex_n^2 \equiv \varphi_2(x_n). \end{cases}$$

Then, the cyclic composite is given by

$$u_{n+1} = \varphi(u_n) = \varphi_1 \circ \varphi_2(u_n) = cdu_n - c^2eu_n^2.$$

Since the fixed point of φ is given by $r = (cd - 1)/c^2e$, we have $\varphi'(r) = 2 - cd$. we set $c = 1$ and $d = e = 3.8$. Then, $u_{n+1} = \varphi(u_n) = 3.8u_n(1 - u_n)$. From the result of Marotto (1978)¹, it follows that the fixed point r is a snap-back repeller for this specific one-dimensional system. Then, it follows from Theorem 1 that there exists an open neighborhood $U(\varepsilon^\#)$ of $\varepsilon^\# \equiv (0, 0, 1, 3.8, 3.8, 0)$ in R^6 such that for any $\varepsilon \in U(\varepsilon^\#) \cap R_{++}^6$, the original system given by (1.4) yields a snap-back repeller and,

¹ Marotto (1978, EXAMPLE 4.1) numerically proved that if $a > 3.68$, the fixed point given by $r = 1 - (1/a)$ is a snap-back repeller of $u_{n+1} = au_n(1 - u_n)$.

therefore, topological chaos. That is, if a , b , and h are sufficiently small positive real numbers, c is close to 1, and d and e are close to 3.8, then the system given by (1.4) has topological chaos. We confirm this by performing a numerical simulation. We set $\varepsilon = \tilde{\varepsilon} = (0.06, 0.08, 1.01, 3.77, 3.82, 0.07)$. Part (1) of Figure 1 shows a chaotic attractor of the system given by (1.4). Since topological chaos is not always observable, computer simulations are necessary to make sure the observability of chaotic dynamics. Moreover, Part (2) of Figure 1 shows that the Lyapunov number of the path shown in Part (1) is larger than 1. ■

Figure 1 about here.

2. Application

In this section, we apply the result obtained in Section 1 to the interdependent consumer model constructed by Gaertner and Jungeilges (1988, 1993). We consider three consumers. The utility functions of the consumers are of the Cobb-Douglas type:

$$u_1(x_1, y_1) = x_1^r y_1^{1-r}, \quad u_2(x_2, y_2) = x_2^s y_2^{1-s}, \quad u_3(x_3, y_3) = x_3^t y_3^{1-t},$$

where x_k ($k \in \{1, 2, 3\}$) is the consumption of goods x by person k and y_k ($k \in \{1, 2, 3\}$) is the consumption of goods y by person k ; further, $0 < r, s, t < 1$. The budget constraint of person k (i.e., $I_k \geq p_x x_k + p_y y_k$, $k \in \{1, 2, 3\}$) yields the subsequent demand functions at time n :

$$\begin{aligned} x_{1,n} &= rI_1 / p_x, & y_{1,n} &= (1-r)I_1 / p_y, & x_{2,n} &= sI_2 / p_x, \\ y_{2,n} &= (1-s)I_2 / p_y, & x_{3,n} &= tI_3 / p_x, & y_{3,n} &= (1-t)I_3 / p_y. \end{aligned}$$

We assume a situation in which each individual's elasticity of preference depends on his/her own consumption and the consumption of the other persons in the immediate past:

$$\begin{aligned} r_{n+1} &= \theta_r(x_{1,n}, x_{2,n}, x_{3,n}, y_{1,n}, y_{2,n}, y_{3,n}), \\ s_{n+1} &= \theta_s(x_{1,n}, x_{2,n}, x_{3,n}, y_{1,n}, y_{2,n}, y_{3,n}), \\ t_{n+1} &= \theta_t(x_{1,n}, x_{2,n}, x_{3,n}, y_{1,n}, y_{2,n}, y_{3,n}). \end{aligned}$$

