Title: A Growth-Cycle Model of the Solow-Swan Type, I

Subtitle: Growth-Cycle, I

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Abstract: We construct an endogenous growth-cycle model of the Solow-Swan type. The equilibrium point of the growth-cycle model is the same as the steady state of the Solow-Swan growth model. Unlike in the Solow-Swan growth model, the representative household in the growth-cycle model, however, adaptively estimates his/her average income and determines his/her consumption in proportion to average income. We prove that if the steady state is unstable, any non-equilibrium path converges to a limit cycle. However, even if the steady state is stable, growth cycles can emerge. In fact, we prove that the growth-cycle model generates corridor stability. As a result, we prove that there exists an unstable cyclic path such that any path in the interior of the cyclic path converges to the steady state and any path in the exterior of the cyclic path tends toward a limit cycle (growth cycle). We also prove that a high economic growth rate is not compatible with a stable economy.

Keywords: Solow-Swan growth model; Growth cycles; Consumption function; Poincaré-Bendixson theorem; Limit cycles; Corridor stability; Hopf bifurcation.

JEL classification: E2; E3; O4.

1. Introduction

After the Keynesian growth model by Harrod (1939) and Domar (1946), Solow (1956) and Swan (1956) constructed the neoclassical growth model. Harrod and Domar tried to identify the mechanism under which the steady state is unstable. Conversely, from a more long-run viewpoint, Solow and Swan proved the global stability of the steady state.[1] The Solow-Swan growth model has laid the foundation for the advance of growth theory. For a recent empirical study on the Solow-Swan model, see Mankiw, Romer and Weil (1992).

On the other hand, there have existed several economists who have been interested in the occurrence of growth cycles. Of these, remarkable are Hicks (1950) and Goodwin (1955). Models constructed in these works, however, are insufficient to explain growth. Recently, an ingenious model on growth cycles has been devised by Day (1982). The model has made a great impact on many economists who investigate economic growth and fluctuation. By introducing the influence of pollution into the production function or considering a nonlinear saving function, Day showed the possible occurrence of chaos in the Solow-Swan growth model. For the deterministic growth models of the Solow-Swan type, it has been known that persistent fluctuation may arise "only" when allowing for non-convexities or externalities in production and/or a nonlinear aggregate saving function. However, more recently, it has been proved that this understanding is not always correct. In fact, by introducing constant but different savings propensities out of capital and labor income, the insightful paper by Böhm and Kaas (2000) showed that complex dynamics emerge.[2] These models are of the discrete-time type. In the present paper, we consider a "continuous-time"

[1] Solow argued that the Harrod and Domar's models assume that the capital-output ratio is rigidly determined by a fixed-coefficients production function and, therefore, that no mechanism exists to bring the warranted and natural rates into line. Solow excluded from his model the possibility of divergence between the warranted and actual rates of growth. The warranted rate is the growth rate of equilibrium income. Harrod, however, relied on the instability of the warranted rate, which results from such divergence. Moreover, Harrod considered that such instability is caused by *short-run disequilibrium* (discrepancies between savings and planned investment). Thus, the standpoint of Harrod (1939) is essentially different from Solow (1956). For these points, see Boianovsky and Hoover (2009).

[2] With regard to the optimal growth models, there have been many papers on the occurrence of cyclic and chaotic paths. For discrete-time models, see for example Benhabib and Nishimura (1985), Boldrin and Montrucchio (1986), and Deneckere and Pelikan (1986). For continuous-time models, see Benhabib and Nishimura (1979). On the other hand, Commendatore (2005) and Commendatore and Palmisani (2008) showed that chaos emerges in a discrete-time version of the two-class model of growth and distribution proposed by Pasinetti (1962) and Samuelson and Modigliani (1966).

growth-cycle model of the Solow-Swan type. We assume that the representative consumer plans his/her consumption plan from a long-run perspective. We show that such a consumption plan leads us to cyclical dynamics.

The continuous-time growth model we consider is a growth-cycle version of Dohtani *et al*. (2007). The growth-cycle model possesses the same steady state as the Solow-Swan growth model. Unlike the Solow-Swan growth model, the growth-cycle model, however, supposes a proportional relation not between consumption and income but between consumption and expected average income. Since expected average income usually changes slowly, in comparison with the Solow-Swan growth model, consumption in the growth-cycle model is usually not very sensitive to income. Despite the slight difference between the constructions of the two growth models, there appears to be a big difference between their dynamics. The growth-cycle model yields a stable limit cycle that implies the emergence of a growth cycle. In this case, one aspect of the economic viewpoint of Keynesian is realized because of the instability of the steady state.[3] Moreover, interesting dynamics emerge. Corridor stability also emerges, which implies that there exists an unstable Hopf cycle such that any path in the interior of the Hopf cycle converges to the steady state. The result implies that the economic viewpoint of the neoclassical school is realized in the interior of the Hopf cycle but not in the exterior of the Hopf cycle. In the exterior of the Hopf cycle, the Keynesian economic viewpoint is realized because of the instability of the domain. Thus, the occurrence of corridor stability implies that both viewpoints coexist.[4] The notion of corridor stability was originally introduced by Leijonhufvud (1973).[5] On the

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[3] The economic viewpoint of the Keynesian stated here implies the instability of the steady state. Like the Solow-Swan model, we exclude from our model the possibility of short-run disequilibrium that is an important Keynesian viewpoint. See Footnote 1.

[4] For ideological or cosmological consideration of the stability of the market equilibrium, see Leijonhufvud (1973). He presented the corridor stability as a new type of economic cosmology.

[5] For the notion of corridor stability, see Rosser (1991, Section 6.1), Owase (1991), and Lorenz (1993, Section 3.2). See also Semmler and Sieveking (1993), Kind (1999), and Flaschel (2001). Leijonhufvud (1973) suggested the relation between permanent income and corridor stability. The notion of average income in the present paper is closely related to that of permanent income. Harrod (1973, 33) introduced an economic view that is roughly similar to the notion of corridor stability:

> *There are likely to be some deviations all the time. But if they are of moderate dimensions, I would not suppose that they would bring the instability principle into operation. That is why I so much object to the knife-edge idea. It requires a fairly large deviation, such as might be caused by a revision of assessments across the board in some important industry, like the motor car industry, to produce a deviation sufficient to bring the instability principle into play.*

To the best of my knowledge, Yoshida (1999) was the first to point out this fact.

other hand, Benhabib and Miyao (1981) related such a notion to the situation where an unstable Hopf cycle emerges.

The purpose of the present paper is to complete the analysis of global dynamics and show the endogenous emergence of stable limit cycles (i.e., growth cycles). As a method of detecting limit cycles, we have the Hopf Bifurcation Theorem. The Hopf Bifurcation Theorem is a local bifurcation analysis and gives no information on global dynamics. Therefore, in order to complete the global dynamic analysis and to prove the occurrence of business cycles, the present paper uses the phase diagram analysis and the Poincaré-Bendixson Theorem.

The paper is organized as follows. In Section 2, by introducing a linear relation between consumption and expected average income into the capital accumulation equation that appears in the Solow-Swan growth model, we construct a growth-cycle model of the Solow-Swan type. In section 3, we use the phase diagram analysis to analyze the global dynamics of the growth-cycle model and the Poincaré-Bendixson Theorem to prove the occurrence of limit cycles. Moreover, we use the Hopf Bifurcation Theorem to prove the occurrence of corridor stability. Section 4 gives the conclusions and the final remarks. The appendix gives the proofs of some results.

2. The Modified Growth Model

By modifying the growth model of Dohtani *et al.* (2007),[6] we construct a growth-cycle version of the continuous-time Solow-Swan growth model. Throughout this paper, any function defined in the present paper is assumed to be of class C^1. Apart from the decision making on consumption, the capital accumulation equation in the present paper is the same as that in the Solow-Swan growth model. That is,

$$\dot{k} = f(k) - (n+\delta)k - c,$$

where k is the capital stock per capita, c is the per capita consumption, n is the growth rate of population, δ is the depreciation rate, and f is a production function. Henceforth, we omit the term "per capita." In the sense that the growth-cycle model

[6] Dohtani *et al.* (2007) tried to show the possible occurrence of an unstable Hopf cycle in a growth-cycle model of the Solow-Swan type. The calculation of the stability formula in Dohtani *et al.* (2007, Proof of Theorem 3 in Appendix) is, however, incorrect. In this paper, we correct the error.

contains the capital accumulation equation, the model is of the Solow-Swan type. We here introduce the following standard assumptions:

Assumption 1: $f(0)=0$, $f'(k)>0$, and $f''(k)<0$ for any $k>0$;
Assumption 2: $\lim_{k\to 0} f'(k)=+\infty$ and $\lim_{k\to +\infty} f'(k)=0$.

By assuming the proportional relation between income and consumption, Solow and Swan proved the stability of the steady state. In what follows, by assuming a proportional relation between expected average income and consumption, we will prove the endogenous emergence of a growth cycle.

We assume a representative household. The representative household is assumed to estimate adaptively the following expected average (per capita) income y_A:

$$\dot{y}_A = \omega(y - y_A), \tag{2.1}$$

where ω is a positive adjustment coefficient. Dohtani *et al*. (2007) called the expected average income the expected permanent income, whose consideration is based on Friedman (1957). To be sure, in the case where ω is a constant, the expected average income in this paper has a feature in common with the original notion of permanent income. In fact, the dynamic equation (2.1) can be derived from the notion of permanent income by Friedman. We briefly see this. Consider that the permanent income of the household is determined by the following weighted average (or distributed lag) of y:

$$y_A(t) = \int_{(-\infty,t]} \omega y(\tau)\exp(\omega(\tau - t))d\tau, \tag{2.2}$$

where ω is a constant weight. For the expression (2.2), see Friedman (1957, p. 268). To transform equation (2.2) into a differential equation, we consider its time derivative. System (2.1) is obtained as follows:

$$\begin{aligned}\dot{y}_A(t) &= -\omega\exp(-\omega t)\int_{(-\infty,t]} \omega y(\tau)\exp(\omega\tau)d\tau + \exp(-\omega t)\omega y(t)\exp(\omega t)\\ &= \omega\{y(t) - y_A(t)\}.\end{aligned}$$

We later consider an adjustment coefficient that is a non-constant function. In such a case, in the sense that ω is not a constant, it may be inappropriate to connect equation

(2.1) with the notion of permanent income. As such, we assume equation (2.1) and not (2.2).

Equation (2.1) represents the situation that the representative household slowly and gradually changes his/her expected average income by referring to income at each time. Since unlike the Solow-Swan growth model, we assume a linear relation between expected average income and consumption, we obtain a dynamic equation of consumption. We explain this. We assume that the household determines his/her consumption by

$$c = \alpha y_A,$$

where $1 > \alpha > 0$. This equation shows that the representative household plans his/her consumption plan from a long-run perspective. The proportional relation is similar to the linear consumption function used in the Solow-Swan growth model. Hence, we simply call it the consumption function. If ω is small, then the change in expected average income is slow. As a result, in comparison with the Solow-Swan growth model, consumption determined by the above consumption function is not very sensible to income. In the following, we prove that insensibility becomes an important cause of the endogenous occurrence of cyclic paths (i.e., growth cycles). Differentiating both sides of the consumption function and substituting equation (2.1) yield

$$\dot{c} = \alpha \dot{y}_A = \alpha\omega(y - y_A) = \omega\{\alpha f(k) - c\}. \tag{2.3}$$

We call equation (2.3) the dynamic consumption equation.[7]

Since the household is representative, it is plausible that he/she does not spend all of his/her income. We later provide an empirical support for this situation. In order to consider such a situation, we introduce some assumptions. Due to these assumptions, we can avoid complete divergence and lock growth paths in a plausible domain.

