SOLVABLE LIE A-ALGEBRAS

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Abstract

A finite-dimensional Lie algebra L over a field F is called an A-algebra if all of its nilpotent subalgebras are abelian. This is analogous to the concept of an A-group: a finite group with the property that all of its Sylow subgroups are abelian. These groups were first studied in the 1940s by Philip Hall, and are still studied today. Rather less is known about A-algebras, though they have been studied and used by a number of authors. The purpose of this paper is to obtain more detailed results on the structure of solvable Lie A-algebras.

It is shown that they split over each term in their derived series. This leads to a decomposition of L as $L = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_0$ where A_i is an abelian subalgebra of L and $L^{(i)} = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_i$ for each $0 \leq i \leq n$. It is shown that the ideals of L relate nicely to this decomposition: if K is an ideal of L then $K = (K \cap A_n) \dot{+} (K \cap A_{n-1}) \dot{+} \ldots \dot{+} (K \cap A_0)$. When L^2 is nilpotent we can locate the position of the maximal nilpotent subalgebras: if U is a maximal nilpotent subalgebra of L then $U = (U \cap L^2) \oplus (U \cap C)$ where C is a Cartan subalgebra of L.

If L has a unique minimal ideal W then $N = Z_L(W)$. If, in addition, L is strongly solvable the maximal nilpotent subalgebras of L are L^2 and the Cartan subalgebras of L (that is, the subalgebras that are complementary to L^2.) Necessary and sufficient conditions are given for such an algebra to be an A-algebra. Finally, more detailed structure results are given when the underlying field is algebraically closed.
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1 Introduction

A finite-dimensional Lie algebra L over a field F is called an A-algebra if all of its nilpotent subalgebras are abelian. This is analogous to the concept of an A-group, which is a finite group with the property that all of its Sylow subgroups are abelian. These groups were first studied in the 1940s by Philip Hall as soluble A-groups, and are still studied today. A great deal is known about their structure. Rather less is known about A-algebras, though they have been studied and used by a number of authors, including Bakhturin and Semenov [1], Dallmer [2], Drensky [3], Sheina [8] and [9], Premet and Semenov [6], Semenov [7] and Towers and Varea [13], [14].

They arise in the study of constant Yang-Mills potentials. Every non-abelian nilpotent Lie algebra admits a non-trivial solution of the constant Yang-Mills equations. Moreover, if a subalgebra of a Lie algebra L admits a non-trivial solution of the Yang-Mills equations then so does L. It is therefore useful to know if a given non-nilpotent Lie algebra has a non-abelian nilpotent subalgebra (see [2] for more details). They have also been particularly important in relation to the problem of describing residually finite varieties (see [1], [8], [9], [7] and [6]).

The Frattini ideal of L, $\phi(L)$, is the largest ideal of L contained in all maximal subalgebras of L. The Lie algebra L is called *ϕ-free* if $\phi(L) = 0$, and *elementary* if $\phi(B) = 0$ for every subalgebra B of L. We say that L is an E-algebra if $\phi(B) \leq \phi(L)$ for all subalgebras B of L. Following Jacobson [4], we say that a linear Lie algebra $L \leq \mathrm{gl}(V)$ is *almost algebraic* if L contains the nilpotent and semisimple Jordan components of its elements. Every algebraic Lie algebra is almost algebraic. An abstract Lie algebra L is called almost algebraic if $\mathrm{ad}L \leq \mathrm{gl}(L)$ is almost algebraic. The classes of elementary Lie algebras, E-algebras, almost algebraic Lie algebras and A-algebras are related, as is shown in [13] and [14]. The *centre* of L is $Z(L) = \{x \in L : [x, y] = 0 \text{ for all } y \in L\}$. We summarise below some of the known results for Lie A-algebras.

Theorem 1.1 *Let L be a Lie A-algebra over a field F.*

(i) If F has characteristic zero, then

(a) L is almost algebraic if and only if it is elementary; in this case L splits over each of its ideals;

(b) L is elementary whenever $L/R(L)$ and $R(L)$ are elementary, where $R(L)$ is the solvable radical of L; and

(c) *L is an E-algebra.*

(ii) *If F has characteristic $\neq 2, 3$, then $Q(L) = \{c \in L : (\mathrm{adc})^2 = 0\}$is the unique maximal abelian ideal in L.*

(iii) *If F has characteristic $\neq 2, 3$ and cohomological dimension ≤ 1, then*

(a) *$L^2 \cap Z(L) = 0$; and*

(b) *L has a Levi decomposition and every Levi subalgebra is representable as a direct sum of simple ideals, each one of which splits over some finite extension of the ground field into a direct sum of ideals isomorphic to $sl(2)$.*

Proof. (i) See Towers and Varea, [14].
(ii), (iii) See Premet and Semenov, [6].

The purpose of this paper is to obtain more detailed results on the structure of solvable Lie A-algebras. Some of the development is suggested by [5], but more is possible for Lie algebras.

In section two we collect together the preliminary results that we need, including the fact that for Lie A-algebras the derived series coincides with the lower nilpotent series. We also see that Lie A-algebras need not be metabelian.

Section three contains the basic structure theorems for solvable Lie A-algebras. First they split over each term in their derived series. This leads to a decomposition of L as $L = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_0$ where A_i is an abelian subalgebra of L and $L^{(i)} = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_i$ for each $0 \leq i \leq n$. It is shown that the ideals of L relate nicely to this decomposition: if K is an ideal of L then $K = (K \cap A_n) \dot{+} (K \cap A_{n-1}) \dot{+} \ldots \dot{+} (K \cap A_0)$; moreover, if N is the nilradical of L, $Z(L^{(i)}) = N \cap A_i$. We also see that the result in Theorem 1.1 (iii)(a) holds when L is solvable without any restrictions on the underlying field.

The fourth section looks at Lie A-algebras in which L^2 is nilpotent. These are metabelian and so the results of section three simplify. In addition we can locate the position of the maximal nilpotent subalgebras: if U is a maximal nilpotent subalgebra of L then $U = (U \cap L^2) \oplus (U \cap C)$ where C is a Cartan subalgebra of L.

