Some Fibonacci & Lucas identities via the Chebyshev polynomials

journal or publication title	東京商船大学研究報告. 自然科学
volume	47
page range	53-61
year	1997
URL	http://id.nii.ac.jp/1342/00000549/

Some Fibonacci & Lucas identities via the Chebyshev polynomials

Shigeru NAKAMURA

Abstract

There exists a deep relationship between the Chebyshev polynomials and the Fibonacci & Lucas numbers. In this short note, I will show some new Fibonacci & Lucas identities via the Chebyshev polynomials.

1. Introduction

It is well-known that there are close relationships between the Chebyshev polynomials and the Fibonacci & Lucas numbers. In an earlier paper [6], written in Japanese, I revealed some of these relations, and got some Fibonacci & Lucas identities.

In this note, I will first summarize some of main results of my Japanese paper, and then I will show further Fibonacci & Lucas identities via the Chebyshev polynomials.

Definition The Chebyshev polynomials of the 1st & 2nd kinds are defined respectively by

$$T_n(x) = \cos n\theta,\ U_n(x) = \sin(n+1)\theta / \sin\theta,$$

where $x = \cos\theta$. Note that once we get polynomial expressions of these functions, x varies in **C**, the field of all the complex numbers.

In the following, I will use mostly the "modified" Chebyshev polynomials of the 1st & 2nd kinds:

$$t_n(x) = 2T_n(x/2),\ u_n(x) = U_n(x/2),$$

which are monic polynomials of degree n $(n > 0)$.

Note that these polynomials are the same as the "Vieta polynomials" (cf. N. Robbins[8]).

The following recurrent relations provide the easiest way to calculate these polynomials:

$$t_{n+2}(x) = xt_{n+1}(x) - t_n(x);\ t_0(x) = 2,\ t_1(x) = x. \qquad \cdots\cdots(A1)$$

$$u_{n+2}(x) = xu_{n+1}(x) - u_n(x);\ u_0(x) = 1,\ u_1(x) = x. \qquad \cdots\cdots(A2)$$

Here, (A1) corresponds to (1) in [6]. Similarly, the Theorem A1 & Proposition A1 correspond to the Theorem 1 & Proposition 1 in [6], respectively, and so on.

From the definition, the Chebyshev polynomials can be expressed by

$$t_n(x) = \left(\frac{x+\sqrt{x^2-4}}{2}\right)^n + \left(\frac{x-\sqrt{x^2-4}}{2}\right)^n \qquad \cdots\cdots(A3)$$

$$u_n(x) = \left\{\left(\frac{x+\sqrt{x^2-4}}{2}\right)^{n+1} - \left(\frac{x-\sqrt{x^2-4}}{2}\right)^{n+1}\right\} / \sqrt{x^2-4} \qquad \cdots\cdots(A4)$$

Two kinds of these polynomials are related each other by the following equations:

$$t_n(x) = u_n(x) - u_{n-2}(x), \qquad \cdots\cdots(A7)$$

$$(x^2-4)u_n(x) = t_{n+2}(x) - t_n(x), \qquad \cdots\cdots(A8)$$

which are readily obtained from trigonometric identities:

$\sin(n+1)\theta - \sin(n-1)\theta = 2\sin\theta \cdot \cos n\theta$,

and $\cos(n+2)\theta - \cos n\theta = -2\sin\theta \cdot \sin(n+1)\theta$.

From these identities and some trigonometric identities, we can easily get the following Propositions:

Proposion A1

(a) $\sum_{k=1}^{n} t_{2k-1}(x) = u_{2n-1}(x)$,

(b) $\sum_{k=1}^{n} t_{2k}(x) = u_{2n}(x) - 1$,

(c) $\sum_{k=1}^{n} t_k(x) = u_n(x) + u_{n-1}(x) - 1$,

(d) $(x^2-4)\sum_{k=1}^{n} u_{2k-1}(x) = t_{2n+1}(x) - x$,

(e) $(x^2-4)\sum_{k=1}^{n} u_{2k}(x) = t_{2n+2}(x) - x^2 + 2$,

(f) $(x^2-4)\sum_{k=1}^{n} u_k(x) = t_{n+2}(x) + t_{n+1}(x) - x^2 - x + 2$,

(a') $\sum_{k=1}^{n} (-1)^{k-1} t_{2k-1}(x) = \dfrac{(-1)^{n-1} t_{2n}(x) + 2}{x}$,