We assume that the adjustment coefficient of the dynamic consumption equation depends on the propensity to save. Define

$$\omega = \beta\psi(\xi), \quad \xi = \text{propensity to save} = 1 - c / f(k) = s / f(k),$$

where β is a positive constant, s is savings, and ψ is assumed to satisfy

Assumption 3: $\psi(\xi) > 0$ and $\psi'(\xi) \le 0$ for any ξ,
Assumption 4: $\beta\psi(0) > (n + \delta)/(1 - \alpha)$.

[7] Dohtani *et al.* (2007) called equation (2.3) the consumption equation.

The parameter β will be used later as a (Hopf) bifurcation parameter. We call ψ the adjustment function.

In the following, we explain the economic implication of Assumptions 3 and 4. From the viewpoint of behavioral economics, it has been known that households follow the prudent rule that they live within their income. See Thaler (1990). As a reason, we have liquidity constraints. Thaler stresses that an important source of liquidity constraints is self-imposed rules used by households who simply do not like to be in debt. See also Thaler (1992, Chapter 9). In this paper, we consider that the representative household follows such a prudent rule. We specify ψ to explain the economic implication. We set

(2.4) $$\psi(\xi) = \begin{cases} 1 & \text{if} \quad \xi > a, \\ b(a-\xi)^{\sigma} + 1 & \text{if} \quad a \geq \xi \geq 0, \end{cases}$$

Figure 1 about here.

where a and b are positive constants and $\sigma > 1$. We assume $a < 1-\alpha$. It is easy to see that function (2.4) satisfies Assumptions 3 and 4. Clearly, we have $\psi(1-\alpha) = 1$. This function is a typical adjustment function that satisfies Assumptions 3 and 4. The specified ψ is utilized later in numerical simulations. The adjustment coefficient (i.e., the adjustment function) depends on the propensity to save, ξ. Moreover, under the above specification, the adjustment coefficient is monotonously decreasing in a small neighborhood of $\xi = 0$ and constant outside of that neighborhood. See Figure 1. We explain why the adjustment coefficient varies with the propensity to save merely in a small neighborhood of $\xi = 0$ Consider the case where the economy is in a sufficiently small neighborhood of $\xi = s/y = 0$ so that for a sufficiently small $\tilde{a} < a$, we have $s = \tilde{a}y$, and hence, $c = (1-\tilde{a})y$. Then, in such a case, it follows from $a < 1-\alpha$ that

(2.5) $$y = s/\tilde{a} = c/(1-\tilde{a}) < c/(1-a) < c/\alpha = y_A.$$

That is, the expected average income is larger than the actual income and the consumption planned based on the expected average income is so large that the savings is nearly zero. As stated above, such a situation is undesirable for the representative household who follows the prudent rule and does not like to be in debt. Therefore, it is natural to suppose that the households judge such a value of expected average income to be “overestimated” and try to revise largely the expected average income by reducing the difference between the expected average income and the actual income. That is, the households reduce the expected average income more largely than in the outside of the neighborhood of $\xi = 0$. Since the inequality (2.5) yields $\dot{y}_A < 0$, this implies that the adjustment speed of the expected average income is larger than that in the outside. The form of the adjustment function in the neighborhood of $\xi = 0$ describes such a situation. See Figure 1. Using the typical adjustment function (2.4), we explain the economic implication of Assumption 3. On the other hand, although Assumption 4 is related to Assumption 3, it is a mathematical condition. As shown in the next section, Assumption 4 implies that the adjustment coefficient at $\xi = 0$ (or $y = c$) is so large that the growth paths are included in the domain in which $\xi > 0$ (or $y > c$) is satisfied. We might suppose directly the typical adjustment function (2.4) instead of assuming Assumptions 3 and 4. It, however, suffices to assume Assumptions 3 and 4 when detecting periodic paths by the Poincaré-Bendixson theorem.

By combining the capital accumulation equation and the dynamic consumption equation, we now obtain the following continuous-time growth-cycle model:

$$\Lambda : \begin{cases} \dot{k} = f(k) - (n+\delta)k - c, \\ \dot{c} = \beta\psi(1 - c/f(k))\{\alpha f(k) - c\}. \end{cases}$$

The equilibrium point of System Λ is obtained by solving the equations $f(k) = (n+\delta)k/(1-\alpha)$ and $c = \alpha f(k)$ for (k, c). We will see later that System Λ possesses a unique equilibrium point in the positive quadrant. It is now clear that the equilibrium point is equal to the neoclassical steady state. We use (k^*, c^*) to denote the steady state.

We are now in a position to analyze the dynamic behavior of System Λ. Assumptions 1 to 4 are necessary for the argument below. As such, throughout the paper, we work under Assumptions 1 to 4 without any notification.

3. Dynamic Analysis

As stated above, in the present paper, we stress the phase diagram analysis that we use to understand the global dynamics of System Λ completely. As a result, we observe the endogenous occurrence of a growth cycle. However, first we prove several preliminary lemmas.

Lemma 1: $f(k)/k > f'(k)$ for any $k > 0$. ■

Proof: See Appendix. ■

Using Lemma 1, we can prove the following lemma that plays the most important role in proving the occurrence of growth cycles.

Lemma 2: The steady state is uniquely determined in the positive quadrant $R^2_{++} = \{(x, y) \in R^2 : x > 0, \ \ y > 0\}$. Moreover, the vector field on the graph of $c = f(k)$ except for (0,0) points downward on the graph. ■

Proof: Define $D(k) = (1-\alpha) f(k) - (n+\delta)k$. Then, it follows from Assumption 2 that $\lim_{k \to 0} D'(k) = +\infty$ and $\lim_{k \to +\infty} D'(k) = -(n+\delta)$. This implies that $D(k)$ intersects the k −axis in R^2_{++}. Since $D''(k) = (1-\alpha) f''(k) < 0$, the intersection is uniquely determined in R^2_{++}. Since $\psi(\xi) > 0$ for any $\xi > 0$, this proves the first half of the lemma. We now prove the second half. It follows directly from Assumptions 3 and 4, and Lemma 1 that on the curve $c = f(k)$,

$$\frac{\dot{c}}{\dot{k}} = \frac{\beta\psi(0)\{\alpha f(k) - c\}}{f(k) - (n+\delta)k - c} = \frac{\beta\psi(0)(1-\alpha) f(k)}{(n+\delta)k} > \frac{f(k)}{k} > f'(k),$$

so that we obtain that the slope of vector field on $c = f(k)$ is steeper than the slope of $c = f(k)$. Thus, the vector field on the graph of $c = f(k)$ points downward on the graph. This proves the latter half. ■

The proof of Lemma 2 explains the reason why we need Assumption 4. As stated in Section 2, we consider the representative household who follows the prudent rule that households live within their income. The proof of Lemma 2 shows that Assumption 4 guarantees that the vector field on the graph $\{(k,c) \in R^2 : y = f(k) = c\}$ points downward on the graph. See Figure 2. Thus, the representative household avoids the situation of spending all of his/her income, so that for any path, $f(k) = y > c$ is satisfied. Thus, the closed connected set $\Theta \equiv \{(k,c) \in R^2 : k \geq 0, c \geq 0, f(k) = y \geq c\}$ plays an important role when investigating the dynamics of Λ. See Figure 2. We can now easily obtain the following important lemma.

Figure 2 about here.

Lemma 3: The closed domain Θ includes the steady state and is positively invariant (i.e., any path with the initial point in Θ stays in Θ).■

Proof: Since $1 > \alpha > 0$ and $n + \delta > 0$, we obtain that the graphs of $\dot{k} = 0$ and $\dot{c} = 0$ are lower than that of $c = f(k)$. This implies that the set Θ includes the steady state. On the other hand, we see from Lemma 2 that the phase diagram of System Λ is given as Figure 2, so that the set Θ is positively invariant. This completes the proof.■

From Lemma 3, we see that Assumptions 3 and 4 prevent complete divergence and lock growth paths in the plausible domain.

Although we consider a general production function, a typical example of production function is of the Cobb-Douglass type that has a constant elasticity. In the argument below, elasticity plays an important role. In order to develop a more general argument, it is convenient to use $e(k) = kf'(k)/f(k)$ to denote the elasticity of the general production function. Lemma 1 shows that $1 > e(k) > 0$ for any $k > 0$. We introduce the following assumption:

Assumption 5: $1 > e(k^*) > 1 - \alpha$.

Henceforth, we work under Assumptions 1 to 5 without any notification. Since $c^* / f(k^*) = \alpha,$ we have

(3.1) $$\pi = \psi(1 - c^* / f(k^*)) = \psi(1-\alpha).$$

We obtain the stability condition of the steady state.

Lemma 4: In the case where $(n+\delta)\{e(k^*)+\alpha-1\}/(1-\alpha)-\beta\pi$ is positive, the steady state is completely unstable. On the other hand, in the case where it is negative, the steady state is locally asymptotically stable.■

Proof: See Appendix.■

Now, using the well-known Poincaré-Bendixson Theorem we detect stable limit cycles of System Λ. The following lemma is necessary for the construction of a positively invariant compact set to which we can apply the Poincaré-Bendixson Theorem.

Lemma 5: The point $k^{\#}(>0),$ at which the curve $c = f(k)-(n+\delta)k \quad (k>0)$ and the $k-$axis intersect, is uniquely determined. Moreover, $k^{\#} > k^*$.■

Proof: See Appendix.■

Using Lemma 5, we can define the set

$$\Omega = \{(k,c) \in R^2 : k^{\#} \geq k \geq 0, \quad c \geq 0, \quad f(k) \geq c\}.$$

We see that the compact set Ω contains the steady state and is positively invariant. See Figure 3.

Figure 3 about here.

The boundary condition of Ω and the instability of the steady state are insufficient to prove the emergence of limit cycles because the compact set Ω

contains the equilibrium point (0,0). It should be noted here that the derivative of the production function at $k = 0$ does not exist. Therefore, the linearization of System Λ cannot be used to analyze the dynamic behavior in a neighborhood of (0,0). Thus, care should be taken when analyzing it. Rather than analyzing it, we try to detect a positively invariant compact set in Ω, which excludes (0,0).