Section five is devoted to Lie A-algebras having a unique minimal ideal W. These have played a significant part in the study of varieties of residually finite Lie algebras. Again some of the results of sections three and four simplify. In particular, $N = Z_L(W)$, and if L is strongly solvable the maximal nilpotent subalgebras of L are L^2 and the Cartan subalgebras of L (that is, the subalgebras that are complementary to L^2.) We also give necessary and sufficient conditions for a Lie algebra with a unique minimal ideal to be a strongly solvable A-algebra.

The final section is devoted to more detailed structure results when the underlying field is algebraically closed.

Throughout L will denote a finite-dimensional Lie algebra over a field F. Algebra direct sums will be denoted by $\oplus$, whereas vector space direct sums will be denoted by $\dot{+}$.

2 Preliminary results

First we note that the class of Lie A-algebras is closed with respect to subalgebras, factor algebras and direct sums. Also that there is always a unique maximal abelian ideal, and it is the nilradical (which is equal to $Q(L)$ if F has characteristic $\neq 2, 3$, by Theorem 1.1 (ii)).

Lemma 2.1 *Let L be a Lie A-algebra and let N be its nilradical. Then*

(i) N is the unique maximal abelian ideal of L; and

(ii) every subalgebra and every factor algebra of L is an A-algebra.

Proof. (i) Clearly N is abelian and contains every abelian ideal of L.

(ii) It is easy to see that L is subalgebra closed; that it is factor algebra closed is [6, Lemma 1].

Lemma 2.2 *Let B, C be ideals of the Lie algebra L.*

(i) If L/A, L/B are A-algebras, then $L/(B \cap C)$ is an A-algebra.

(ii) If $L = B \oplus C$, where B, C are A-algebras, then L is an A-algebra.

Proof. (i) Let $U/(B \cap C)$ be a nilpotent subalgebra of $L/(B \cap C)$. Then $(U + B)/B$ is a nilpotent subalgebra of L/B, which is an A-algebra. It follows that $U^2 \subseteq B$. Similarly, $U^2 \subseteq C$, whence the result.

(ii) This follows from (i).

We define the *nilpotent residual*, $\gamma_\infty(L)$, of L be the smallest ideal of L such that $L/\gamma_\infty(L)$ is nilpotent. Clearly this is the intersection of the terms of the lower central series for L. Then the *lower nilpotent series* for L is the sequence of ideals $N_i(L)$ of L defined by $N_0(L) = L$, $N_{i+1}(L) = \gamma_\infty(N_i(L))$ for $i \geq 0$. The *derived series* for L is the sequence of ideals $L^{(i)}$ of L defined by $L^{(0)} = L$, $L^{(i+1)} = [L^{(i)}, L^{(i)}]$ for $i \geq 0$; we will also write L^2 for $L^{(1)}$. If $L^{(n)} = 0$ but $L^{(n-1)} \neq 0$ we say that that L has *derived length n*.

For Lie A-algebras we have the following result.

Lemma 2.3 *Let L be a Lie A-algebra. Then the lower nilpotent series coincides with the derived series.*

Proof. Since $L/L^{(1)}$ is nilpotent we have $N_1(L) \subseteq L^{(1)}$. Also $L/N_1(L)$ is nilpotent and hence abelian, by Lemma 2.1 (ii), so $L^{(1)} \subseteq N_1(L)$. Repetition of this argument gives $N_i(L) = L^{(i)}$ for each $i \geq 0$.

If F has characteristic zero, then every Lie A-algebra over F is metabelian, since L^2 is nilpotent. This is not the case, however, when F is any field of characteristic $p > 0$, as the following example, which is taken from [4, pages 52, 53], shows.

Example **2.1** *Let*

$$e = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ \vdots & & & & & & \vdots \\ 0 & . & . & . & . & 0 & 1 \\ 1 & 0 & . & . & . & . & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & p-1 \end{bmatrix},$$

let F be a field of prime characteristic p and put $L = Fe + Ff + F^p$ with product $[a + \mathbf{x}, b + \mathbf{y}] = [a, b] + (\mathbf{x}b - \mathbf{y}a)$ for all $a, b \in Fe + Ff$, $\mathbf{x}, \mathbf{y} \in F^p$.

Then L is a solvable Lie algebra and $L^2 = Fe + F^p$ is not nilpotent. Moreover, F^p is a minimal ideal of L so the maximal subalgebras are either isomorphic to $Fe + Ff$, which is solvable but not nilpotent, or of the form $F(\alpha e + \beta f) + F^p$ for some $\alpha, \beta \in F$. It is straightforward to calculate that the characteristic polynomial of $\alpha e + \beta f$ is $x^p - \beta^{p-1}x - \alpha^p$. This is never divisible by x^2 and is divisible by x if and only if $\alpha = 0$. It follows that the nilpotent subalgebras of L are one-dimensional, $Ff + F\mathbf{x}_1$ where

$\mathbf{x}_1 = (1, 0, 0, \ldots, 0)$, or inside F^p; in particular, all of them are abelian so this is a Lie A-algebra.

Note that L is also ϕ-free but not elementary. For let $B = Fe + F^p$. Then it is easy to see that $F\mathbf{x}_1 + \cdots + F\mathbf{x}_p$ (where $\mathbf{x}_i$ is the i^{th} standard basis vector for F^p) is an ideal of B, and is, in fact, $\phi(B)$. Therefore this is an example of a Lie A-algebra that is not an E-algebra.

If B is a subalgebra of L, the *centraliser* of B in L is $Z_L(B) = \{x \in L : [x, B] = 0\}$. We shall also need the following simple result.

Lemma 2.4 *Let L be any solvable Lie algebra with nilradical N. Then $Z_L(N) \subseteq N$*

Proof. Suppose that $Z_L(N) \not\subseteq N$. Then there is a non-trivial abelian ideal $A/(N \cap Z_L(N)$ of $L/(N \cap Z_L(N)$ inside $Z_L(N)/(N \cap Z_L(N)$. But now $A^3 \subseteq [A, N] = 0$, so A is a nilpotent ideal of L. It follows that $A \subseteq N \cap Z_L(N)$, a contradiction.

3 Decomposition results

Here we have the basic structure theorems. First we see that L splits over the terms in its derived series.