(b') $\sum_{k=1}^{n} (-1)^{k-1} t_{2k}(x) = \dfrac{(-1)^{n-1} t_{2n+1}(x) + x}{x}$,

(c') $\sum_{k=1}^{n} (-1)^{k-1} t_k(x) = (-1)^{n-1}(u_n(x) - u_{n-1}(x)) + 1$,

(d') $\sum_{k=1}^{n} (-1)^{k-1} u_{2k-1}(x) = \dfrac{(-1)^{n-1} u_{2n}(x) + 1}{x}$,

(e') $\sum_{k=1}^{n} (-1)^{k-1} u_{2k}(x) = \dfrac{(-1)^{n-1} u_{2n+1}(x) + x}{x}$,

(f') $\sum_{k=1}^{n} (-1)^{k-1} u_k(x) = (-1)^{n-1}(t_{n+2}(x) - t_{n+1}(x)) - x + 2$.

Proposition A2

(a) $t_n(x)^2 - (x^2-4)u_{n-1}(x)^2 = 4$,

(b) $t_{2n-1}(x) = t_n(x)t_{n-1}(x) - x$,

(c) $t_{2n}(x) = t_n(x)^2 - 2$,

(d) $u_{2n-1}(x) = t_n(x)u_{n-1}(x)$,

(e) $u_{2n}(x) = t_n(x)u_n(x) - 1$,

(f) $t_{n+k}(x)t_{n-k}(x) - t_n(x)^2 = t_k(x)^2 - 4$,

(g) $u_{n+k}(x)u_{n-k}(x) - u_n(x)^2 = -u_{k-1}(x)^2$.

Derivatives of the Chebyshev polynomials are as follows:

$t'_n(x) = nu_{n-1}(x)$, ...(A11)

$(x^2-4)u'_n(x) = (n+1)t_{n+1}(x) - xu_n(x)$. ......(A12)

Then the following Proposition is an easy consequence of the Proposition A1.

Proposition A4

(a) $\sum_{k=1}^{n} kt_{2k-1}(x) = nu_{2n-1}(x) - \frac{t_{2n-1}(x)-x}{x^2-4}$,

(b) $\sum_{k=1}^{n} kt_{2k}(x) = nu_{2n}(x) - u_{n-1}(x)^2$,

(c) $\sum_{k=1}^{n} kt_k(x) = \frac{nt_{n+1}(x)-(n+1)t_n(x)+2}{x-2}$,

(d) $(x^2-4)\sum_{k=1}^{n} ku_{2k-1}(x) = \frac{(2n+1)t_{2n+1}(x)-xu_{2n}(x)}{2}$,

(e) $(x^2-4)\sum_{k=1}^{n} ku_{2k}(x) = nxu_{2n+1}(x)-(2n+1)u_{2n}(x)+1$,

(f) $(x^2-4)\sum_{k=1}^{n} ku_k(x) = (x+2)\{nu_{n+1}(x)-(n+1)u_n(x)+1\}$,

(a') $\sum_{k=1}^{n}(-1)^{k-1}kt_{2k-1}(x) = \frac{(-1)^{n-1}(nxt_{2n}(x)+t_{2n-1}(x))+x}{x^2}$,

(b') $\sum_{k=1}^{n}(-1)^{k-1}kt_{2k}(x) = \frac{(-1)^{n-1}(nxt_{2n+1}(x)+t_{2n}(x))+2}{x^2}$,

(c') $\sum_{k=1}^{n}(-1)^{k-1}kt_k(x) = n(-1)^{n-1}(u_n(x)-u_{n-1}(x)) + \frac{(-1)^{n-1}t_n(x)+2}{x+2}$,