Theorem 1: Any path in the closed domain Θ enters the compact set Ω. Suppose that $(n+\delta)\{e(k^*)+\alpha-1\}/(1-\alpha) > \beta\pi$. Then, any path in $\Omega \setminus \{(k^*, c^*), (0,0)\}$ tends toward a nontrivial (i.e., non-equilibrium) $\omega-$limit cycle[8] in Ω. ■

Proof: From Figure 3 we can easily get the first half of the theorem. In what follows, we prove the second half. Consider the path $\phi(t) = (\phi_k(t), \phi_c(t))$ with $(\phi_k(0), \phi_c(0)) = (k^\#, 0)$. Since the compact set Ω is positively invariant and contains $(k^\#, 0)$, the path $\phi(t)$ $(t > 0)$ is included in Ω. On the other hand, from the continuity of ψ and f and Assumption 4, we see that there exists a $\zeta \in (\alpha, 1)$ such that

(3.2) $$\beta\psi(1-\zeta) > (n+\delta)/(\zeta-\alpha).$$

Therefore, we obtain the following sublemma.

Sublemma 1: Define $\Omega_\theta = \{(k,c) \in R^2 : k^\# \geq k \geq 0,\ \ c \geq 0,\ \ \theta f(k) \geq c\}$. Then, for any $\theta \in (\zeta, 1)$, the vector field points inward on the boundary of Ω_θ. ■

Proof: Choose $\theta \in (\zeta, 1)$ arbitrarily. From (3.2) and Assumption 3 we get

(3.3) $$\beta\psi(1-\theta) > \beta\psi(1-\zeta) > \frac{n+\delta}{\zeta-\alpha} > \frac{n+\delta}{\theta-\alpha}.$$

On the other hand, the positive intersection of the equations $c = f(k) - (n+\delta)k$ and $c = \theta f(k)$ is unique. Let the intersection be u (> 0). Then, we obtain that in $\{k \in R^1 : k > 0\}$,

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[8] There are multiple definitions of the $\omega-$limit cycle. In the paper, we employ the definition given by Smale and Hirsch (1974, Section 11.5), which is as follows. If a closed orbit γ satisfies that there is a point $x \notin \gamma$ such that the trajectory starting at x spirals toward γ, then the closed orbit is called an $\omega-$limit cycle. It should be noted here that, under the definition, an $\omega-$limit cycle is not always stable.

(3.4) $(n+\delta)/(1-\theta) > f(k)/k \quad \Leftrightarrow \quad k > u$.

Since $\theta f(k) > \alpha f(k)$, the positive intersection of the equations $c = f(k)-(n+\delta)k$ and $c = \theta f(k)$ is smaller than that of the equations $c = f(k)-(n+\delta)k$ and $c = \alpha f(k)$. This shows that $u < k^{*} < k^{\#}$. See Figure 4. It follows directly from (3.4) that

(3.4) $0 < (1-\theta) f(k)/(n+\delta)k < 1$ for any $k \in (u, k^{\#}]$.

Therefore, Lemma 1, and (3.3) and (3.4) yield that on the curve $c = \theta f(k)$

$$\frac{\dot{c}}{\dot{k}} = \frac{\beta\psi(1-c/f(k))\{\alpha f(k)-c\}}{f(k)-(n+\delta)k-c} = \frac{\beta\psi(1-\theta)(\theta-\alpha)f(k)}{(n+\delta)k-(1-\theta)f(k)}$$
$$= \frac{\beta\psi(1-\theta)(\theta-\alpha)}{n+\delta} \times \frac{f(k)/k}{1-(1-\theta)f(k)/(n+\delta)k} > \frac{f(k)}{k} > f'(k) > \theta f'(k)$$

for any $k \in (u, k^{\#}]$. Thus, we obtain that for any $k \in [u, k^{\#}]$, the slope of the vector field on $c = \theta f(k)$ is steeper than the slope of $c = \theta f(k)$. Now, it follows from the phase diagram that the vector field points inward on the boundary of Ω_{θ}. See Figure 4. This completes the proof.■

Since $\phi(0) = (k^{\#}, 0) \in \Omega_{\theta}$, from Sublemma 1 and the phase diagram, we see that there is a point that gives the first intersection of the path $\phi(t)$ $(t > 0)$ and the curve of $\dot{k} = 0$. See Figure 4. We use $(k_1, c_1) = \phi(\tilde{t}) = (\phi_k(\tilde{t}), \phi_c(\tilde{t}))$ to denote the intersection. Moreover, we define

$$\Phi \equiv \{\text{the compact set which is sandwiched between the curve } \{\phi(t) : \tilde{t} \geq t \geq 0\} \text{ and the } k-\text{axis}\}.$$

Then, we get that the compact set Φ is positively invariant. See Figure 5. We now prove the second half. Choose a point $P \in \Omega \setminus \{(k^{*}, c^{*}), (0,0)\}$ arbitrarily. If P is included in the boundary of Ω, then Lemma 2 shows that the path starting at P enters in the interior of Ω. Therefore, without loss of generality we assume that P is included in the interior of Ω. Then, there exists a $\theta \in (\zeta, 1)$ such that Ω_{θ} contains the point P. From Sublemma 1 and the phase diagram, the path staring at P intersects the curve of $\dot{k} = 0$. See Figure 4. Therefore, the path enters in Φ. The steady state is the only equilibrium point in the compact set Φ. From Lemma 4 and the assumption of Theorem 1, we see that the steady state is completely unstable. Since the set Φ is positively invariant, the Poincaré-Bendixson Theorem shows that the path starting at P tends toward a nontrivial $\omega-$ limit cycle. Since the initial point P is chosen in $\Omega \setminus \{(k^*, c^*), (0,0)\}$ arbitrarily, we complete the proof.■

Figures 4 and 5 about here.

Theorem 1 shows that the continuous-time system Λ yields a growth cycle. Moreover, Theorem 1 implies that some kind of nonlinearity of ψ is necessary for the occurrence of limit cycles. We ensure this. The assumptions yield $e(k^*) + \alpha < 2$. Therefore, from Assumption 4 and the assumption of Theorem 1, (3.1) yields

(3.5) $$\beta\psi(0) > \frac{n+\delta}{1-\alpha} > \frac{n+\delta}{1-\alpha}\{e(k^*) + \alpha - 1\} > \beta\pi = \beta\psi(1-\alpha),$$

so that the adjustment coefficient evaluated at a point in $\{(k,c) : \xi = 1 - c / f(k) = 0\}$ is larger than that evaluated at the steady state. Since $\dot{c} < 0$ on the graph $\{(k,c) : s = f(k) - c = 0\}$, this also implies that the representative household increases his/her saving as his/her saving is close to zero. From Lemma 3, due to the rapid adjustment, we can lock growth paths in the plausible domain Θ. See Figure 2. On the other hand, Lemma 4 shows that if the adjustment coefficient of the expected average income is slow, the steady state is unstable. Growth cycles result from both situations. Thus, we see that ψ must be nonlinear to yield growth cycles.

We now see the occurrence of a Hopf bifurcation. By correcting Dohtani *et al*. (2007, Proof of Theorem 3 in Appendix), we get that corridor stability emerges. We introduce the following assumption that plays the most important role in guaranteeing the occurrence of an unstable Hopf cycle:

Assumption 6: $f'''(k^*) > 0$,
Assumption 7: ψ is constant in an open neighborhood V of the steady state.

A standard example of production function that satisfies Assumption 6 is presented later.[9] Moreover, as shown later, the typical adjustment function (2.4) satisfies Assumption 7. As a bifurcation parameter, we choose the adjustment coefficient β. Define

[9] For another example of the production function, see Dohtani *et al*. (2007).

$$\beta^{\#} = \frac{(n+\delta)\{e(k^*)+\alpha-1\}}{(1-\alpha)\pi}.$$

Assumption 5 shows that $\beta^{\#} > 0$. Lemma 4 shows that if $\beta^{\#} < \beta$, then the steady state is completely unstable. We will prove that a Hopf bifurcation occurs at $\beta = \beta^{\#}$. The point $\beta^{\#}$ is called a bifurcation point. By the Hopf Bifurcation Theorem and the phase diagram analysis, we can prove the following result.

Theorem 2: Suppose that Assumptions 6 and 7 are satisfied. Then, a Hopf bifurcation occurs at $\beta = \beta^{\#}$.[10] There exists a $\tilde{\beta} > \beta^{\#}$ such that for any $\beta \in (\beta^{\#}, \tilde{\beta})$, System Λ has an unstable Hopf cycle in the compact set Φ. Moreover, for any $\beta \in (\beta^{\#}, \tilde{\beta})$, any path in the exterior of the Hopf cycle tends toward a nontrivial $\omega-$limit cycle[11] that surrounds the Hopf cycle and that is included in the set Φ.■

Proof: See Appendix.■

Figure 6 about here.

In Theorem 2, the growth-cycle model Λ is asymptotically stable in the interior of the unstable Hopf cycle. Therefore, Theorem 2 says that the growth-cycle model generates corridor stability, so that any path in the exterior of the Hopf cycle tends toward an $\omega-$limit cycle (i.e., one aspect of the Keynesian economic viewpoint is realized) and any path in the interior of the Hopf cycle converges to the neoclassical steady state (i.e., the neoclassical economic viewpoint is realized). See Figure 6. Medio and Lines (2001, Chapter 5.3) called an unstable Hopf cycle a stability threshold. In the present paper, the unstable Hopf cycle is surely a threshold in the sense that, in the exterior of the unstable Hopf cycle, the neoclassical viewpoint must be replaced by the Keynesian viewpoint. In other words, the neoclassical and the Keynesian features

[10] For the Hopf bifurcation, see Guckenheimer and Homes (1983, pp.151-153). By a similar analytical approach, Grasman and Wentzel (1994) showed that Kaldor's business cycle model yields corridor stability. That is, they showed the coexistence of an unstable Hopf cycle and an $\omega-$limit cycle that attracts the paths in the exterior of the Hopf cycle.

[11] It should be noted here that Theorem 2 does not ensure the uniqueness of the $\omega-$limit cycle that does not correspond with the Hopf cycle.

coexist in the growth-cycle model. As can be seen from the argument stated in Section 2, the most important point is that the marked dynamic property of the growth-cycle model is generated by the rapid adjustment of expected average income merely in a neighborhood of $c = f(k)$ and the slow adjustment of expected average income in a neighborhood of the steady state.

In what follows, we consider a typical example of the growth-cycle model Λ. We consider a production function of the Cobb-Douglass type $f(k) = dk^m$, where $d > 0$ and $1 > m > 0$. Clearly, Assumptions 1, 2, and 6 are satisfied for the production function. The above growth-cycle model becomes

$$\Pi : \begin{cases} \dot{k} = dk^m - (n+\delta)k - c, \\ \dot{c} = \beta\psi(1 - c/dk^m)(\alpha dk^m - c). \end{cases}$$

We set the adjustment function as (2.4). Then, $\psi(\xi) = 1$ for any $\xi > a$. Since $a < 1-\alpha$, we have $\pi = \psi(\xi) = 1$ in a neighborhood of $\xi = 1-\alpha$. This implies that Assumption 7 is satisfied. Therefore, we set $\alpha = 0.81$, $e(k) = m = 0.35$, and $\sigma = 1.6$. Then, Assumptions 3 and 5 are satisfied. Thus, all assumptions except Assumption 4 are satisfied. Since $\psi(0) = ba^{1.6} + 1$, Assumption 4 is satisfied in the case where

$$\beta > \frac{n+\delta}{\psi(0)(1-\alpha)} = \frac{n+\delta}{0.19ba^{1.6} + 0.19}.$$

The set of the parameter β at which Assumption 4 is satisfied is now given by

$$I = (\frac{n+\delta}{0.19ba^{1.6} + 0.19}, \infty).$$

On the other hand, since $\pi = 1$, the bifurcation point is given by

$$\beta^{\#} = \frac{(n+\delta)\{e(k^*) + \alpha - 1\}}{(1-\alpha)} = \frac{16(n+\delta)}{19}.$$

Generally speaking, System Π with $\beta = \beta^{\#}$ does not always satisfy Assumption 4. We need a condition for guaranteeing it. The condition is given in the following lemma:

Lemma 6: If $b > 21/4a^{1.6}$, then $\beta^{\#} \in I$. Therefore, we have $(0, \beta^{\#}) \cap I = \phi$. ■

Proof: If $b > 21/4a^{1.6}$, then we have $0.19ba^{1.6} + 0.19 > 0.19 \times 25/4 = 19/16$. This shows that

$$\frac{n+\delta}{0.19ba^{1.6} + 0.19} < \frac{16(n+\delta)}{19} = \beta^{\#}.$$

This completes the proof.■

Note here that Lemma 6 shows that I contains a small open neighborhood of $\beta^{\#}$. We have the following result.