Theorem 3.1 *Let L be a solvable Lie A-algebra. Then L splits over each term in its derived series. Moreover, the Cartan subalgebras of $L^{(i)}/L^{(i+2)}$ are precisely the subalgebras that are complementary to $L^{(i+1)}/L^{(i+2)}$ for $i \geq 0$.*

Proof. Suppose that $L^{(n+1)} = 0$ but $L^{(n)} \neq 0$. First we show that L splits over $L^{(n)}$. Clearly we can assume that $n \geq 2$. Let C be a Cartan subalgebra of $L^{(n-1)}$ (see, for example, [15, Corollary 4.4.1.2]) and let $L = L_0 \dot{+} L_1$ be the Fitting decomposition of L relative to ad$C$. Then $L_1 = \cap_{k=1}^{\infty} L(\text{ad}C)^k \subseteq L^{(n)}$, and so L_1 is an abelian ideal of L. Also $L^{(n-1)} = L_1 \dot{+} L_0 \cap L^{(n-1)}$ and $L_0 \cap L^{(n-1)} = (L^{(n-1)})_0 = C$, which is abelian. It follows that $L^{(n-1)}/L_1 = N_{n-1}(L)/L_1$ is abelian, whence $L_1 \subseteq N_n(L) = L^{(n)}$ and $L = L_0 \dot{+} L^{(n)}$.

So we have that $L = L^{(n)} \dot{+} B$ where $B = L_0$ is a subalgebra of L. Clearly $B^{(n)} = 0$, so, by the above argument, B splits over $B^{(n-1)}$, say

$B = B^{(n-1)} \dot{+} D$. But then $L = L^{(n)} \dot{+} (B^{(n-1)} \dot{+} D) = L^{(n-1)} \dot{+} D$. Continuing in this way gives the desired result.

This gives us the following fundamental decomposition result.

Corollary 3.2 *Let L be a solvable Lie A-algebra of derived length $n+1$. Then*

(i) $L = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_0$ *where A_i is an abelian subalgebra of L for each $0 \leq i \leq n$; and*

(ii) $L^{(i)} = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_i$ *for each $0 \leq i \leq n$*

Proof. (i) By Theorem 3.1 there is a subalgebra B_n of L such that $L = L^{(n)} \dot{+} B_n$. Put $A_n = L^{(n)}$. Similarly $B_n = A_{n-1} \dot{+} B_{n-1}$ where $A_{n-1} = (B_n)^{(n-1)}$. Continuing in this way we get the claimed result. Note, in particular, that it is apparent from the construction that $A_k \cap (A_{k-1} + ... + A_0) = 0$ for each $1 \leq k \leq n$, and that it is easy to see from this that the sum is a vector space direct sum.

(ii) We have that $L^{(n)} = A_n$. Suppose that $L^{(k)} = A_n \dot{+} \ldots \dot{+} A_k$ for some $1 \leq k \leq n$. Then $L = L^{(k)} \dot{+} B_k$ and $A_{k-1} = B_k^{(k-1)}$ by the construction in (i). But now $L^{(k-1)} \subseteq L^{(k)} + B_k^{(k-1)} \subseteq L^{(k-1)}$, whence $L^{(k-1)} = A_n \dot{+} \ldots \dot{+} A_{k-1}$ and the result follows by induction.

Now we show that the result in Theorem 1.1 (iii)(a) holds when L is solvable without any restrictions on the underlying field. We say that L is *monolithic* with *monolith* W if W is the unique minimal ideal of L.

Theorem 3.3 *Let L be a solvable Lie A-algebra. Then $Z(L) \cap L^2 = 0$.*

Proof. Let L be a minimal counter-example and let $z \in Z(L) \cap L^2$. Put $Z(L) = U \dot{+} Fz$. Then U is an ideal of L and

$$U \neq z + U \in (Z(L) \cap L^2 + U)/U = Z(L/U) \cap (L/U)^2.$$

The minimality of L implies that $U = 0$, so $Z(L) = Fz$. But now if K is an ideal of L which does not contain $Z(L)$, then $K \neq z + K \in Z(L/K) \cap (L/K)^2$ similarly, contradicting the minimality of L. It follows that L is monolithic with monolith $Z(L)$.

Now let M be a maximal ideal of L. Then $Z(M) \cap M^2 = 0$ by the minimality of L, so $Z(L) \not\subseteq M^2$, whence $M^2 = 0$. It follows that $L = M\dot{+}Fx$ for some $x \in L$ and M is abelian. Let $L = L_0\dot{+}L_1$ be the Fitting decomposition of L relative to ad$x$. Then $L_1 = \cap_{i=1}^{\infty} L(\text{ad}x)^i \subseteq M$, and $[L_0, L_1] \subseteq L_1$, so L_1 is an ideal of L. But this implies that $Z(L) \subseteq L_1 \cap L_0 = 0$, a contradiction. The result follows.

Next we aim to show the relationship between ideals of L and the decomposition given in Corollary 3.2. First we need the following lemma.

Lemma 3.4 *Let L be a solvable Lie A-algebra of derived length $\leq n+1$, and suppose that $L = B\dot{+}C$ where $B = L^{(n-1)}$ and C is a subalgebra of L. If D is an ideal of L then $D = (B \cap D)\dot{+}(C \cap D)$.*

Proof. Let L be a counter-example for which dim$L$ + dim$D$ is minimal. Suppose first that $D^2 \neq 0$. Then $D^2 = (B\cap D^2)\dot{+}(C\cap D^2)$ by the minimality of L. Moreover, since

$$L/D^2 = (B + D^2)/D^2 \ \dot{+}\ (C + D^2)/D^2$$

we have

$$D/D^2 = (B \cap D + D^2)/D^2 \ \dot{+}\ (C \cap D + D^2)/D^2$$

whence

$$D = B \cap D + C \cap D + D^2 = B \cap D + C \cap D.$$

We therefore have that $D^2 = 0$.

Put $E = C^{(n-1)}$. Then $D + E \subseteq N$, the nilradical of L, which is abelian, so $[D, E] = 0$; that is, $D \subseteq Z_L(E)$. But $Z_L(E) = Z_B(E) + Z_C(E)$. For, suppose that $x = b + c \in Z_L(E)$, where $b \in B$, $c \in C$. Then $0 = [x, E] = [b, E] + [c, E]$, so $[b, E] = -[c, E] \in B \cap C = 0$. This implies that $Z_L(E) \subseteq Z_B(E) + Z_C(E)$. But the reverse inclusion is clear, so equality follows.