(d') $\sum_{k=1}^{n}(-1)^{k-1}ku_{2k-1}(x) = \frac{(-1)^{n-1}(nxu_{2n}(x)+u_{2n-1}(x))}{x^2}$,

(e') $\sum_{k=1}^{n}(-1)^{k-1}ku_{2k}(x) = \frac{(-1)^{n-1}(nxu_{2n+1}(x)+u_{2n}(x))+1}{x^2}$,

(f') $\sum_{k=1}^{n}(-1)^{k-1}ku_k(x) = \frac{(-1)^{n-1}\{nu_{n+1}(x)+(n+1)u_n(x)\}+1}{x+2}$.

Now, $\cos n\theta$ is zero at $n\theta=(2k-1)\pi/2$, $\theta=(2k-1)\pi/2n$, or $\cos\theta=\cos\{(2k-1)\pi/(2n)\}$, $(k=1, 2, 3, \cdots, n)$.

It readily turns to a Chebyshev relation:

$$t_n(x)=0 \Leftrightarrow x=2\cos\frac{2k-1}{2n}\pi,\ (k=1, 2, \cdots, n).$$

Since $\cos x$ is decreasing in $[0,\pi]$, all of these values are different. So we can factorize $t_n(x)$, as follows:

$$t_n(x)=\prod_{k=1}^{n}(x-2\cos\frac{2k-1}{2n}\pi).$$

After some routine transformations, we have the following Proposition:

Proposition A5

(a) $t_n(x)=x^{n-2[n/2]}\cdot\prod_{k=1}^{[n/2]}(x^2-2\cos\frac{2k-1}{n}\pi-2)$,

(b) $u_{n-1}(x)=x^{n-1-2[(n-1)/2]}\cdot\prod_{k=1}^{[(n-1)/2]}(x^2-2\cos\frac{2k}{n}\pi-2)$.

2. Some of main results of our earlier paper

It is well-known that the Fibonacci & Lucas numbers can be expressed by the forms

$$F_n=(\alpha^n-\beta^n)/\sqrt{5}, \qquad \cdots\cdots(A15)$$

and $$L_n=\alpha^n+\beta^n. \qquad \cdots\cdots(A16)$$

where $\alpha=(1+\sqrt{5})/2$, $\beta=(1-\sqrt{5})/2$. Then we can get easily the following identities:

$$\alpha^n=\frac{L_n+\sqrt{5}F_n}{2},\quad \beta^n=\frac{L_n-\sqrt{5}F_n}{2} \qquad \cdots\cdots(A22)$$

and then

$$F_{mn}=\frac{1}{\sqrt{5}}\{(\frac{L_n+\sqrt{5}F_n}{2})^m-(\frac{L_n-\sqrt{5}F_n}{2})^m\} \qquad \cdots\cdots(A15)$$

$$L_{mn}=(\frac{L_n+\sqrt{5}F_n}{2})^m+(\frac{L_n-\sqrt{5}F_n}{2})^m \qquad \cdots\cdots(A16')$$

All of these formulae are well-known (cf. Kelisky[5] and Castellanos[3]). Comparing these identities with (A3) & (A4), we can conclude that

$$\text{if} \quad x=\begin{cases}\sqrt{5}F_n \text{ or } iL_n & (n\text{:odd}),\\ i\sqrt{5}F_n \text{ or } L_n & (n\text{:even}),\end{cases} \qquad (\text{where } i=\sqrt{-1})$$

then the Chebyshev polynomials express (simple ratios of) the Fibonacci & Lucas numbers. All but 2 cases are got by Castellanos[3]. He missed (d) and (j) of the following Theorem.