Proposition 1: Suppose that $b > 21/4a^{1.6}$ is satisfied. System Π yields a Hopf bifurcation at $\beta = \beta^{\#}$. If $\beta \in (0, \beta^{\#}) \cap I$, then the steady state of System Π is completely unstable and any nontrivial path in the interior of Θ tends toward an $\omega-$ limit cycle in Φ. Moreover, there exists a $\tilde{\beta} > \beta^{\#}$ such that for any $\beta \in (\beta^{\#}, \tilde{\beta}) \cap I$, System Π has an unstable Hopf cycle such that any path in the interior of the Hopf cycle converges to the steady state and any path in the exterior of the Hopf cycle tends toward an $\omega-$limit cycle.■

Proof: The proof follows directly from Theorems 1 and 2.■

We now provide a numerical example.

Numerical Example 1: We give an example that possesses an unstable steady state and generates a growth cycle around the steady state. Set

$$\alpha = 0.81, \ \ m = 0.35, \ \ n = 0.05, \ \ \delta = 0.03, \ \ b = 200, \ \text{ and } \ d = 1.$$

Now, set $a = 0.13$. Then, the inequality $a < 1 - \alpha$ is satisfied. Therefore, $\pi = 1$ and $\beta^{\#} \doteqdot 0.06736842$. On the other hand, $21/4a^{1.6} = 21/(4 \times 0.13^{1.6}) < 140 < b$. From Lemma 6, this inequality indicates that $\beta^{\#} \in I$. Moreover, since $I \supseteq (0.0488, \infty)$, we have $J \equiv (0.0488, 0.0673) \subseteq (0, \beta^{\#}) \cap I$. Thus, under the above specification, Proposition 1 shows that for any $\beta \in J$, the steady state is completely unstable and any nontrivial path in the interior of Θ tends toward a limit cycle (i.e., a growth cycle) in Φ. Figure 7 describes the dynamic behavior of System Π with $\beta = 0.067$. Figure 7 also shows that there exists a stable limit cycle. Part (1) of Figure 7 describes a path in the exterior of the stable limit cycle, which tends toward the limit cycle. Part (2) of Figure 7 describes a path in the interior of the stable limit cycle, which tends toward the limit cycle. The path of Part (2) starts at a point near the steady state.■

Figure 7 about here.

We numerically see the occurrence of an unstable Hopf cycle and corridor stability.

Numerical Example 2: Suppose that all parameter values except β are the same as in Numerical Example 1. From Proposition 1, we see that there exists a $\tilde{\beta} > \beta^{\#} \fallingdotseq 0.06736842$ such that for any $\beta \in (\beta^{\#}, \tilde{\beta}) \cap I = (\beta^{\#}, \tilde{\beta})$, an unstable Hopf cycle emerges. Figure 8 shows that System Π with $\tilde{\beta} = 0.068$ yields corridor stability and therefore, there exist stable and unstable limit cycles. Part (1) of Figure 8 describes the dynamics of the exterior of the outer stable limit cycle. Part (1) shows that any path in the exterior of the outer stable limit cycle tends toward the outer stable limit cycle. Part (2) describes the dynamics of the domain that is sandwiched between the two limit cycles. Part (2) shows that any path, which starts at a point near the inner unstable limit cycle, tends slowly toward the outer stable limit cycle. Part (3) describes the dynamics of the interior of the inner unstable limit cycle. Part (3) shows that any path in the interior of the unstable limit cycle tends very slowly toward the steady state. It should be noted here that $\tilde{\beta}$ is very near to the bifurcation point $\beta^{\#} \fallingdotseq 0.06736842$. From these observations and Proposition 1, the inner unstable limit cycle may coincide with the unstable Hopf cycle.■

Figure 8 about here.

Finally, we make one important remark. We consider System Π that satisfies the conditions of Proposition 1. In the system, macro-variables in the steady state grow at the growth rate of population. On the other hand, from $\pi = 1$, the trace of the Jacobian matrix of Π is given by

$$\mathrm{Tr}J(\beta) = e(k^*)(n+\delta)/(1-\alpha) - (n+\delta) - \beta = (n+\delta)\frac{(m+\alpha-1)}{1-\alpha} - \beta.$$

See (A.1) in the appendix. From Assumption 5, we observe that the larger the population growth rate n (i.e., the higher the economic growth rate), the more unstable the steady state. See Lemma 4. The result is important in the sense that it implies that a high population growth rate (therefore, high economic growth rate) is not compatible with a stable economy. See Figure 9. Figure 9 describes the two phase diagrams of System Π with all parameters except n being the same as in Figure 7. In Parts (1) and (2) of Figure 9, we set $n = 0.04$ and $n = 0.055$, respectively. We have

$$\mathrm{Tr}J(\beta) = (n+\delta)\frac{(m+\alpha-1)}{1-\alpha} - \beta = \frac{16n}{19} + \frac{16\times 0.03}{19} - 0.067$$
$$= \frac{16n}{19} - \frac{793}{19000} \equiv \rho(n).$$

Since we have $\rho(0.04) < 0,$ the steady state of Part (1) is asymptotically stable. On the other hand, we have $\rho(0.055) > 0,$ and hence the steady state of Part (2) is completely unstable. Therefore, System Π possesses a growth cycle.

Figure 9 about here.

4. Conclusions and Final Remarks

In this paper, we constructed an endogenous growth-cycle model that gives a modified version of the Solow-Swan growth model. The growth-cycle model possesses the same equilibrium point (i.e., the same steady state) as the Solow-Swan growth model. In the growth-cycle model, the consumption function is given by a proportional relation between expected average income and consumption. Therefore, in comparison with the Solow-Swan growth model, consumption in the growth-cycle model is not very sensible to income.

From the viewpoint of behavioral economics, we supposed that the households follow the prudent rule that they live within their income because they simply do not like to be in debt. Based on our supposition, the adjustment of expected average income in a small neighborhood of the set in which consumption equals income is more rapid than that in the other domain. Under such an assumption, we can avoid divergence and lock paths in a plausible domain. We proved that if the adjustment of expected average income in a small neighborhood of the steady state is slow, then the steady state is completely unstable and, as a result, growth cycles emerge.

Moreover, we showed that even if the steady state is stable, growth cycles can emerge. In such a situation, the growth-cycle model generates corridor stability. That is, there is a cyclic path such that any path in the interior of the cyclic path converges to the steady state and any path in the exterior of the cyclic path tends toward a limit cycle (growth cycle). These results indicate that even under a convex production function and a linear consumption function the growth model can yield a growth cycle. This also implies that neoclassical and Keynesian (like Harrod and Domar) features coexist in the growth-cycle model.

We also showed that the larger the population growth rate n (i.e., the higher the economic growth rate), the more unstable the steady state. Since the growth rate of the growth-cycle model in this paper depends perfectly on the population growth rate, the result implies that a high economic growth rate is not compatible with a stable economy.

Appendix

In the appendix, we prove several of the results given in Section 3.

Proof of Lemma 1: The Mean Value Theorem yields that for any $k > 0,$ there exists a $\tilde{k} \in (0,k)$ such that $f(k)/k = f'(\tilde{k}).$ On the other hand, it follows from Assumption 1 that $f'(\tilde{k}) > f'(k).$ This completes the proof.■

Proof of Lemma 4: From the definition, ψ is constant in the neighborhood of the steady state. Without loss of generality, we may suppose $\pi = 1.$ Suppose $(n+\delta)\{e(k^*)+\alpha-1\}/(1-\alpha)-\beta > 0.$ From the definition of $k^*,$ we have $e(k^*) = k^* f'(k^*)/f(k^*) = f'(k^*)(1-\alpha)/(n+\delta),$ so that (3.1) yields that the trace of the Jacobian matrix of System Λ is given by

(A.1) $$\mathrm{Tr}J(\beta) = f'(k^*) - (n+\delta) - \beta = e(k^*)(n+\delta)/(1-\alpha) - (n+\delta) - \beta > 0.$$

On the other hand, Lemma 1 shows that the determinant of the Jacobian matrix is given by

(A.2) $\quad \text{Det}J(\beta) = \beta\{-(1-\alpha)f'(k^*) + (n+\delta)\} > \beta\{\dfrac{-(1-\alpha)f(k^*) + (n+\delta)k^*}{k^*}\} = 0.$

(A.1) and (A.2) complete the proof.■

Proof of Lemma 5: Let $H(k) = f(k) - (n+\delta)k$. Then, it follows from Assumption 2 that $\lim_{k\to 0} H'(k) = +\infty$ and $\lim_{k\to+\infty} H'(k) = -(n+\delta)$. This implies that $H(k)$ and the $k-$axis intersect. Since $H''(k) = f''(k) < 0$, the intersection is uniquely determined in R^2_{++}. Since $k^\#$ is the positive solution of $H(k) = 0$, we complete the proof of the first half of the lemma. Since k^* is the positive solution of $D(k) = 0$ (see the proof of Lemma 2) and $D(k) < H(k)$, we obtain $k^* < k^\#$. This completes the proof of the second half.■