Now $L^{(n-2)} \subseteq B + E \subseteq L^{(n-2)}$, so $B = L^{(n-1)} = (B + E)^2 = [B, E]$. Let $L^{(n-2)} = L_0\dot{+}L_1$ be the Fitting decomposition of $L^{(n-2)}$ relative to ad$E$. Then $B \subseteq L_1$ so that $Z_B(E) \subseteq L_0 \cap L_1 = 0$, whence $D \subseteq Z_L(E) = Z_C(E) \subseteq C$ and the result follows.

Theorem 3.5 *Let L be a solvable Lie A-algebra of derived length $n+1$ with nilradical N, and let K be an ideal of L and A a minimal ideal of L. Then, with the same notation as Corollary 3.2,* *(i)* $K = (K \cap A_n) \dot{+} (K \cap A_{n-1}) \dot{+} \ldots \dot{+} (K \cap A_0)$*;*

(ii) $N = A_n \oplus (N \cap A_{n-1}) \oplus \ldots \oplus (N \cap A_0)$*;*

(iii) $Z(L^{(i)}) = N \cap A_i$ *for each* $0 \leq i \leq n$*; and*

(iv) $A \subseteq N \cap A_i$ *for some* $0 \leq i \leq n$*.*

Proof. (i) We have that $L = A_n \dot{+} B_n$ where $A_n = L^{(n)}$ from the proof of Corollary 3.2. It follows from Lemma 3.4 that $K = (K \cap A_n) + (K \cap B_n)$. But now $K \cap B_n$ is an ideal of B_n and $B_n = A_{n-1} \dot{+} B_{n-1}$. Applying Lemma 3.4 again gives $K \cap B_n = (K \cap A_{n-1}) \dot{+} (K \cap B_{n-1})$. Continuing in this way gives the required result.

(ii) This is clear from (i), since $A_n = L^{(n)} = N \cap A_n$.

(iii) We have that $L^{(i)} = L^{(i+1)} \dot{+} A_i$ from Corollary 3.2, and $Z(L^{(i)}) \cap L^{(i+1)} = 0$ from Theorem 3.3. Thus, using Lemma 3.4,

$$Z(L^{(i)}) = (Z(L^{(i)}) \cap L^{(i+1)}) + (Z(L^{(i)}) \cap A_i) = Z(L^{(i)}) \cap A_i \subseteq N \cap A_i.$$

It remains to show that $N \cap A_i \subseteq Z(L^{(i)})$; that is, $[N \cap A_i, L^{(i)}] = 0$. Let L be a minimal counter-example. Then $B_n = A_{n-1} + \cdots + A_0$ is a solvable Lie A-algebra with nilradical $A_{n-1} \oplus N \cap A_{n-2} \oplus \ldots \oplus N \cap A_0$, so $[N \cap A_i, B_n^{(i)}] = [N \cap A_i, A_{n-1} + \cdots + A_i] = 0$ by the minimality assumption. But $[N \cap A_i, A_n] = [N \cap A_i, L^{(n)}] \subseteq [N, N] = 0$, whence $[N \cap A_i, L^{(i)}] = [N \cap A_i, A_n + A_{n-1} + \cdots + A_i] = 0$.

(iv) We have $A \subseteq L^{(i)}$, $A \not\subseteq L^{(i+1)}$ for some $0 \leq i \leq n$. Now $[L^{(i)}, A] \subseteq [L^{(i)}, L^{(i)}] = L^{(i+1)}$, so $[L^{(i)}, A] \neq A$. It follows that $[L^{(i)}, A] = 0$, whence $A \subseteq Z(L^{(i)}) = N \cap A_i$, by (ii).

The final result in this section shows when two ideals of a Lie A-algebra centralise each other.

Proposition 3.6 *Let L be a Lie A-algebra and let B, D be ideals of L. Then $B \subseteq Z_L(D)$ if and only if $B \cap D \subseteq Z(B) \cap Z(D)$.*

Proof. Suppose first that $B \subseteq Z_L(D)$. Then $(B \cap D)D = (B \cap D)B = 0$, whence $B \cap D \subseteq Z(B) \cap Z(D)$.

Conversely, suppose that $B \cap D \subseteq Z(B) \cap Z(D)$. Then $BD \subseteq B \cap D \subseteq Z(B + D)$ which yields that $BD \subseteq (B + D)^2 \cap Z(B + D) = 0$, by Theorem 3.3. Hence $B \subseteq Z_L(D)$.

4 Strongly solvable Lie A-algebras

A Lie algebra L is called *strongly solvable* if L^2 is nilpotent. Over a field of characteristic zero every solvable Lie algebra is strongly solvable. Clearly strongly solvable Lie A-algebras are metabelian so we would expect stronger results to hold for this class of algebras. First the decomposition theorem takes on a simpler form.

Theorem 4.1 *Let L be a strongly solvable Lie A-algebra with nilradical N. Then $L = L^2 \dot{+} B$, where L^2 is abelian and B is an abelian subalgebra of L, and $N = L^2 \oplus Z(L)$.*

Proof. We have that $L = L^2 \dot{+} B$, where B is an abelian subalgebra of L, by Theorem 3.1. Also, L^2 is nilpotent and so abelian. Clearly $L^2 + Z(L) \subseteq N$. Moreover, $N = L^2 + N \cap B$ and $[N \cap B, L] = [N \cap B, L^2 + B] = 0$ so that $N \cap B \subseteq Z(L)$, giving the reverse inclusion.

Next we see that the minimal ideals are easy to locate.

Theorem 4.2 *Let $L = L^2 \dot{+} B$ be a strongly solvable Lie A-algebra and let A be a minimal ideal of L. Then*

(i) $A \subseteq L^2$ or $A \subseteq B$;

(ii) $A \subseteq B$ if and only if $A \subseteq Z(L)$ (in which case dim $A = 1$); and

(iii) $A \subseteq L^2$ if and only if $[A, L] = A$.

Proof. (i) We have that $A = (A \cap L^2) \dot{+} (A \cap B)$, by Theorem 3.5 (i). Since A is minimal, either $A \cap L^2 = A$ or $A \cap L^2 = 0$. The former implies that $A \subseteq L^2$, and the latter that $A \subseteq B$.