If the above dependence is nonlinear, then chaotic sequences of consumption can emerge. For example, to illustrate this fact, we consider a case in which the dependence is given by the following relations:

$$\begin{aligned}
r_{n+1} &= \alpha_{11}x_{1,n}y_{1,n} + \alpha_{12}x_{2,n}y_{2,n} + \alpha_{13}x_{3,n}y_{3,n}, \\
s_{n+1} &= \alpha_{21}x_{1,n}y_{1,n} + \alpha_{22}x_{2,n}y_{2,n} + \alpha_{23}x_{3,n}y_{3,n}, \\
t_{n+1} &= \alpha_{31}x_{1,n}y_{1,n} + \alpha_{32}x_{2,n}y_{2,n} + \alpha_{33}x_{3,n}y_{3,n},
\end{aligned}$$

where α_{kj} ($k, j \in \{1, 2, 3\}$) is a parameter showing the influence of person j 's consumption pattern on person k 's consumption pattern. This type of nonlinear experience-dependent function was introduced by Gaertner and Jungeilges (1988). It generalizes the experience-dependent function originally introduced by Benhabib and Day (1981). Here, we assume $\alpha_{kj} \geq 0$ ($k, j \in \{1, 2, 3\}$). Then, a simple calculation yields the three-dimensional discrete-time interdependent consumer model given by the following relations:

$$(2.1) \quad \begin{cases} v_{1,n+1} = \pi_{11}v_{1,n}(1-v_{1,n}) + \pi_{12}v_{2,n}(1-v_{2,n}) + \pi_{13}v_{3,n}(1-v_{3,n}), \\ v_{2,n+1} = \pi_{21}v_{1,n}(1-v_{1,n}) + \pi_{22}v_{2,n}(1-v_{2,n}) + \pi_{23}v_{3,n}(1-v_{3,n}), \\ v_{3,n+1} = \pi_{31}v_{1,n}(1-v_{1,n}) + \pi_{32}v_{2,n}(1-v_{2,n}) + \pi_{33}v_{3,n}(1-v_{3,n}), \end{cases}$$

where

$$\begin{aligned}
(v_{1,n}, v_{2,n}, v_{3,n}) &\equiv \left(\frac{p_x x_{1,n}}{I_1}, \frac{p_x x_{2,n}}{I_2}, \frac{p_x x_{3,n}}{I_3} \right), \quad \pi_{jk} \equiv \frac{\alpha_{jk} I_k^2}{p_x p_y} \quad (j, k \in \{1, 2, 3\}), \\
(\pi_{11}, \pi_{12}, \pi_{13}, \pi_{21}, \pi_{22}, \pi_{23}, \pi_{31}, \pi_{32}, \pi_{33}) &= \varepsilon \in [0, \infty)^9 \equiv \Omega.
\end{aligned}$$

We define $g_{ij}(v_j) \equiv \pi_{ij}v_j(1-v_j)$ ($i, j \in \{1, 2, 3\}$). It is well known that the one-dimensional map $u_{n+1} = au_n(1-u_n)$ shows chaotic dynamics (Example 2). Therefore, the nonlinearity of the map g_{jj} ($j \in \{1, 2, 3\}$) can be a direct source of chaos in the system given by (2.1). Such an expectation is essentially based on the remarkable findings of Benhabib and Day (1981), which show that consumer behavior is chaotic when the preference of a consumer depends on his/her own past consumption decisions. However, in this paper, we consider the cyclic composites of the nonlinear interdependent consumer decisions of different persons. Especially, we focus on the cyclic composites $g_{12} \circ g_{23} \circ g_{31}$. In this section, we demonstrate that the cyclic composites can be sources of chaos in the interdependent consumer model. By using Theorem 1, we now obtain several results for the system given by (2.1).

Result 1. Let $\varepsilon^\# \equiv (0, 3.6, 0, 0, 0, 3.1, 3.9, 0, 0)$. There is an open neighborhood $U(\varepsilon^\#)$ of $\varepsilon^\#$ in R^9 such that for any $\varepsilon \in U(\varepsilon^\#) \cap \Omega$, the interdependent consumer model given by (2.1) possesses a snap-back repeller. ■

Proof. See Appendix. ■

Result 1 shows that for any $\varepsilon \in U(\varepsilon^\#) \cap \Omega$, the consumer model (2.1) has topological chaos. Now, we set $\tilde{\varepsilon} \equiv (0.08, 3.58, 0.1, 0.1, 0.06, 3.12, 3.8, 0.07, 0.06) \in U(\varepsilon^\#) \cap \Omega$. Part (1) of Figure 2 describes a chaotic attractor in the case where $\varepsilon = \tilde{\varepsilon}$. Moreover, Part (2) of Figure 2 shows that the Lyapunov number of the path shown in Part (1) is larger than 1.