Proof of Theorem 2: From Assumption 7, ψ is constant in the open neighborhood V. Without loss of generality, we may suppose $\psi(1 - c/f(k)) = 1$ for any $(x, y) \in V$. We use $\text{Re}(\beta)$ and $\text{Im}(\beta)$ to denote the real part and the imaginary part of the Jacobian matrix $J(\beta)$ evaluated at the steady state, respectively. The definition of $\beta^\#$ and (A.1) show that $\text{Tr}J(\beta^\#) = 0$. We see from (A.2) that, in a neighborhood of $\beta^\#$, $\text{Im}(\beta) \neq 0$ and $\text{Re}(\beta) = \text{Tr}J(\beta)$. Therefore, (A.1) yields $d\,\text{Re}(\beta^\#)/d\beta = d\text{Tr}J(\beta^\#)/d\beta = -1.$ Thus, by the Hopf Bifurcation Theorem, we obtain the emergence of a Hopf cycle. Moreover, the diameter of the Hopf cycle tends to zero as $\beta \to \beta^\#$. For this point, see Guckenheimer and Homes (1983, Theorem 3.4.2). Therefore, we can choose a small $\tilde{\beta}$ such that for any $\beta \in (\beta^\#, \tilde{\beta})$, the Hopf cycle is included in the interior of the compact set Φ, $\text{Int}\Phi$. We then prove the instability of the Hopf cycle. This is proved by calculating the stability formula (see Guckenheimer and Holmes (1983, pp.151-153)). To calculate this, we merely need the information on the higher order derivatives estimated at the steady state. Therefore, it suffices to consider System Λ that is restricted to V. For simplicity, we here define $\omega = \alpha f'(k^*)$ and $e^* = e(k^*)$. Since $0 = \text{Tr}J(\beta^\#) = f'(k^*) - (n+\delta) - \beta^\#$, the Jacobian matrix evaluated at $\beta = \beta^\#$ is given by

$$J^\# \equiv \begin{bmatrix} f'(k^*) - (n+\delta) & -1 \\ \alpha\beta^\# f'(k^*) & -\beta^\# \end{bmatrix} = \begin{bmatrix} \beta^\# & -1 \\ \beta^\#\omega & -\beta^\# \end{bmatrix}.$$

The definitions of $\beta^\#$ and e^* yield

(A.3) $\quad \beta^\# = \dfrac{(n+\delta)(e^* + \alpha - 1)}{1-\alpha}.$

Since $e^* = f'(k^*)k^* / f(k^*)$ and $(1-\alpha) f(k^*) = (n+\delta)k^*$, we have

(A.4) $$f'(k^*) = \frac{e^*(n+\delta)}{1-\alpha}.$$

Equations (A.3) and (A.4) yield the following equation:

$$\frac{f'(k^*)}{\beta^\#} = \frac{e^*}{e^* + \alpha - 1}.$$

From this equation and Assumption 5, we obtain

$$\frac{\omega}{\beta^\#} - 1 = \frac{\alpha f'(k^*)}{\beta^\#} - 1 = \frac{\alpha e^* - (e^* + \alpha - 1)}{e^* + \alpha - 1} = \frac{(1-e^*)(1-\alpha)}{e^* + \alpha - 1} > 0.$$

We now define

$$D \equiv \begin{bmatrix} 1 & -a \\ \omega & 0 \end{bmatrix} \text{ where } a \equiv \left(\frac{\omega}{\beta^\#} - 1 \right)^{1/2}.$$

Then, since $a^2 \beta^\# \omega = (\omega / \beta^\# - 1)\beta^\# \omega = \omega(\omega - \beta^\#),$ we obtain

$$D^{-1} J^\# D = \begin{bmatrix} 0 & a \\ -\omega & 1 \end{bmatrix} \begin{bmatrix} \beta^\# & -1 \\ \beta^\# \omega & -\beta^\# \end{bmatrix} \begin{bmatrix} 1 & -a \\ \omega & 0 \end{bmatrix} / a\omega$$

$$= \begin{bmatrix} 0 & -a^2 \beta^\# \omega \\ \omega(\omega - \beta^\#) & 0 \end{bmatrix} / a\omega = \begin{bmatrix} 0 & -a\beta^\# \\ a\beta^\# & 0 \end{bmatrix}.$$

Consider the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = D^{-1} \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix} \quad \left(\Rightarrow \begin{bmatrix} k \\ c \end{bmatrix} = D \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} k^* \\ c^* \end{bmatrix} = \begin{bmatrix} x - ay + k^* \\ \omega x + c^* \end{bmatrix} \right).$$

We use $(F(k,c), G(k,c))$ to denote the vector field of Λ. From the transformation, we have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = D^{-1} \begin{bmatrix} F(D(x,y)^T + (k^*, c^*)^T) \\ G(D(x,y)^T + (k^*, c^*)^T) \end{bmatrix},$$

where $\bullet^T$ denotes the transpose of $\bullet$. Now, we have

(A.5) $$\begin{bmatrix} H(x,y) \\ L(x,y) \end{bmatrix} \equiv D^{-1} \begin{bmatrix} F(D(x,y)^T + (k^*, c^*)^T) \\ G(D(x,y)^T + (k^*, c^*)^T) \end{bmatrix} - D^{-1} J^\# D \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a \\ -\omega & 1 \end{bmatrix} \begin{bmatrix} f(x-ay+k^*) \\ \alpha\beta^{\#} f(x-ay+k^*) \end{bmatrix} / a\omega + \text{Linear Equation}$$

$$= \begin{bmatrix} a\alpha\beta^{\#} f(x-ay+k^*) \\ (-\omega+\alpha\beta^{\#}) f(x-ay+k^*) \end{bmatrix} / a\omega + \text{Linear Equation.}$$

Then, we see that

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = D^{-1} J^{\#} D \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} H(x,y) \\ L(x,y) \end{bmatrix} = \begin{bmatrix} 0 & -\alpha\beta^{\#} \\ \alpha\beta^{\#} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} H(x,y) \\ L(x,y) \end{bmatrix}.$$

Clearly, we have

$$\begin{bmatrix} H(0,0) \\ L(0,0) \end{bmatrix} = D^{-1} \begin{bmatrix} F(k^*, c^*) \\ G(k^*, c^*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Moreover, differentiating (A.5) yields

$$\begin{bmatrix} H_x(0,0) & H_y(0,0) \\ L_x(0,0) & L_y(0,0) \end{bmatrix} = D^{-1} J^{\#} D - D^{-1} J^{\#} D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, we are now in a position to calculate the stability formula of Guckenheimer and Holmes (1983, (3.4.11)). In the following, all third and second derivatives of H and L are estimated at $(0,0)$. We define

$$\Xi_1 = H_{xxx} + H_{xyy} + L_{xxy} + L_{yyy},$$

$$\Xi_2 = H_{xy}(H_{xx} + H_{yy}) - L_{xy}(L_{xx} + L_{yy}) - H_{xx}L_{xx} + H_{yy}L_{yy}.$$

Then, the stability formula is given by $d \equiv \Xi_1/16 + \Xi_2/16\xi$, where $\xi = \alpha\beta^{\#} > 0$. Simple calculations yield

$$H_{xxx} = a\alpha\beta^{\#} f'''/a\omega, \quad H_{xyy} = a^3\alpha\beta^{\#} f'''/a\omega,$$

$$L_{xxy} = -(-\omega+\alpha\beta^{\#})af'''/a\omega, \quad L_{yyy} = -(-\omega+\alpha\beta^{\#})a^3 f'''/a\omega,$$

$$H_{xx} = a\alpha\beta^{\#} f''/a\omega, \quad H_{xy} = -a^2\alpha\beta^{\#} f''/a\omega, \quad H_{yy} = a^3\alpha\beta^{\#} f''/a\omega,$$

$$L_{xx} = (-\omega+\alpha\beta^{\#}) f''/a\omega, \quad L_{xy} = -(-\omega+\alpha\beta^{\#})af''/a\omega,$$

$$L_{yy} = (-\omega+\alpha\beta^{\#})a^2 f''/a\omega,$$

where all higher order derivatives of f are evaluated at $k = k^*$. Since

$a > 0$, $\omega > 0$, and $f'''(k^*) > 0$ (Assumption 6), we have

$$\Xi_1 = (a + a^3)\omega f''' / a\omega = (1 + a^2) f''' > 0.$$

Moreover,

$$\Xi_2 = \{-a^2(a + a^3)\alpha^2\beta^{\#2} + a(1 + a^2)(-\omega + \alpha\beta^{\#})^2 + a^5\alpha\beta^{\#}(-\omega + \alpha\beta^{\#}) - a\alpha\beta^{\#}(-\omega + \alpha\beta^{\#})\} f''^2 /(a\omega)^2$$

$$= \{\frac{\omega}{\alpha\beta^{\#}} - (1 + a^2)\}\alpha\beta^{\#}\omega a(1 + a^2) f''^2 /(a\omega)^2$$

$$= \{\frac{f'(k^*)}{\beta^{\#}} - (1 + a^2)\}\alpha\beta^{\#}\omega a(1 + a^2) f''^2 /(a\omega)^2.$$

Here, from Assumption 1 and the definition of $\beta^{\#}$, we get

$$\frac{f'(k^*)}{\beta^{\#}} - (1 + a^2) = \frac{f'(k^*) - \omega}{\beta^{\#}} = \frac{(1 - \alpha)}{\beta^{\#}} f'(k^*) > 0.$$

This yields $\Xi_2 > 0$. Thus, we see that the stability formula is positive. This implies that the existing Hopf cycle is unstable. For this point, see Guckenheimer and Holmes (1983, pp.151-153). This proves the first half of the theorem. We now prove the second half. We use P and Ext$P$ to denote the Hopf cycle and the exterior of the Hopf cycle, respectively. As stated above, we have $P \subseteq \mathrm{Int}\Phi$. Choose arbitrarily a path in $\Theta \cap \mathrm{Ext}P$. From the proof of Theorem 1, we see that the path enters the set $W = \Phi \cap \mathrm{Ext}P$. Clearly, the set W is positively invariant and contains no equilibrium point. Therefore, the Poincaré-Bendixson Theorem proves that the path must tend toward a nontrivial $\omega-$limit cycle in W. Since the Hopf cycle is unstable, the nontrivial $\omega-$limit cycle does not correspond with the Hopf cycle. It is also clear that the $\omega-$limit cycle surrounds the Hopf cycle. This completes the proof of the second half.■

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References

Benhabib, J., Miyao, T., 1981. Some new results on the dynamics of the generalized Tobin model. International Economic Review 22, 589-596.

Benhabib, J., Nishimura, K., 1979. The Hopf bifurcation and the existence and stability of closed orbits in multisector models of optimal economic growth. Journal of Economic Theory 21, 421-444.

Benhabib, J., Nishimura, K., 1985. Competitive equilibrium cycles. Journal of Economic Theory 35, 284-306.

Boianovsky, M., Hoover, K. D., 2009. The neoclassical growth model and twentieth-century economics. History of Political Economy 41, 1-23.

Boldrin, M., Montrucchio, L., 1986. On the indeterminacy of capital accumulation paths. Journal of Economic Theory 40, 26-39.

Böhm, V., Kaas, L., 2000. Differential savings, factor shares, and endogenous growth cycles. Journal of Economic Dynamics and Control 24, 965-980.

Commendatore, P., 2005. Complex dynamics in a Pasinetti-Solow model of growth and distribution. Macroeconomics 0511006, EconWPA.

Commendatore, P., Palmisani, C., 2009. The Pasinetti-Solow growth model with optimal saving behavior: A local bifurcation analysis. In: Skiadas, C.H., Dimotikalis, I., Skiadas, C. (Eds.). Topics on Chaotic Systems: Selected Papers from CHAOS 2008 International Conference. Singapore: World Scientific, 87-95.

Day, R. H., 1982. Irregular growth cycles. American Economic Review 72, 406-414.

Deneckere, R., Pelikan, S., 1986. Competitive chaos. Journal of Economic Theory 40, 13-25.

Domar, E. D., 1946. Capital expansion, rate of growth, and employment. Econometrica 14, 137-147.

Dohtani, A., Inaba, T., Osaka, H., 2007. Corridor stability of the neoclassical steady state. In: Asada, T., Ishikawa, T. (Eds.). Time and Space in Economics. Tokyo: Springer, 129-143.