(ii) Suppose first that $A \subseteq B$. Then $[A, L] = [A, L^2 + B] = [A, L^2]$. Now $[A, L^2] = A$ implies that $A \subseteq L^2$, a contradiction. It follows that $[A, L^2] = 0$ and so $A \subseteq Z(L)$.

Now suppose that $A \subseteq Z(L)$. Then $A \cap L^2 \subseteq Z(L) \cap L^2 = 0$, by Theorem 3.3.

(iii) Suppose that $A \subseteq L^2$. Then $[A, L] \neq 0$ from (ii), so $[A, L] = A$. The converse is clear.

Corollary 4.3 *Let L be a strongly solvable Lie A-algebra. Then L is ϕ-free if and only if $L^2 \subseteq AsocL$.*

Proof. Suppose first that L is ϕ-free. Then $L^2 \subseteq N = \text{Asoc}L$, by [11, Theorem 7.4].

So suppose now that $L^2 \subseteq \text{Asoc}L$. Then L splits over $\text{Asoc}L$ by Theorem 3.1. But now L is ϕ-free by [11, Theorem 7.3].

Finally we can identify the maximal nilpotent subalgebras of L. First we need the following lemma.

Lemma 4.4 *Let L be a metabelian Lie algebra, and let U be a maximal nilpotent subalgebra of L. Then $U \cap L^2$ is an abelian ideal of L and $L^2 = (U \cap L^2) \oplus K$ where K is an ideal of L and $[U, K] = K$.*

Proof. Let $L = L_0 \dot{+} L_1$ be the Fitting decomposition of L relative to $\text{ad}U$. Then $L_1 = \cap_{i=1}^{\infty} L(\text{ad}U)^i \subseteq L^2$, and so L_1 is an abelian ideal of L. Moreover, $L^2 = (L_0 \cap L^2) \dot{+} L_1$ and

$$[L, L_0 \cap L^2] = [L_0 + L_1, L_0 \cap L^2] \subseteq (L_0 \cap L^2) + L^{(3)} = L_0 \cap L^2,$$

so $L_0 \cap L^2$ is an ideal of L. It follows that $U + (L_0 \cap L^2)$ is nilpotent and so $L_0 \cap L^2 \subseteq U \cap L^2$. The reverse inclusion is clear. Finally put $K = L_1$.

Theorem 4.5 *Let L be a strongly solvable Lie A-algebra, and let U be a maximal nilpotent subalgebra of L. Then $U = (U \cap L^2) \oplus (U \cap C)$ where C is a Cartan subalgebra of L.*

Proof. Put $U = (U \cap L^2) \oplus D$, so D is an abelian subalgebra of L. Let $L = L_0 \dot{+} L_1$ be the Fitting decomposition of L relative to $\text{ad}D$, and let $L_0 = L_0^2 \dot{+} E$ where E is an abelian subalgebra of L_0. Put $L^2 = (U \cap L^2) \oplus K$ as given by Lemma 4.4. Then

$$K = [U, K] = [D, K] \text{ so } K \subseteq L_1 \text{ and } U \cap L^2 \subseteq L_0 \cap L^2.$$

Hence

$$L_0 \cap L^2 = (U \cap L^2) + (L_0 \cap L^2 \cap K) = U \cap L^2.$$

But $L^2 = L_0^2 \oplus [L_0, L_1]$, so $L_0 \cap L^2 = L_0^2$ and

$$U = L_0^2 \oplus (E \cap U) = (U \cap L^2) \oplus (E \cap U).$$

Now put $E = (E \cap L^2) \oplus C$ where $E \cap U \subseteq C$. Then

$$L = L_1 + L_0 = L^2 + L_0 = L^2 + E = L^2 \dot{+} C,$$

so C is a Cartan subalgebra of L, by Theorem 3.1. Moreover, $E \cap U \subseteq C \cap U$, whence

$$C \cap U = (E \cap U) \oplus (C \cap U \cap L^2) = E \cap U,$$

since $C \cap U \cap L^2 \subseteq C \cap (E \cap L^2) = 0$.

5 Monolithic solvable Lie A-algebras

Monolithic algebras play a part in the application of A-algebras to the study of residually finite varieties, so it seems worthwhile to investigate what extra properties they might have.

Theorem 5.1 *Let L be a monolithic solvable Lie A-algebra of derived length $n+1$ with monolith W. Then, with the same notation as Corollary 3.2,*

(i) W is abelian;

(ii) $Z(L) = 0$ and $[L, W] = W$;

(iii) $N = A_n = L^{(n)}$;

(iv) $N = Z_L(W)$; and

(v) L is ϕ-free if and only if $W = N$.

Proof. (i) Clearly $W \subseteq L^{(n)}$, which is abelian.

(ii) If $Z(L) \neq 0$ then $W \subseteq Z(L) \cap L^2 = 0$, by Theorem 3.5, a contradiction. Hence $Z(L) = 0$. It follows from this that $[L, W] \neq 0$, whence $[L, W] = W$.

(iii) We have $N = A_n \oplus N \cap A_{n-1} \oplus \ldots \oplus N \cap A_0$ by Theorem 3.5(i). Moreover, $N \cap A_i$ is an ideal of L for each $0 \leq i \leq n-1$, by Theorem 3.5(ii). But if $N \cap A_i \neq 0$ then $W \subseteq A_n \cap N \cap A_i = 0$ if $i \neq n$. This contradiction yields the result.

(iv) We have that $L = N \dot{+} B$ for some subalgebra B of L, by Theorem 3.1 and (iii). Put $C = Z_L(W)$ and note that $N \subseteq C$. Suppose that $N \neq C$. Then $C = N \dot{+} B \cap C$. Choose A to be a minimal ideal of $B \cap C$, so that A

is abelian, and let $L = L_0 \dot{+} L_1$ be the Fitting decomposition of L relative to ad$A$. Then

$$L_1 = \bigcap_{i=1}^{\infty} L(\mathrm{ad}A)^i \subseteq [[[L, A], A], A] \subseteq [[C, A], A] \subseteq [N + A, A] \subseteq N,$$

which is abelian. It follows that L_1 is an ideal of L and so $L_1 = 0$, since otherwise $W \subseteq L_1 \cap L_0 = 0$. This yields that $N + A$ is nilpotent and thus abelian, whence $A \subseteq Z_L(N) \subseteq N$, by Lemma 2.4. This contradiction implies that $N = C$.