Figure 2 about here.

3. Conclusions and Final Remark

We defined the notion of cyclic composites of interdependent relations between different variables and considered high-dimensional systems in which a cyclic composite can be constituted for specific parameters. By using the cyclic composite, a high-dimensional system with specific parameters is reduced to a one-dimensional system. By reducing the number of dimensions and employing the perturbation method introduced by Marotto (1979), we proved a general result on the existence of chaos in the original high-dimensional system. By the application of the result to some examples, we showed that the cyclic composites of different variables are sources of chaotic dynamics in high-dimensional economic systems. In particular, we applied the results to an interdependent consumer model in which each individual's consumption pattern depended not only on his/her own past consumption decisions but also on the consumption decisions of other persons; moreover, we showed that the cyclic composites of the nonlinear interdependent relations between different persons can be sources of chaotic consumption patterns. We expect that by paying attention to the cyclic composites, new insights into chaotic dynamics are obtained.

4. Appendix

In the appendix, we prove Theorem 1 and Result 1. For simplicity, we denote $W_{loc}^S(p)$ as W_{loc}^S and $W_{loc}^U(p)$ as W_{loc}^U .

We define the orbit $O = \{\Psi^n(a_1, \dots, a_s) \equiv (b_{1n}, \dots, b_{sn}) : n = 0, 1, \dots\}$. Now, let us prove that O is a homoclinic orbit of the system given by (1.2) for p (i.e., for some M , $(b_{1M}, \dots, b_{sM}) = p$). Since $a_{1N} = \varphi^N(a_1) = r$, the equations given by (4.1) and (4.4) show that

$$(4.5) \quad \Psi^{sN}(a_1, \varphi_{2\dots s}(a_1), \dots, \varphi_s(a_1)) = (a_{1N}, \varphi_{2\dots s}(a_{1N}), \dots, \varphi_s(a_{1N})) = p.$$

Then, it follows from (4.3) that O is a homoclinic orbit of the system given by (1.2). Next, we prove that p is a snap-back repeller. Since $\varphi = \varphi_{1\dots s}$, we see from (4.2) that

$$\begin{aligned} 0 \neq \frac{d\varphi^N(a_1)}{du} &= \prod_{j=0}^{N-1} \frac{d\varphi(\varphi^j(a_1))}{du} = \prod_{j=0}^{N-1} \frac{d\varphi(a_{1j})}{du} \\ &= \prod_{j=0}^{N-1} \frac{d\varphi_1(\varphi_{2\dots s}(a_{1j}))}{du} \bullet \frac{d\varphi_2(\varphi_{3\dots s}(a_{1j}))}{du} \bullet \dots \bullet \frac{d\varphi_s(a_{1j})}{du}. \end{aligned}$$

Here, note that $a_1 = a_{10}$. Thus, we can conclude that for any $n \in \{0, \dots, N-1\}$,

$$(4.6) \quad \varphi_1'(a_{1n}) \neq 0, \dots, \varphi_s'(a_{1n}) \neq 0.$$

On the other hand, since r is a snap-back repeller for φ , we see that $0 \neq \varphi'(r) = \varphi_1'(\varphi_{2\dots s}(r)) \cdots \varphi_s'(r)$. Since $a_{1N+k} = \varphi^{N+k}(a_1) = \varphi^k(r) = r$, for any $k \geq 0$,

$$(4.7) \quad \varphi_1'(\varphi_{2\dots s}(a_{1N+k})) = \varphi_1'(\varphi_{2\dots s}(r)) \neq 0, \dots, \varphi_s'(a_{1N+k}) = \varphi_s'(r) \neq 0.$$

Consequently, by combining (4.6) with (4.7), we can obtain the following expression for any nonnegative integer n :

$$(4.8) \quad \varphi_1'(\varphi_{2\dots s}(a_{1n})) \neq 0, \dots, \varphi_s'(a_{1n}) \neq 0.$$