Flaschel, P., 2001. Viability and corridor stability in Keynesian supply-driven growth. Metroeconomica 52, 26-48.

Friedman, M., 1957. A Theory of the Consumption Function. Princeton: Princeton University Press.

Goodwin, R. M., 1955. A model of cyclical growth. In: Lundberg, E. (Ed.). The Business Cycle in the Post-War World. Proceedings of the Conference held by the International Economic Association (Oxford, September, 1952). London:

Macmillan, 203-221.
Grasman, J., Wentzel, J. J., 1994. Co-existence of a limit cycle and an equilibrium in Kaldor's business cycle model and its consequences. Journal of Economic Behavior and Organization 24, 369-377.
Guckenheimer, J., Holmes, P., 1983. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. New York-Berlin-Heidelberg: Springer-Verlag.
Harrod, R. F., 1939. An essay in dynamic theory. Economic Journal 49, 14-33.
Harrod, R. F., 1973. Economic Dynamics. London and Basingstoke: Macmillan.
Hicks, J. R.,1950. A Contribution to the Theory of the Trade Cycle. Oxford: Clarendon Press.
Kind, C., 1999. Remarks on the economic interpretation of Hopf bifurcations. Economics Letters 62, 147-154.
Leijonhufvud, A., 1973. Effective demand failures. Swedish Journal of Economics 75, 27-48.
Lorenz, H.-W., 1993. Nonlinear Dynamical Economics and Chaotic Motion. Berlin-Heidelberg-New York: Springer-Verlag.
Mankiw, N. G., Romer, D., Weil, N. D., 1992. A contribution to the empirics of economic growth. Quarterly Journal of Economics 107, 407-437.
Medio, A., Lines, M., 2001. Nonlinear Dynamics: A Primer. Cambridge, England: Cambridge University Press.
Owase. T., 1991. Nonlinear dynamical systems and economic fluctuations: A brief historical survey. IEICE Transactions E74, 1393-1400.
Pasinetti, L. L., 1962. Rate of profit and income distribution in relation to the rate of economic growth. Review of Economic Studies 29, 267-279.
Rosser JB, Jr., 1991. From Catastrophe to Chaos: A General Theory of Economic Discontinuities. Boston-Dordrecht-London: Kluwer Academic Publishers.
Samuelson, P. A., Modigliani, F., 1966. The Pasinetti paradox in neoclassical and more general models. Review of Economic Studies 33, 269-301.
Semmler, W., Sieveking, M., 1993. Nonlinear liquidity-growth dynamics with corridor-stability. Journal of Economic Behavior and Organization 22, 189-208.
Smale, S., Hirsch, M. W., 1974. Differential Equations, Dynamical Systems, and Linear Algebra. New York: Academic Press.
Solow, R. M., 1956. A contribution to the theory of economic growth. Quarterly Journal of Economics 70, 65-94.
Swan, T. W., 1956. Economic growth and capital accumulation. Economic Record 32, 334-361.

Thaler, R. H., 1990. Anomalies: Saving, fungibility, and mental accounts. Journal of Economic Perspectives 4, 193-205.

Thaler, R. H., 1992. The Winner's Curse: Paradoxes and Anomalies of Economic Life. New York: The Free Press.

Yoshida, H., 1999. Harrod's 'knife-edge' reconsidered: An application of the Hopf bifurcation theorem and numerical simulations. Journal of Macroeconomics 21, 537-562.

Captions of Figures

Caption of Figure 1: Adjustment function ψ.

Caption of Figure 2: Phase diagram and the set Θ.

Caption of Figure 3: Positively invariant set Ω.

Caption of Figure 4: Positively invariant set Ω_θ

Caption of Figure 5: Positively invariant set Φ.

Caption of Figure 6: Coexistence of stable and unstable limit cycles.

Caption of Figure 7: Numerical example with a stable limit cycle

Caption of Figure 8: Numerical example with stable and unstable limit cycles.

Caption of Figure 9: As the growth rate of population increases, the steady state becomes unstable and a stable cycle emerges.