(v) Clearly $W = \mathrm{Asoc}L$. Suppose first that L is ϕ-free. Then $W = \mathrm{Asoc}L = N$, by [11, Theorem 7.4]. So suppose now that $\mathrm{Asoc}L = W = N$. Then L splits over Asoc$L$ by Theorem 3.1 and (iii). But now L is ϕ-free by [11, Theorem 7.3].

Note that Example 2.1 is monolithic, so monolithic solvable A-algebras are not necessarily metabelian. However, when the Lie A-algebra is strongly solvable the situation is more straightforward.

Theorem 5.2 *Let L be a monolithic strongly solvable Lie A-algebra. Then the maximal nilpotent subalgebras of L are L^2 and the Cartan subalgebras of L (that is, the subalgebras that are complementary to L^2.)*

Proof. Let U be a maximal nilpotent subalgebra of L and let W be the monolith of L. Then $L^2 = (U \cap L^2) \oplus K$ where $U \cap L^2, K$ are ideals of L and $[U, K] = K$, by Lemma 4.4. Either $W \subseteq U \cap L^2$ and $K = 0$ or else $W \subseteq K$ and $U \cap L^2 = 0$.

In the former case $N = L^2 \subseteq U$, by Theorem 5.1. But then $U \subseteq Z_L(N) \subseteq N$, by Lemma 2.4, so $U = L^2$. In the latter case U is a Cartan subalgebra of L, by Theorem 4.5.

Finally we give necessary and sufficient conditions for a monolithic algebra to be a strongly solvable Lie A-algebra. The next two results are essentially Lemma 3 of [8], though the proofs are somewhat different.

Lemma 5.3 *Let $L = L^2 \dot{+} B$ be a metabelian Lie algebra, where B is a subalgebra of L, and suppose that $[L^2, b] = L^2$ for all $b \in B$. Then L is a strongly solvable A-algebra.*

Proof. Let U be a maximal nilpotent subalgebra of L. We have $L^2 = (U \cap L^2) \oplus K$ where K is an ideal of L and $[U, K] = K$, by Lemma 4.4. Let $u = x + b \in U$, where $x \in L^2$, $b \in B$. Then $L^2 = [L^2, b] = [L^2, u]$, so $L^2 = L^2(\text{ad}u)^i$ for all $i \geq 1$. It follows that $L^2 = K$ from which $U^2 \subseteq U \cap L^2 = 0$ and L is an A-algebra.

Theorem 5.4 *Let L be a monolithic Lie algebra. Then L is a strongly solvable A-algebra if and only if $L = L^2 \dot{+} B$ is metabelian, where B is a subalgebra of L and $[L^2, b] = L^2$ for all $b \in B$ (or, equivalently, ad$b$ acts invertibly on L^2).*

Proof. Suppose first that L is a strongly solvable A-algebra. Then $L = L^2 \dot{+} B$ is metabelian, where B is a subalgebra of L, by Theorem 3.1. Let $b \in B$ and let $L = L_0 \dot{+} L_1$ be the Fitting decomposition of L relative to ad$b$. It is easy to see, as in Lemma 4.4, that $L^2 = (L^2 \cap L_0) \dot{+} L_1$ and $L^2 \cap L_0$ and L_1 are ideals of L, so $L^2 = L^2 \cap L_0$ or $L^2 = L_1$ as L is monolithic. The former implies that $[L^2, b] = 0$, but then L^2 and Fb are ideals of L, which is impossible. It follows that $L^2 = L_1$, whence $[L^2, b] = L^2$. If $\theta = \text{ad}b|_{L^2}$ then $L^2 = \text{Ker}\theta \dot{+} \text{Im}\theta$, so $\text{Ker}\theta = \{0\}$ and θ is invertible.

The converse follows from Lemma 5.3.

6 Solvable A-algebras over an algebraically closed field

First we need the following lemma.

Lemma 6.1 *Let L be a solvable Lie A-algebra over a perfect field F of characteristic $p > 0$. Let K be an ideal of L, A a minimal ideal of L with $A \subseteq Z(K)$, and N an ideal of L containing K and such that $N/K \subseteq N(L/K)$, the nilradical of L/K. Then dim $(N/Z_N(A)) \leq 1$.*

Proof. Put $\bar{L} = L/K$ and for each $x \in L$ write $\bar{x} = x + K$. Then A is an irreducible $\bar{L}$-module, and hence an irreducible U-module, where U is the universal enveloping algebra of $\bar{L}$. Let ϕ be the corresponding representation of U and let $\bar{x} \in \bar{L}$, $n \in N$. Then $[[\bar{x}, \bar{n}], \bar{n}] = \bar{0}$, whence $[\bar{x}, \bar{n}^p] = 0$ and so $\bar{n}^p \in Z = Z(U)$.

Let $n_1, n_2 \in N$. Then $\bar{n}_1^p, \bar{n}_2^p \in Z$, so $\alpha_1\bar{n}_1^p + \alpha_2\bar{n}_2^p \in \ker(\phi)$, for some $\alpha_1, \alpha_2 \in F$, since dim $\phi(Z) \leq 1$, by Schur's Lemma. Since F is perfect, there are $\beta_1, \beta_2 \in F$ such that $\alpha_1 = \beta_1^p, \alpha_2 = \beta_2^p$, so $(\beta_1\bar{n}_1 + \beta_2\bar{n}_2)^p = \beta_1^p\bar{n}_1^p + \beta_2^p\bar{n}_2^p \in \ker(\phi)$, since $[\bar{n}_1, \bar{n}_2] = \bar{0}$. It follows that $A + F(\beta_1 n_1 + \beta_2 n_2)$ is a nilpotent subalgebra of L and hence abelian. Thus $\beta_1\bar{n}_1 + \beta_2\bar{n}_2 \in \ker(\phi)$ and so dim $\phi(\bar{N}) \leq 1$. Hence $Z_N(A)$ has codimension at most 1 in N.

The following result was proved by Drensky in [3]. We include a proof since, as far as we know, no English translation of the proof has appeared.

Theorem 6.2 *Let L be a solvable Lie A-algebra over an algebraically closed field F. Then the derived length of L is at most* 3.