On the other hand, we have

$$\begin{aligned} &\det J\Psi(b_{1sn+k}, b_{2sn+k}, \dots, b_{ssn+k}) \\ &= \det \begin{bmatrix} 0 & \varphi_1'(b_{2sn+k}) & 0 & \cdots & 0 \\ 0 & 0 & \varphi_2'(b_{3sn+k}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \varphi_{s-1}'(b_{ssn+k}) \\ \varphi_s'(b_{1sn+k}) & 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= (-1)^{s+1} \varphi_1'(b_{2sn+k}) \varphi_2'(b_{3sn+k}) \cdots \varphi_{s-1}'(b_{ssn+k}) \varphi_s'(b_{1sn+k}). \end{aligned}$$

From (4.4) and (4.8), it follows that $\det J\Psi(b_{1sn+k}, b_{2sn+k}, \dots, b_{ssn+k}) \neq 0$ for any $k \in \{0, 1, \dots, s-1\}$ and $n \in \{0, 1, \dots\}$. That is,

$$(4.9) \quad \det J\Psi(b_{1j}, b_{2j}, \dots, b_{3j}) \neq 0 \quad \text{for any } j \in \{0, \dots\}.$$

As proved above, we have $|\lambda| > 1$. Therefore, from (4.3), (4.5), and (4.9), we conclude that the fixed point p is a snap-back repeller. This proves the first half of Theorem 1. The latter half follows directly from the fact that any snap-back repeller persists under small perturbations. ■

Proof of Result 1: It is sufficient to prove that the cyclic auxiliary system with $\varepsilon = \varepsilon^\#$ satisfies all the conditions of Theorem 1. The cyclic auxiliary system is given by the following relations:

$$(4.10) \quad \begin{cases} v_{1,n+1} = 3.6v_{2,n}(1-v_{2,n}) = g_{12}(v_{2,n}), \\ v_{2,n+1} = 3.1v_{3,n}(1-v_{3,n}) = g_{23}(v_{3,n}), \\ v_{3,n+1} = 3.9v_{1,n}(1-v_{1,n}) = g_{31}(v_{1,n}). \end{cases}$$

Consider the following cyclic composite system:

$$(4.11) \quad \begin{aligned} u_{n+1} = \xi(u_n) &\equiv g_{12} \circ g_{23} \circ g_{31}(u_n) \\ &= 3.6 \times 3.1 \times 3.9 u_n (1 - u_n) \{1 - 3.9 u_n (1 - u_n)\} \\ &\quad \times [1 - 3.1 \times 3.9 u_n (1 - u_n) \{1 - 3.9 u_n (1 - u_n)\}]. \end{aligned}$$

Numerical calculation yields the following: $\xi(0.77881) > 0.77881$ and $\xi(0.77882) < 0.77882$. Therefore, the system given by (4.11) has a fixed point that exists in $(0.77881, 0.77882)$. We denote the fixed point by r . By numerical calculation, we see that $|d\xi(u)/du| > 1$ for any $u \in (0.77881, 0.77882)$, so that

$$(4.12) \quad |d\xi(r)/du| > 1.$$

Theorem 1 shows that p , given by $(r, g_{23} \circ g_{31}(r), g_{31}(r))$, is a fixed point of the system given by (4.10). We define $I \equiv (0.763, 0.764)$. Then, by numerical calculation, we obtain $r \in \xi^3(I)$. This implies that there exists $z \in I$ such that

$$(4.13) \quad \xi^3(z) = r.$$

Here, we denote a set of critical values by H . Then, $d\xi(u)/du = 0$ for any $u \in H$. By numerical calculation, we obtain $H \cap \bigcup_{k=0}^3 \xi^k(I) = \emptyset$, so that

$$(4.14) \quad d\xi(\xi^k(z))/du \neq 0 \quad \text{for any } k \in \{0, 1, 2\}.$$

We define $J \equiv (0.73, 0.8) \supset I \cup \{z\}$ and $\pi \equiv \xi|_J$. Then, by numerical calculation, we

obtain that $d\pi(u)/du < -1.1$ for any $u \in J$ and $\pi^{-1}(J) \subseteq J$. Thus, we see that $\lim_{n \rightarrow \infty} \pi^{-n}(z) = r$. Therefore, for a sufficiently large positive N , we have $\pi^{-N}(z) \in W_{loc}^u(r)$. By combining this result with (4.12) to (4.14), we see that r is a snap-back repeller of the cyclic composite ξ . Thus, Result 1 has been proved. ■