Theorem A1 For any non-negative integer m and n, the following equalities hold:

(a) $t_{2n-1}(\sqrt{5}F_{2m-1})=\sqrt{5}F_{(2m-1)(2n-1)}$,

(b) $t_{2n-1}(i\sqrt{5}F_{2m})=i\sqrt{5}(-1)^{n-1}F_{2m(2n-1)}$,

(c) $t_{2n}(\sqrt{5}F_{2m-1})=L_{(2m-1)\cdot 2n}$,

(d) $t_{2n}(i\sqrt{5}F_{2m})=(-1)^nL_{4mn}$,

(e) $t_n(L_{2m})=L_{2mn}$,

(f) $t_n(iL_{2m-1})=i^nL_{(2m-1)\cdot n}$,

(g) $u_{2n-1}(\sqrt{5}F_{2m-1})=\sqrt{5}F_{(2m-1)\cdot 2n}/L_{2m-1}$,

(h) $u_{2n-1}(i\sqrt{5}F_{2m})=i\sqrt{5}(-1)^{n+1}F_{4mn}/L_{2m}$,

(i) $u_{2n}(\sqrt{5}F_{2m-1})=L_{(2m-1)(2n+1)}/L_{2m-1}$,

(j) $u_{2n}(i\sqrt{5}F_{2m})=(-1)^nL_{2m(2n+1)}/L_{2m}$,

(k) $u_{n-1}(L_{2m})=F_{2mn}/F_{2m}$,

(l) $u_{n-1}(iL_{2m-1})=i^{n-1}F_{(2m-1)n}/F_{2m-1}$.

Note that (e) & (f) are found in Kelisky[5]. For example, let m=0 in (f) & (l), then we have

$$t_n(i)=i^nL_n,\ u_{n-1}(i)=i^{n-1}\cdot F_n. \qquad \cdots\cdots(A23)$$

Let m=0 in (f) & (k), then we have

$$t_n(3)=L_{2n},\ u_{n-1}(3)=F_{2n}.$$

(Bernstein [2] called the latter identity "new formula". Cf. Rivlin[7], 1. 5. 57(d).) And so on.

Combining this theorem with the former Propositions, we can readily get the following Theorems:

Theorem A2 For any positive integers m and n, we have

(a) $\displaystyle\sum_{k=1}^{n}F_{(2m-1)k}=\frac{F_{(2m-1)(n+1)}+F_{(2m-1)n}-F_{2m-1}}{L_{2m-1}}$,

(b) $\displaystyle\sum_{k=1}^{n}F_{2mk}=\frac{L_{2m(n+1)}+L_{2mn}-L_{2m}-2}{5F_{2m}}$,

(c) $\sum_{k=1}^{n} L_{(2m-1)k} = \frac{L_{(2m-1)(n+1)} + L_{(2m-1)n} - L_{2m} - 2}{L_{2m}}$,

(d) $\sum_{k=1}^{n} L_{2mk} = \frac{F_{2m(n+1)} + F_{2mn} - F_{2m}}{F_{2m}}$,

(a') $\sum_{k=1}^{n} (-1)^{k-1} F_{(2m-1)k} = \frac{(-1)^{n-1}(F_{(m-1)(n+1)} - F_{(2m-1)n}) + F_{2m-1}}{L_{2m-1}}$,

(b') $\sum_{k=1}^{n} (-1)^{k-1} F_{2mk} = \frac{(-1)^{n-1}(L_{2m(n+1)} - L_{2mn}) + L_{2m} - 2}{5F_{2m}}$,

(c') $\sum_{k=1}^{n} (-1)^{k-1} L_{(2m-1)k} = \frac{(-1)^{n-1}(L_{(2m-1)(n+1)} - L_{(2m-1)n}) + L_{2m-1} - 2}{L_{2m-1}}$,

(d') $\sum_{k=1}^{n} (-1)^{k-1} L_{2mk} = \frac{(-1)^{n-1}(F_{2m(n+1)} - F_{2mn}) + F_{2m}}{F_{2m}}$.

Note that (a) & (b) are found in Castellanos[3].