Proof. Suppose the result is false. Then there is such an algebra L of derived length 4. First note that we can assume that the ground field is of characteristic $p > 0$, since otherwise L is strongly solvable and so of derived length at most 2. Let A be a minimal ideal of L contained in $L^{(3)}$. Then, putting $K = L^{(3)}$, $N = L^{(2)}$ in Lemma 6.1, we deduce that dim $(L^{(2)}/Z_{L^{(2)}}(A)) \leq 1$.

Suppose that dim $(L^{(2)}/Z_{L^{(2)}}(A)) = 1$. Put $S = L/Z_{L^{(2)}}(A)$. Then $\dim(S^{(2)}) = 1$. It follows that $S/Z_L(S^{(2)}) \subseteq \mathrm{Der}(S^{(2)})$ and so has dimension at most one, giving $[S^{(1)}, S^{(2)}] = 0$. But now $S^{(1)}$ is nilpotent but not abelian. As S must be an A-algebra, this is a contradiction. We therefore have that dim $(L^{(2)}/Z_{L^{(2)}}(A)) = 0$, whence $[A, L^{(2)}] = 0$.

Since the field F is algebraically closed there is a chain of ideals of L, $0 = A_0 \subset A_1 \subset \ldots \subset A_r = L^{(3)}$, where $\dim A_i = i$ for each $1 \leq i \leq r$. By the above we have $[A_i, L^{(2)}] \subseteq A_{i-1}$ for each $1 \leq i \leq r$. It follows that $L^{(3)} + Fx$ is a nilpotent subalgebra of L for each $x \in L^{(2)}$, whence $[L^{(3)}, L^{(2)}] = 0$. This means that $L^{(2)}$ is a nilpotent subalgebra of L and hence abelian. We infer that $L^{(3)} = 0$, a contradiction, and so the result follows.

Using the above we can examine in more detail the structure of monolithic Lie A-algebras.

Theorem 6.3 *Let L be a monolithic solvable Lie A-algebra of dimension greater than one over an algebraically closed field F, with monolith W. Then either*

(i) $L = L^2 \dot{+} Fb$ where L^2 is abelian and $L^2(adb - \lambda 1)^k = 0$ for some $k > 0$ and some $0 \neq \lambda \in F$, and $\dim W = 1$; or

(ii) F has characteristic $p > 0$, $\dim W = p$ and $L = L^{(2)} \dot{+} B$ where $L^{(2)}$ is abelian, $B = Fb + Fn$, $[n, b] = n$, $L^{(2)}(ad\, n - \lambda 1)^k = 0$ and $L^{(2)}((ad\, b)^p - ad\, b - \mu^p 1)^k = 0$ for some $k > 0$ and some $0 \neq \lambda, \mu \in F$.

Proof. Suppose first that L is strongly solvable. Then $L = L^2 \dot{+} B$ where L^2 is abelian, B is an abelian subalgebra and $W \subseteq L^2$. Now W is an irreducible B-module and so one dimensional, by [10, Lemma 5.6]. Now $L/Z_L(W)$ is isomorphic to a subalgebra of $\mathrm{Der}(W)$ and so $N = Z_L(W)$ has codimension at most one in L. It follows that L is abelian (and hence one dimensional) or $\dim B = 1$ and $N = L^2$. Decompose L^2 into $\mathrm{ad}\, B$-invariant subspaces. Each is an ideal of L and so there can be only one. It follows that $L^2(\mathrm{ad}\, b - \lambda 1)^k = 0$ for some $k > 0$ and some $0 \neq \lambda \in F$, where $B = Fb$, giving case (i).

So suppose now that L^2 is not nilpotent. Then F has characteristic $p > 0$, L has derived length 3 and $W \subseteq L^{(2)}$. Let $N/L^{(2)}$ be the nilradical of $L/L^{(2)}$. Then applying Lemma 6.1 with $K = L^{(2)}$ we see that $\dim (N/Z_N(W)) \leq 1$. But $Z_L(W) = L^{(2)} \subseteq N$ by Theorem 5.1(iv), so $Z_N(W) = L^{(2)}$. As $L^{(1)} \subseteq N$ we cannot have $N = L^{(2)}$, so $\dim (N/L^{(2)}) = 1$; say $N = L^{(2)} \dot{+} Fn$.

Put $L = L^{(2)} \dot{+} B$ where B is a subalgebra of L containing n, and let $C = Z_B(Fn)$. Then C is a nilpotent ideal of B and so $C = Fn$. It follows that $B/Fn = B/C$ has dimension at most one, and so $\dim B \leq 2$. As B is not abelian we have $B = Fn + Fb$ where $[n, b] = n$. This algebra has a unique p-map making it into a restricted Lie algebra: namely $b^{[p]} = b, n^{[p]} = 0$ (see [10]). We can decompose $L^{(2)} = \oplus_{\lambda,S} V_{\lambda,S}$ where $\lambda \in (Fn)^*, S \in B^*$ and

$$V_{\lambda,S} = \{x \in L^{(2)} : x(\mathrm{ad}\, n - \lambda(n)1)^k = 0 \text{ and } x((\mathrm{ad}\, b)^p - ad\, b - S(b)^p 1)^k = 0\}$$

by [10, page 236]. As each $V_{\lambda,S}$ is an ideal of L there can be only one of them. The fact that $\dim W = p$ follows from [10, Example 1, page 244], so we have case (ii).

Corollary 6.4 *If, in addition to the hypotheses of Theorem 6.3, L is also ϕ-free, then either*

(i) L is two-dimensional non-abelian; or

(ii) F has characteristic $p > 0$ and L is isomorphic to the algebra in Example 2.1.