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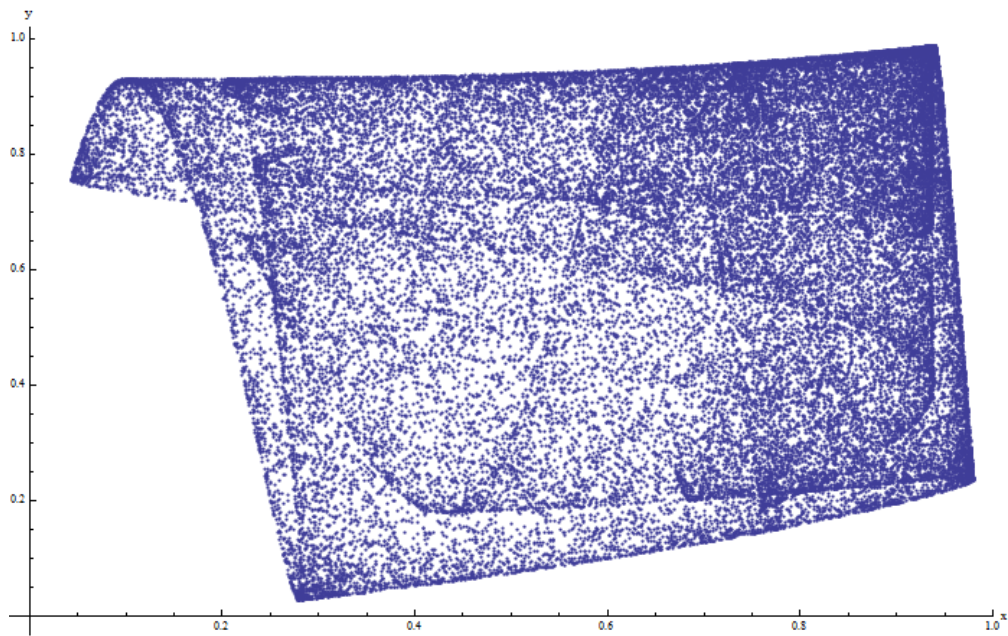
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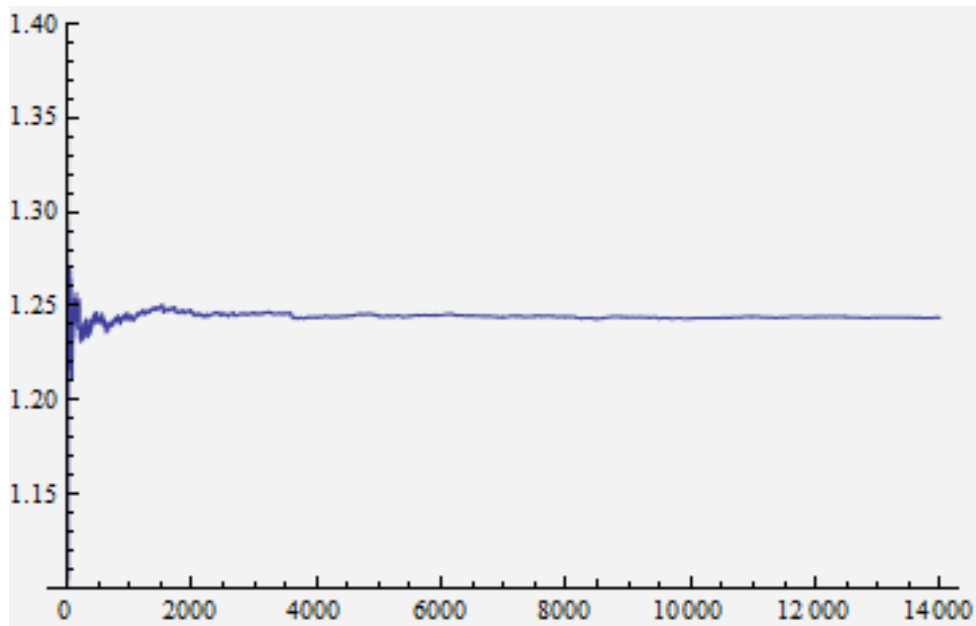
Captions of Figures

Caption of Figure 1: (1)-Chaotic path of the system given by (1.4) where $\varepsilon = \tilde{\varepsilon}$.
(2)- Lyapunov number of the chaotic path of Figure 1.1.