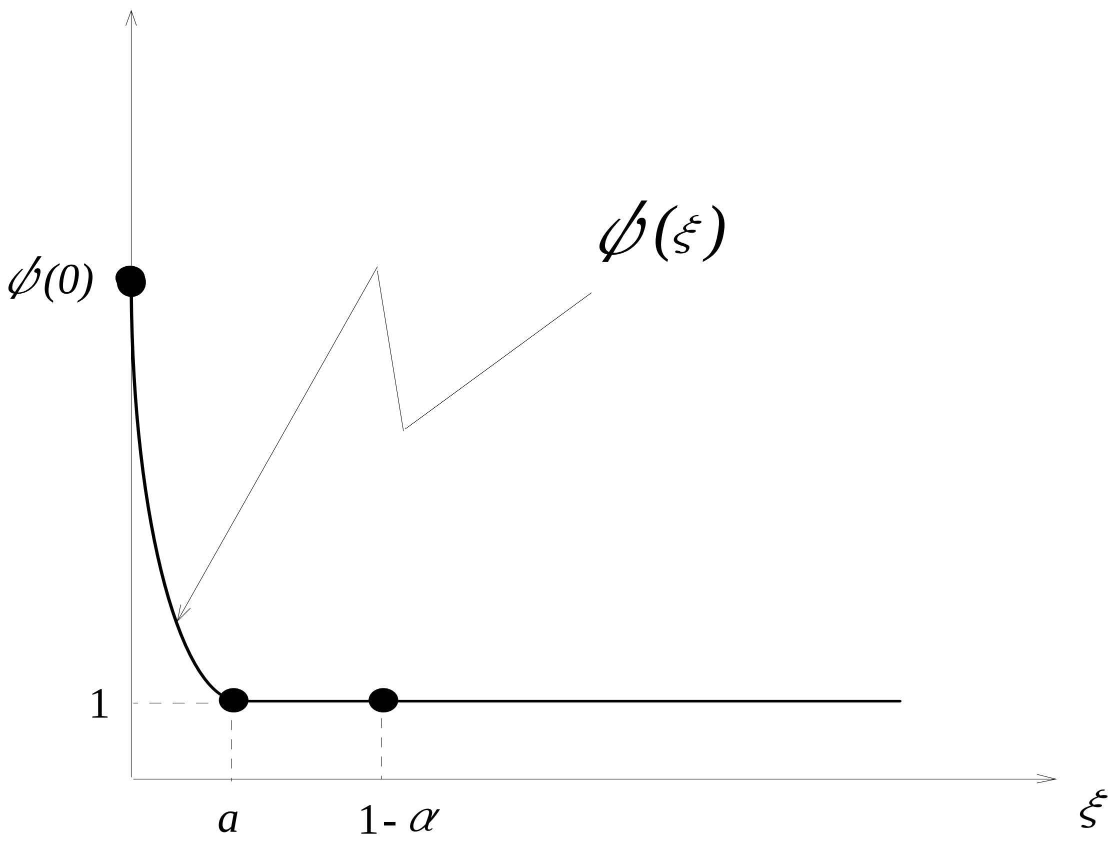


Figure 1

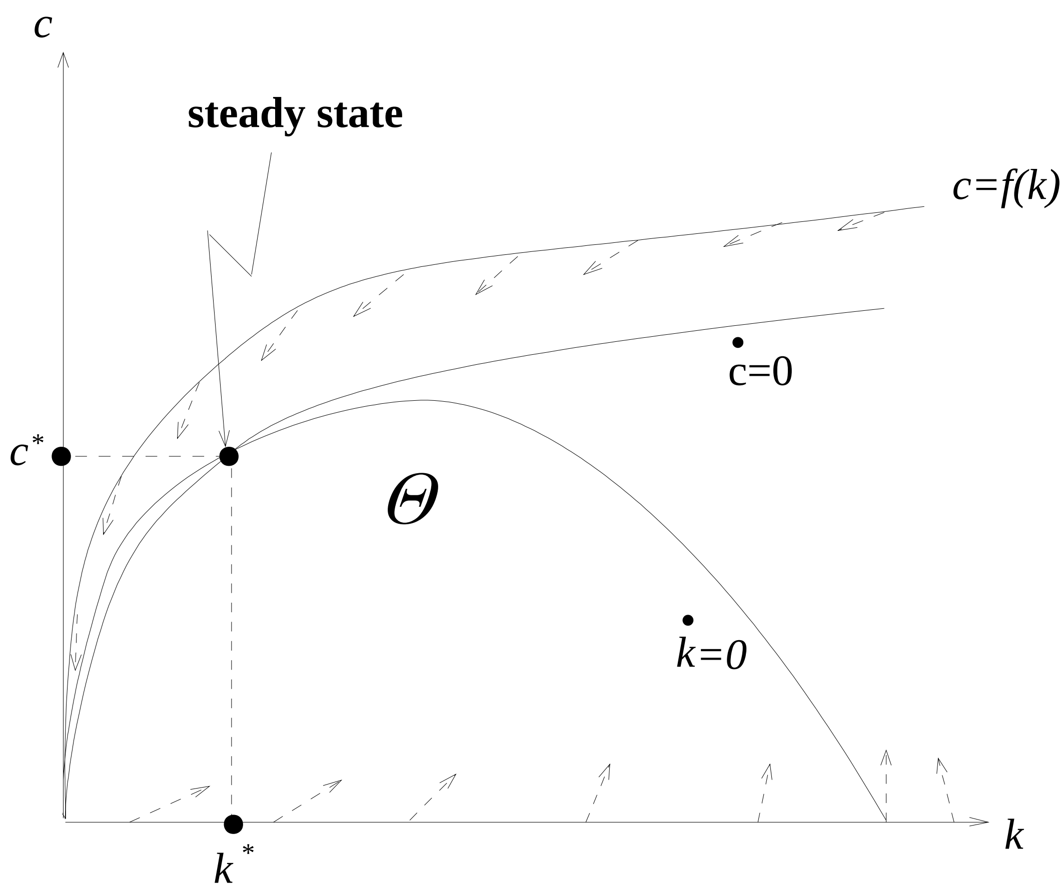


Figure 2

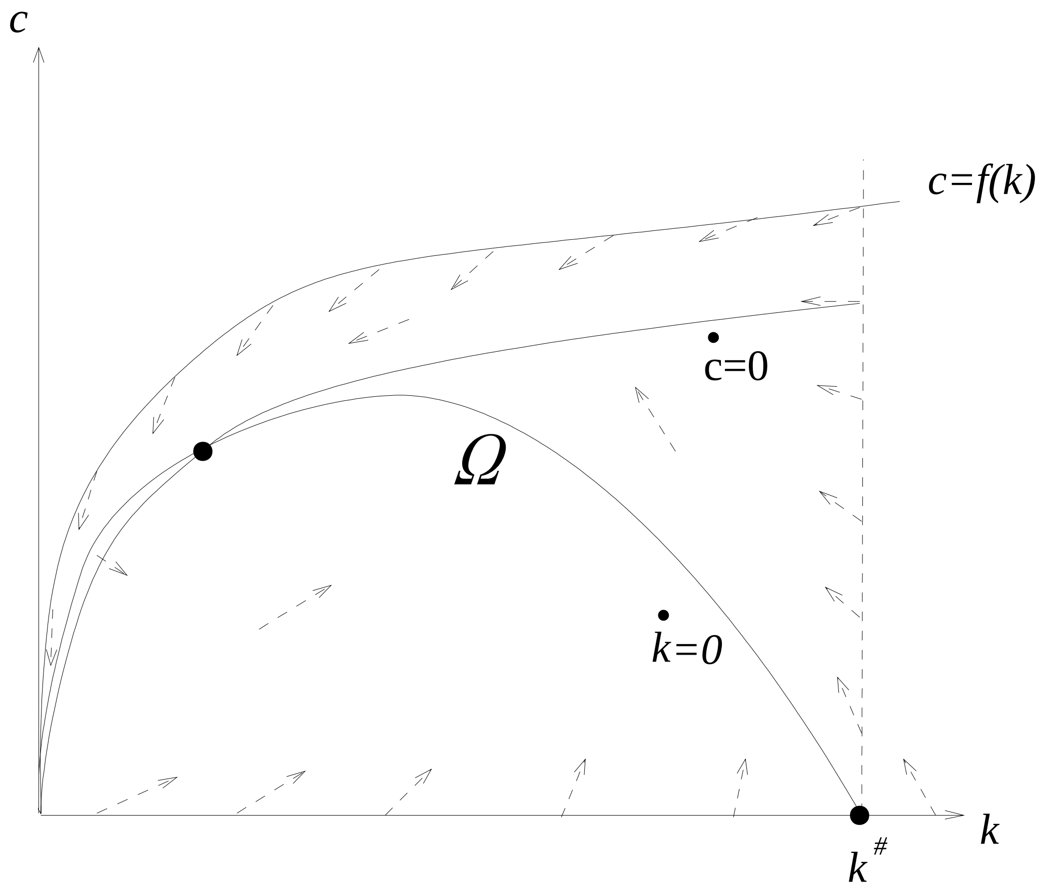
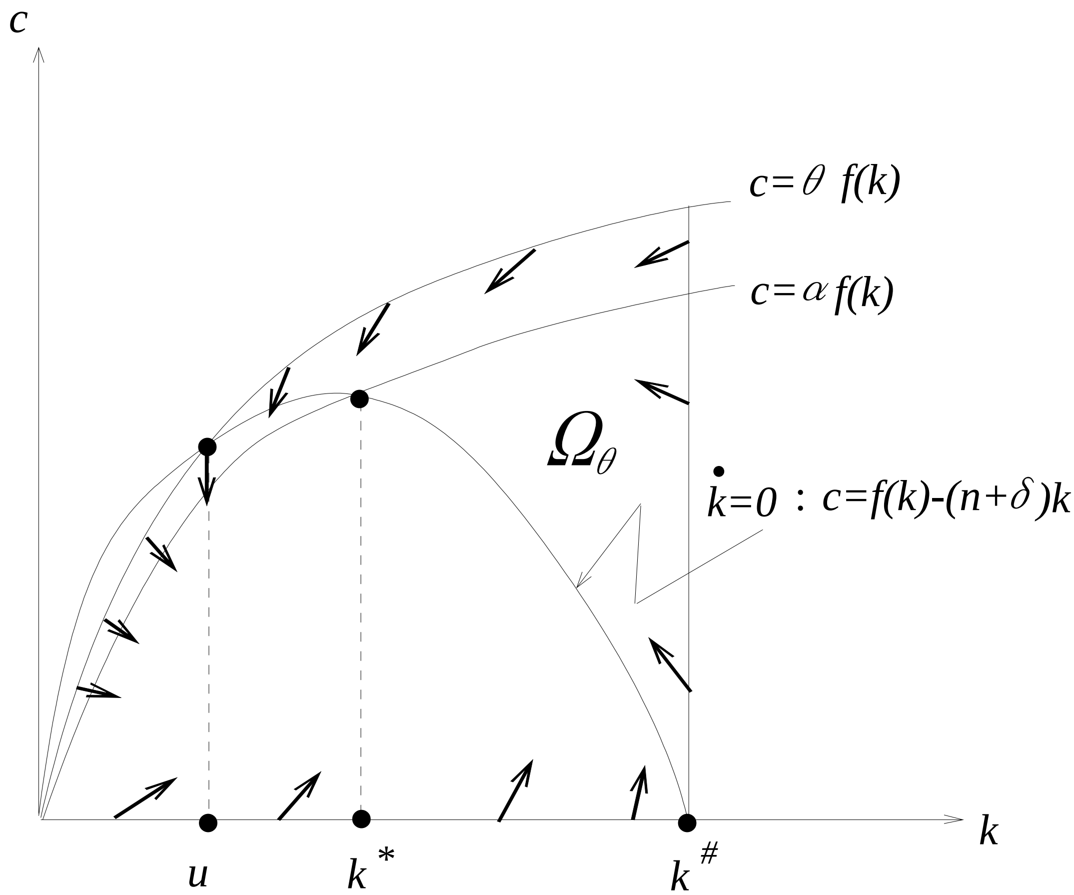


Figure 3



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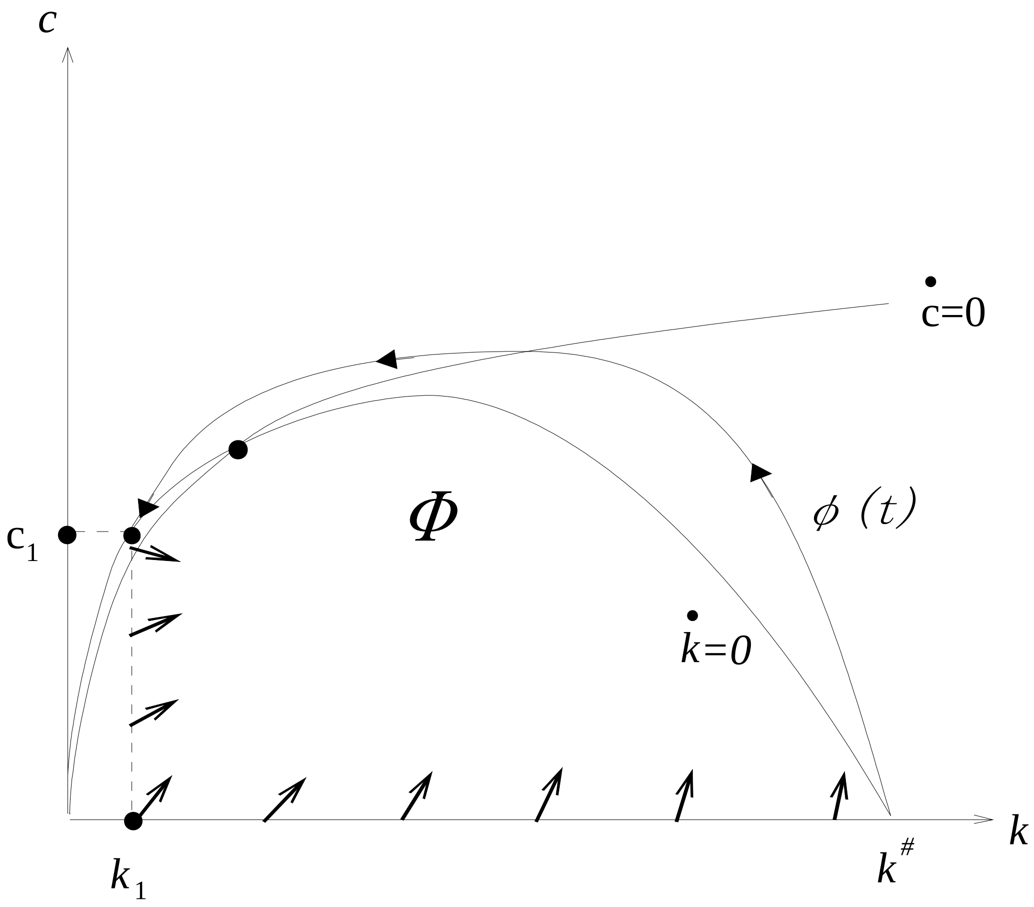


Figure 5

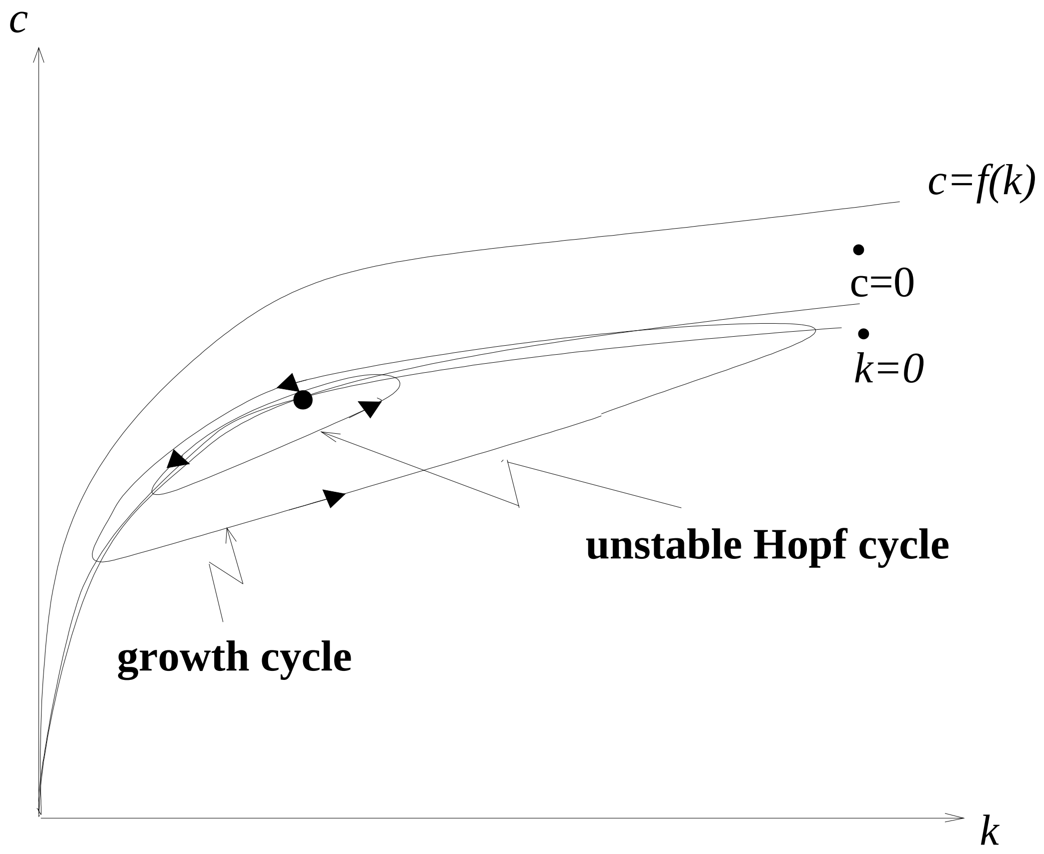
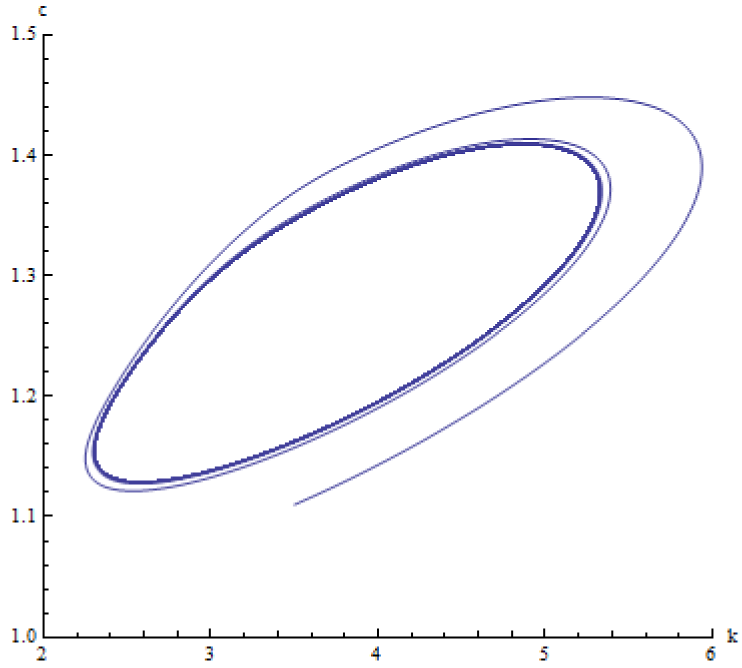