Theorem A4

(a) $\sum_{k=1}^{n} kF_{2k-1} = nF_{2n} - F_{2n-1} + 1$,

(b) $\sum_{k=1}^{n} kF_{2k} = nF_{2n+1} - F_{2n}$,

(c) $\sum_{k=1}^{n} kF_k = (n-1)F_{n+2} - F_{n+1} + 2$,

(d) $\sum_{k=1}^{n} kL_{2k-1} = nL_{2n} - L_{2n-1} - 1$,

(e) $\sum_{k=1}^{n} kL_{2k} = nL_{2n+1} - L_{2n} + 2$,

(f) $\sum_{k=1}^{n} kL_k = (n-1)L_{n+2} - L_{n+1} + 4$,

(a') $\sum_{k=1}^{n} (-1)^{k-1} kF_{2k-1} = \frac{(-1)^{n-1}\{nF_{2n} + (2n+1)F_{2n-1}\} + 1}{5}$,

(b') $\sum_{k=1}^{n} (-1)^{k-1} kF_{2k} = \frac{(-1)^{n-1}\{(2n+1)L_{2n+1} - F_{2n+1}\}}{10}$,

(c') $\sum_{k=1}^{n} (-1)^{k-1} kF_k = (-1)^{n-1}\{(n+1)F_{n-1} - F_n\} + 2$,

(d') $\sum_{k=1}^{n} (-1)^{k-1} kL_{2k-1} = \frac{(-1)^{n-1}(5nF_{2n} + L_{2n-1}) - 1}{5}$,

(e') $\sum_{k=1}^{n} (-1)^{k-1} kL_{2k} = \frac{(-1)^{n-1}(5nF_{2n+1} + L_{2n}) + 2}{5}$,

(f') $\sum_{k=1}^{n} (-1)^{k-1} kL_k = (-1)^{n-1}\{(n+2)L_{n-1} - L_n\} - 4$.

Note that some of these identities are the same as Fibonacci-Lucas identities obtained by G. Wulczyn[10](in slightly different forms).

As a bonus, we can get nice formulae expressing the Fibonacci & Lucas numbers:

Theorem A5

(a) $F_n = \prod_{k=1}^{[\frac{n-1}{2}]} (3 + 2\cos\frac{2k\pi}{n})$,

(b) $L_n = \prod_{k=1}^{[\frac{n}{2}]} (3 + 2\cos\frac{(2k-1)\pi}{n})$,

(c) $F_{mn}=F_m \cdot L_m^{n-1-2[\frac{n-1}{2}]} \prod_{k=1}^{[\frac{n-1}{2}]} (L_{2m}+2(-1)^{m-1}\cos\frac{2k\pi}{n})$,

(d) $L_{mn}=L_m^{n-2[\frac{n}{2}]} \prod_{k=1}^{[\frac{n}{2}]} (L_{2m}+2(-1)^{m-1}\cos\frac{(n-k)\pi}{n})$.

Note that (a) is solved repeatedly. See Bedrosian [1] and solution of advanced problem (H-93) in the Fibonacci Quarterly (cf. (H-111) and (H-466), see also (H-64)).

3. Further results of similar type

We know that a trigonometric identity leads a Chebyshev identity, and then it leads a number of the Fibonacci & Lucas identities.

For example, a simple trigonometric identity

$$\sin(2n+1)\theta-\sin\theta=2\sin n\theta\cdot\cos(n+1)\theta$$

turns to a Chebyshev identity

$$u_{2n}(x)-1=u_{n-1}(x)t_{n+1}(x)$$

and then, it turns (by letting $x=L_{2m}$) to a Fibonacci/Lucas identity:

$$F_{2m(2n+1)}-F_{2m}=F_{2mn}\cdot L_{2m(n+1)}. \quad\cdots\cdots(1)$$

Similarly, starting from a trigonometric identity

$$\sin(n+k)\theta\cdot\cos m\theta-\sin n\theta\cdot\cos(m+k)\theta=\sin k\theta\cdot\cos(n-m)\theta$$

we get in turn

$$u_{n+k-1}(x)t_m(x)-u_{n-1}(x)t_{m+k}(x)=u_{k-1}(x)t_{n-m}(x)$$

and $F_{n+k}L_m-F_nL_{m+k}=(-1)^mF_kL_{n-m}$. ……(2)