Caption of Figure 2: (1)-Chaotic path of the interdependent consumer model given by (2.1) where $\varepsilon = \tilde{\varepsilon}$. (2)-Lyapunov number of the chaotic path of Figure 2.1.

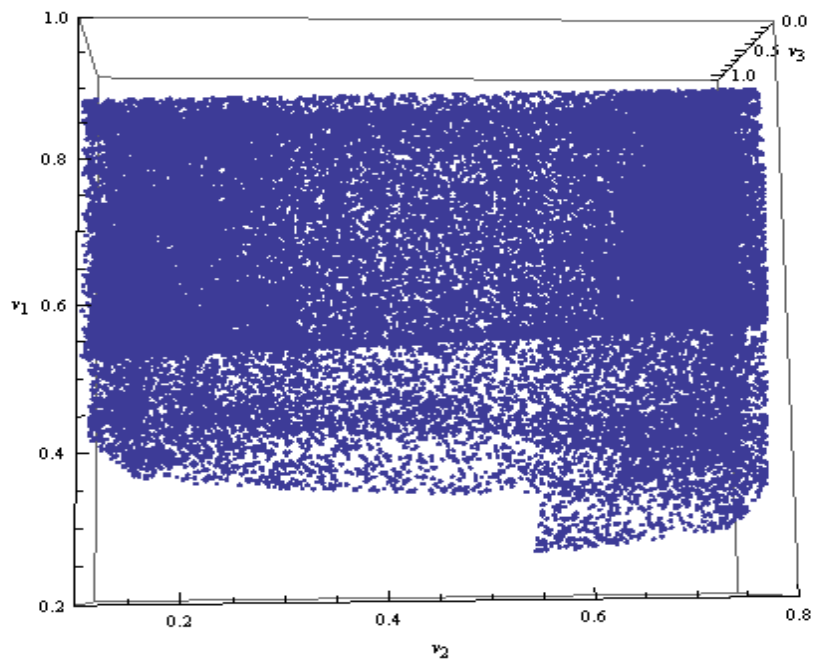


(1)

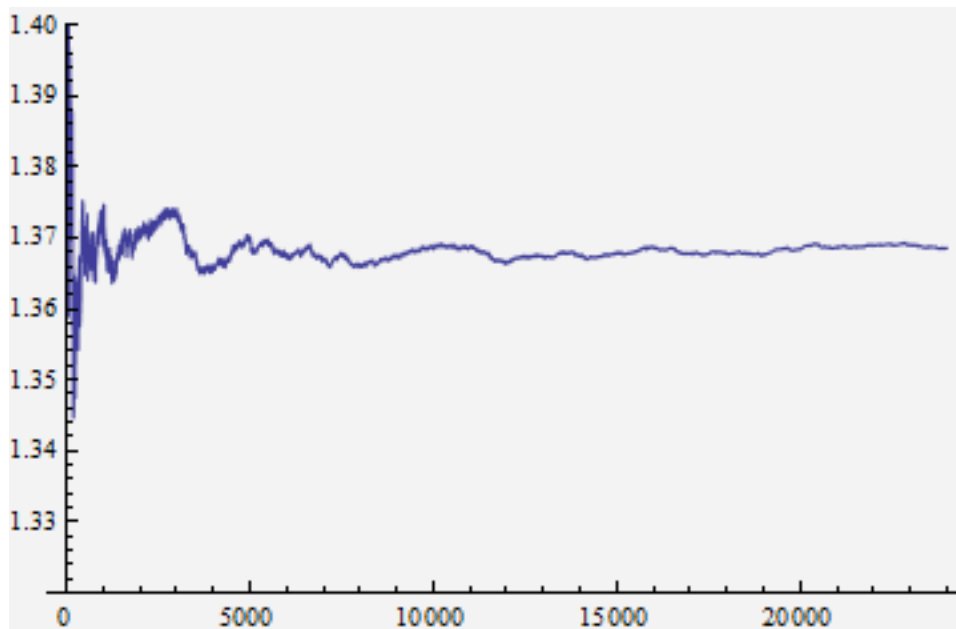


(2)

Figure 1



(1)



(2)

Figure 2