Figure 6



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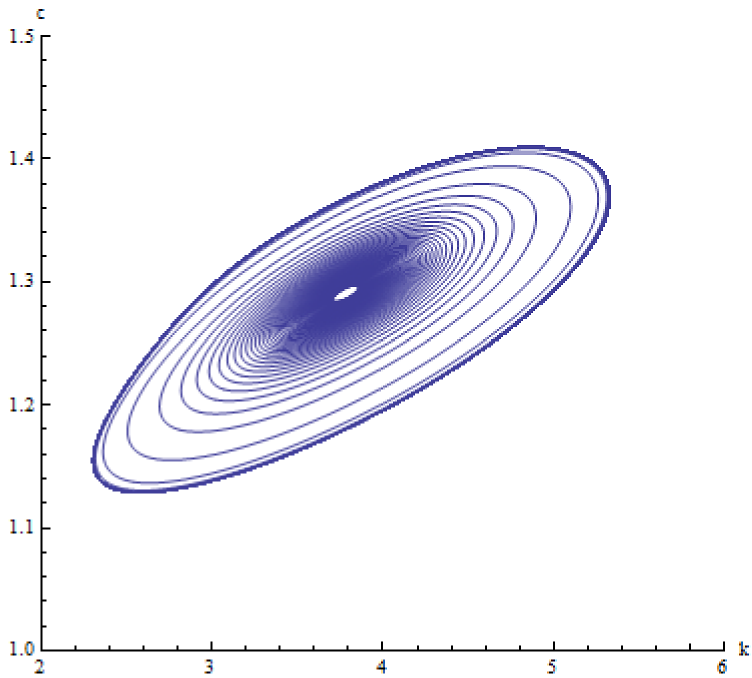

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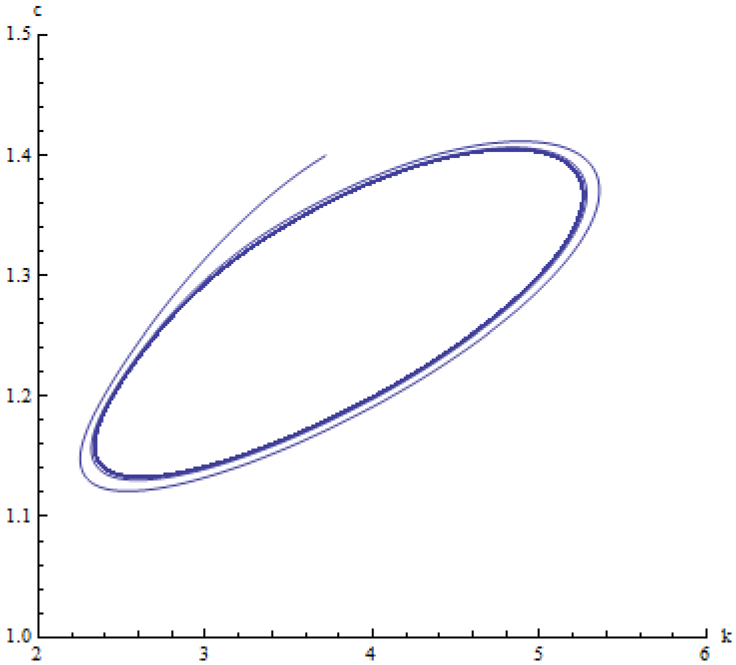
Figure 7



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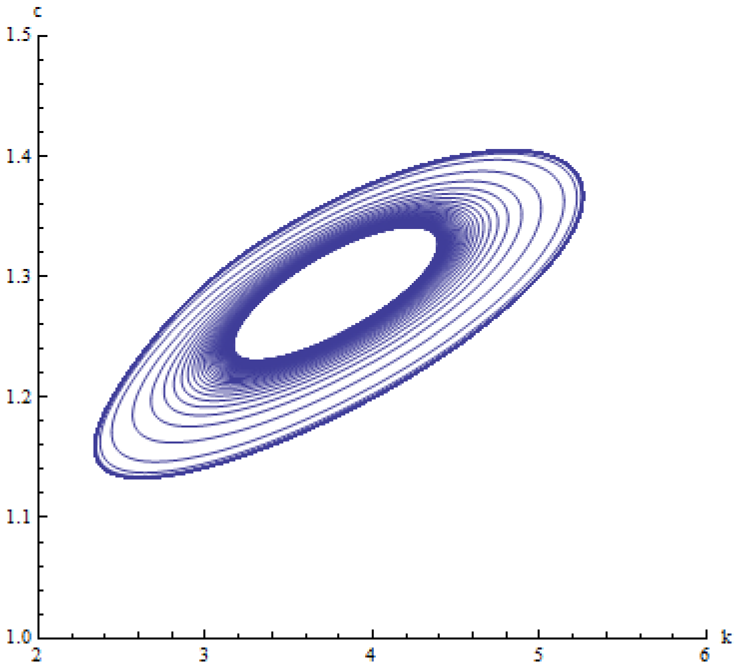

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Figure 8

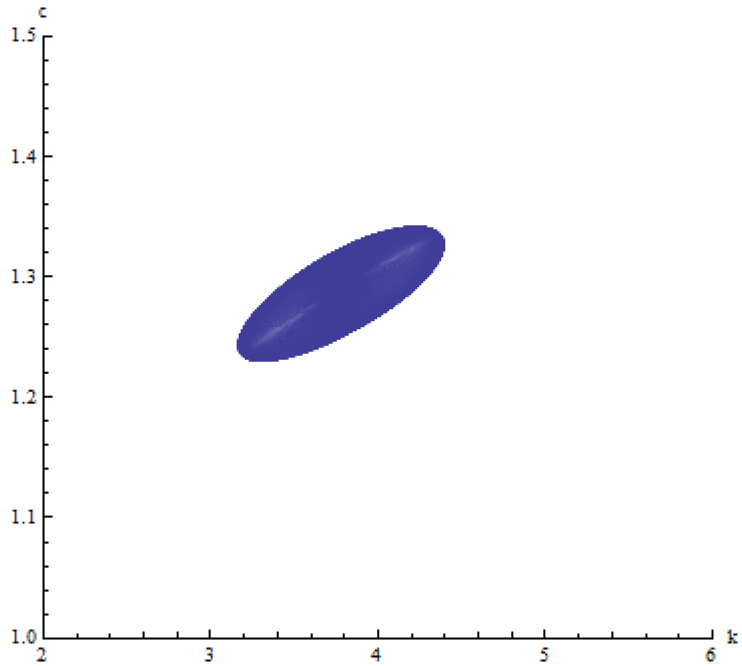

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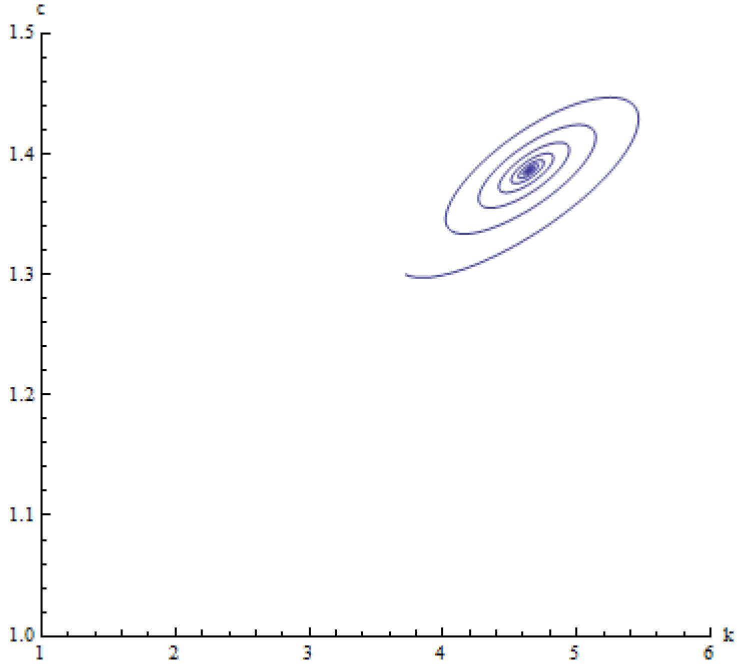
Figure 8



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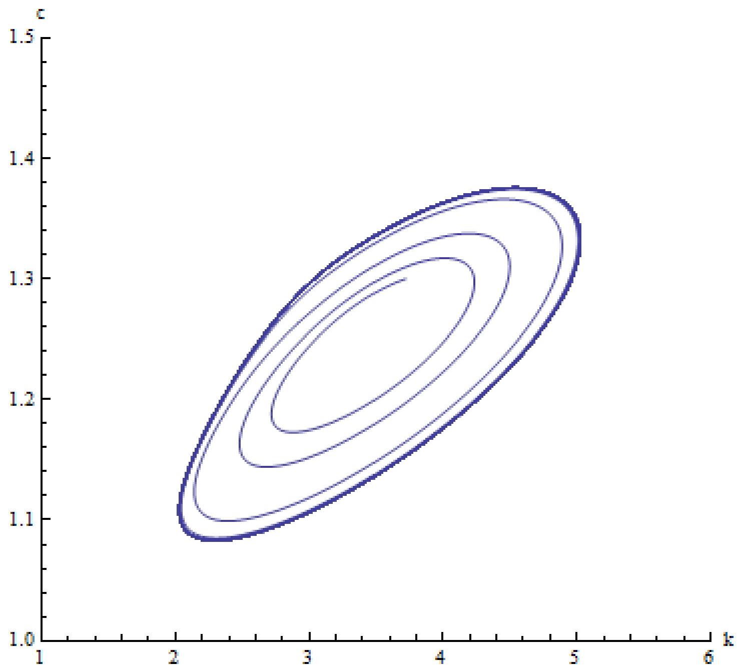

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Figure 9