This is a generalization of (I28) of Hoggatt[4]. In this fashion, we can get the following:

Theorem 1

(a) $\sum_{k=0}^{n}\binom{n}{k}F_{2k}=\begin{cases}5^{(n-1)/2}\cdot L_n, & \text{(n:odd)}\\ 5^{n/2}\cdot F_n, & \text{(n:even)}\end{cases}$

(b) $\sum_{k=0}^{n}\binom{n}{k}L_{2k}=\begin{cases}5^{(n+1)/2}\cdot F_n, & \text{(n:odd)}\\ 5^{n/2}\cdot L_n, & \text{(n:even)}\end{cases}$

(c) $\sum_{k=0}^{n}\binom{n}{k}F_{4k}=3^n\cdot F_{2n}$,

(d) $\sum_{k=0}^{n}\binom{n}{k}L_{4k}=3^n\cdot L_{2n}$,

(e) $\sum_{k=0}^{n}\binom{n}{k}F_{(4m+2)k}=\begin{cases}5^{(n-1)/2}\cdot F_{2m+1}^{n}L_{(2m+1)n}, & \text{(n:odd)}\\ 5^{n/2}\cdot F_{2m+1}^{n}F_{(2m+1)n}, & \text{(n:even)}\end{cases}$

(f) $\sum_{k=0}^{n}\binom{n}{k}L_{(4m+2)k}=\begin{cases}5^{(n+1)/2}\cdot F_{2m+1}^{n}F_{(2m+1)n}, & \text{(n:odd)}\\ 5^{n/2}\cdot F_{2m+1}^{n}L_{(2m+1)n}, & \text{(n:even)}\end{cases}$

(g) $\sum_{k=0}^{n}\binom{n}{k}F_{4mk}=L_{2m}^{n}\cdot F_{2mn}$,

(h) $\sum_{k=0}^{n}\binom{n}{k}L_{4mk}=L_{2m}^{n}\cdot L_{2mn}$.

(proof) From Euler's formula, it's easy to get a trigonometric identity

$$(1+\cos\theta+i\sin\theta)^n=(1+e^{i\theta})^n=\sum_{k=0}^{n}\binom{n}{k}e^{ik\theta}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\cos k\theta+i\sum_{k=0}^{n}\binom{n}{k}\sin k\theta \qquad \cdots\cdots(3)$$

which, in turn, leads a Chebyshev identity:

$$(x+2+\sqrt{x^2-4})^n$$
$$=2^{n-1}\{\sum_{k=0}^{n}\binom{n}{k}t_k(x)+\sqrt{x^2-4}\sum_{k=0}^{n}\binom{n}{k}u_{k-1}(x)\}. \qquad \cdots\cdots(4)$$

Let x=3 in (4), and we have

$$2^{n-1}(\sum_{k=0}^{n}\binom{n}{k}L_{2k}+\sqrt{5}\sum_{k=0}^{n}\binom{n}{k}F_{2k})=(5+\sqrt{5})^n$$

that is

$$\sum_{k=0}^{n}\binom{n}{k}L_{2k}+\sqrt{5}\sum_{k=0}^{n}\binom{n}{k}F_{2k}=2\sqrt{5}^n\alpha^n=\sqrt{5}^n(L_n+\sqrt{5}F_n).$$

Thus we get (a) & (b). Similarly, let $x=7(\&L_{2m})$, and we have (c) & (d) (and (e), (f), (g) & (h), respectively).

Note that (a) & (b) are (69) & (70) of Vajda[9], (appendix;list of formulae). In the sequel, we shall write (**V**69) & (**V**70). □

Starting from identities

$$(-1+\cos\theta+i\sin\theta)^n=(-1+e^{i\theta})^n$$
$$=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\cos k\theta+i\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sin k\theta \qquad \cdots\cdots(5)$$

and

$$(x-2+\sqrt{x^2-4})^n$$
$$=2^{n-1}\{\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}t_k(x)+\sqrt{x^2-4}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}u_{k-1}(x)\} \qquad \cdots\cdots(6)$$

we get the following:

Theorem 2

(a) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}F_{2k}=(-1)^nF_n,$

(b) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}L_{2k}=(-1)^nL_n,$

(c) $\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}F_{4k}=\begin{cases}5^{(n-1)/2}L_{2n}, & \text{(n:odd)}\\ -5^{n/2}F_{2n}, & \text{(n:even)}\end{cases}$

(d) $\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}L_{4k}=\begin{cases}5^{(n+1)/2}F_{2n}, & \text{(n:odd)}\\ -5^{n/2}L_{2n}, & \text{(n:even)}\end{cases}$

(e) $\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}F_{(4m+2)k}=(-1)^{n-1}L^n_{2m+1}F_{(2m+1)n},$

(e') $$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}F_{4mk}=\begin{cases}5^{(n-1)/2}\cdot F^{n}_{nm}L_{2mn}, & \text{(n:odd)}\\ -5^{n/2}\cdot F^{n}_{2m}F_{2mn}, & \text{(n:even)}\end{cases}$$

(f) $$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}L_{(4m+2)k}=(-1)^{n-1}L^{n}_{nm+1}L_{(2m+1)n},$$

(f') $$\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}L_{4mk}=\begin{cases}5^{(n+1)/2}\cdot F^{n}_{2m}F_{2mn}, & \text{(n:odd)}\\ -5^{n/2}\cdot F^{n}_{2m}L_{2mn}, & \text{(n:even)}\end{cases}$$

Note that (a) & (b) are (**V**71) &(**V**72).

Combining Euler's formula and the sum formula of a geometric series, we have, for any real number p,

$$\sum_{k=0}^{n}p^{k}\cos k\theta+i\sum_{k=0}^{n}p^{k}\sin k\theta=\sum_{k=0}^{n}(pe^{i\theta})^{k}$$

$$=\frac{p^{n+1}\{\cos(n+1)\theta+i\sin(n+1)\theta\}-1}{p(\cos\theta+i\sin\theta)-1} \qquad \cdots\cdots(7)$$

$$\therefore \sum_{k=0}^{n}p^{k}t_{k}(x)+\sqrt{x^{2}-4}\sum_{k=0}^{n}p^{k}u_{k-1}(x)$$

$$=\frac{p^{n+1}\{t_{n+1}(x)+\sqrt{x^{2}-4}u_{n}(x)\}-4}{px-2+p\sqrt{x^{2}-4}} \qquad \cdots\cdots(8)$$

Thus we get the following Theorems.

Theorem 3

(a) $$\sum_{k=0}^{n}p^{k}F_{k}=\frac{p^{n+1}(F_{n+1}+pF_{n})-p}{p^{2}+p-1},$$

(b) $$\sum_{k=0}^{n}p^{k}L_{k}=\frac{p^{n+1}(L_{n+1}+pL_{n})+p-2}{p^{2}+p-1},$$

(c) $$\sum_{k=0}^{n}p^{k}F_{2k}=\frac{p^{n+1}\{(p-2)F_{2n}-F_{2n-1}\}+p}{p^{2}-3p+1},$$

(d) $$\sum_{k=0}^{n}p^{k}L_{2k}=\frac{p^{n+1}\{(p-2)L_{2n}-L_{2n-1}\}-3p+2}{p^{2}-3p+1},$$

(proof) Let $x=\sqrt{5}$ and 3 in (8), then the Theorem follows immediately.

Note that (a) is obtained by C. Podilla as a solution to the problem B98. □

Next, we shall extend, only a little bit, the Theorem A4.

Theorem 4

For all positive integers m and n,

(a) $$F^{3}_{2m}\sum_{k=1}^{n}k^{2}L_{2m(2k+1)}=n^{2}F_{2m}F_{4m(n+1)}-2nF_{2m}L_{2m(2n+1)}+L_{2m}F_{2mn}L_{2m(n+1)}$$

(b) $$5\cdot F^{3}_{2m}\sum_{k=1}^{n}k^{2}L_{4mk}=(5n^{2}F^{2}_{2m}+2)F_{2m(2n+1)}-(2n+1)F_{2m}L_{4mn}.$$

(c) $$L^{3}_{2m}\sum_{k=1}^{n}(-1)^{k-1}k^{2}L_{2m(2k+1)}=(-1)^{n-1}\{n(n+1)L^{2}_{2m}L_{4m(n+1)}-5nF_{2m}L_{2m}F_{4m(n+1)}$$
$$-5F_{2m}F_{2m(2n+1)}\}-5F^{2}_{2m}$$

(d) $$L^{3}_{2m}\sum_{k=1}^{n}(-1)^{k-1}k^{2}L_{4mk}=(-1)^{n-1}\{5(2n+1)F_{2m}F_{4mn}-n(nL^{2}_{2m}+4)L_{2m(2n+1)}\}$$

(proof) These are readily obtained by the Chebyshev identities:

$$x^2 \sum_{k=1}^{n} (-1)^{k-1} k^2 u_{2k}(x) = (-1)^{n-1} \{n^2 x u_{2n+1}(x) + 2n u_{2n}(x) - \frac{1}{x} t_{2n+1}(x)\} - 1 \quad \cdots\cdots(9)$$

$$x^3 \sum_{k=1}^{n} (-1)^{k-1} k^2 t_{2k}(x) = (-1)^{n-1} \{(n^2x^2-2) t_{2n+1}(x) + (2n+1) x t_{2n}(x)\} \quad \cdots\cdots(10)$$

$$(x^2-4) \sum_{k=1}^{n} k^2 u_{2k}(x) = n(n+1) t_{2n+2}(x) - n x u_{2n+1}(x) + x(t_{2n+1}(x) - x)/(x^2-4) \quad \cdots\cdots(11)$$

$$(x^2-4) \sum_{k=1}^{n} k^2 t_{2k}(x) = (2n+1) x u_{2n-1}(x) + n(nx^2-4-4n) u_{2n}(x) \quad \cdots\cdots(12)$$

We only have to apply (d) & (j) of Theorem A1. □

References

[1] Bedrosian, S. D. :The Fibonacci numbers via trigonometric expressions, **J. Franklin Inst., 295**, pp. 175-177 (1973)

[2] Bernstein, L. :A formula for Fibonacci numbers from a new approach to generalized Fibonacci numbers, **Fibonacci Quart., 14**, pp. 358-368 (1976)

[3] Castellanos, D. :A generalization of Binet's formula and some of its consequences; **Fibonacci Quart., 27**, pp. 424-438 (1989)

[4] Hoggatt, V. E. Jr. : **Fibonacci and Lucas numbers**, Fibonacci Association, (1969)

[5] Kelisky, R. P. :On formulas involving both the Bernoulli and Fibonacci numbers; **Scripta Math., 23**, pp. 27-35 (1958)

[6] Nakamura, S. :On the Chebyshev polynomials and the Fibonacci and Lucas numbers (Japanese, with English Summary); **J. Tokyo Univ. of Mercantile Marine** (Natural Sci.), **42**, pp. 101-116 (1992)

[7] Rivlin, T. J. : **Chebyshev polynomials**, 2nd ed., Wiley-Interscience (1990)

[8] Robbins, N. :Vieta's triangular array and a related family of polynomials, **Internat. J. Math. & Math. Sci., 14**, pp. 239-244 (1991)

[9] Vajda, S. : **Fibonacci & Lucas numbers, and the golden section: theory and application**, Ellis Horwood (John Wiley), (1989)

[10] Wulczyn, G. : Formation of generalized F-L identities of the form $\sum_{r=1}^{n} r^{\bar{s}} F_{hr+a}$, **A collection of manuscripts related to the Fibonacci sequence**, (ed. by V.E.Hoggatt & M.Bicknell-Johnson), The Fibonacci Association, pp. 192-201 (1980)

AMS Classification Numbers: 11B36, 33C45,