Proof. Case (i) follows from Theorem 6.3 (i) because $W = L^2$ by Theorem 5.1. If case (ii) of Theorem 5.1 holds, then F has characteristic $p > 0$, $\dim W = p$ and $L = W \dot{+} B$ where W is abelian, $B = Fb + Fn$ and $[n, b] = n$. Let λ be an eigenvalue for $(\operatorname{ad} b)|_W$, so $[w, b] = \lambda\, w$ for some $w \in W$. Then $[w(\operatorname{ad} n)^i, b] = (\lambda + i)w(\operatorname{ad} n)^i$ for every i, so putting $w_i = w(\operatorname{ad} n)^i$ we see that $Fw_0 + \cdots + Fw_{p-1}$ is B-stable and hence equal to W. We then have $[w_i, b] = (\lambda + i)w_i$, $[w_i, n] = w_{i+1}$ (indices modulo p). But now the characteristic polynomial of $\operatorname{ad}(b + \alpha\, n)$ is $(x - \lambda)^p - (x - \lambda) - \alpha^p$ and this is divisible by x precisely when $\alpha^p = \lambda - \lambda^p$. It follows that by choosing α satisfying this equation and replacing b by $b + \alpha\, n$ we can take $\lambda = 0$. This gives the algebra in Example 2.1.

Note: alternatively, it can be deduced that W has the form claimed in (ii) by using [10, Example 1, page 244].

Finally we seek describe the structure of ϕ-free solvable Lie A-algebras over an algebraically closed field. The strongly solvable ones are easily described.

Theorem 6.5 *Let L be a ϕ-free strongly solvable Lie A-algebra over an algebraically closed field F. Then*

$$L = \sum_{i=1}^{m} Fa_i + \sum_{i=1}^{n} Fb_i \text{ where } [a_i, b_j] = \lambda_{ij} a_i$$

for all $1 \leq i \leq m, 1 \leq j \leq n$, other products being zero.

Proof. If L is strongly solvable then it is elementary, by [13, Theorem 2.5], and hence as described in (i), by [13, Theorem 3.2 (2)]. (The restriction on the characteristic in that result is not required for the solvable case.)

The ϕ-free solvable Lie A-algebras that are not strongly solvable are more complicated.

Theorem 6.6 *Let L be a ϕ-free solvable Lie algebra, over an algebraically closed field F, that is not strongly solvable. Then L is an A-algebra if and only if the following conditions are satisfied:*

(i) $L = L^{(2)} \dot{+} C \dot{+} B$, *where B, C are abelian subalgebras of L and* $L^{(1)} = L^{(2)} \dot{+} C$;

(ii) $B\dot{+}C$ *is a strongly solvable* ϕ*-free Lie* A*-algebra (and hence given by Theorem 6.5);*

(iii) $L^{(2)} = A_1 \oplus \ldots \oplus A_n$*, where* A_i *is a minimal ideal of* L *of dimension* p *for each* $1 \leq i \leq n$*; and*

(iv) for each $1 \leq i \leq n$*, there exists* $c_i \in C$*,* $b_i \in B$ *and a basis* $a_{i1}, \ldots, a_{ip}$ *for* A_i *such that* $C = Z_C(A_i) \oplus Fc_i$*,* $B = Z_B(A_i) \oplus Fb_i$*,* $[c_i, b_i] = c_i$*,* $[a_{ij}, c_i] = a_{i(j+1)}$ *(indices modulo* p*) and* $[a_{ij}, b] = (\lambda_i + j)a_{ij}$ *for* $1 \leq j \leq p$ *and some* $\lambda_i \in F$*.*

Proof. Suppose first that L is a ϕ-free solvable Lie A-algebra that is not strongly solvable. Then F has characteristic $p > 0$, $L = L^{(2)}\dot{+}C\dot{+}B$ where $L^{(2)}$ is abelian, B, C are abelian subalgebras of L and $L^{(1)} = L^{(2)}\dot{+}C$, by Theorem 6.2 and Corollary 3.2; this is (i). Moreover, $L^{(2)} \subseteq N(L) = \text{Asoc}\, L$, by [11, Theorem 7.4], so we can put $L^{(2)} = A_1 \oplus \ldots \oplus A_n$, where A_i is a minimal ideal of L for each $1 \leq i \leq n$. Put $L_i = A_i\dot{+}C\dot{+}B$. Then $L_i^{(1)} = A_i\dot{+}C$ and $L_i^{(2)} = A_i$, so $[C, B] = C$ and $[A_i, C] = A_i$.

Suppose first that $\dim A_i = 1$. Then $\dim L_i/Z_{L_i}(A_i) \leq 1$. If $A_i\dot{+}C = L_i^{(1)} \subseteq Z_{L_i}(A_i)$ then $A_i = [A_i, C] = 0$, a contradiction; so $C \not\subseteq Z_{L_i}(A_i)$. But $Z_{L_i}(A_i) = A_i\dot{+}(Z_{L_i}(A_i) \cap C)\dot{+}(Z_{L_i}(A_i) \cap B)$, by Theorem 3.5, so $Z_{L_i}(A_i) \cap B = B$, giving $[A_i, B] = 0$. Hence $A_i = [A_i, C] = [A_i, [C, B]] \subseteq [C, [B, A_i]] + [B, [A_i, C]] = 0$, a contradiction again. It follows that $\dim A_i \neq 1$.

Put $Z = Z_C(A_i)\dot{+}Z_B(A_i)$ and $\bar{L}_i = L_i/Z$. We claim that $\bar{L}_i$ is monolithic and ϕ-free.

Let $\bar{D} = D/Z$ be an ideal of $\bar{L}_i$ and suppose that $\bar{A}_i = (A_i + Z)/Z \not\subseteq \bar{D}$. Then $[A_i, D] \subseteq A_i \cap D = 0$, so $D \subseteq Z_{L_i}(A_i) \cap D = (A_i + Z) \cap D = (A_i \cap D) + Z = Z$. It follows that $\bar{L}_i$ is monolithic with monolith $\bar{A}_i$. Let $\bar{U} = U/Z$ be the nilradical of $\bar{L}_i$. Then $\bar{A}_i \subseteq \bar{U}$, so $A_i \subseteq U$ and $[A_i, U] \subseteq A_i \cap Z = 0$. This yields that $U \subseteq A_i + Z$, whence $\bar{U} = \bar{A}_i$. Theorem 5.1(v) now implies that $\bar{L}_i$ is ϕ-free.

Next put $D = C\dot{+}B$. Then D/Z is two dimensional, by Corollary 6.4, and so ϕ-free, whence $\phi(D) \subseteq Z \cap C = Z_C(A_i)$ for each $1 \leq i \leq n$. It follows that $\phi(D)$ is an ideal of L and hence that $\phi(D) \subseteq \phi(L) = 0$, by [11, Lemma 4.1]. This establishes (ii).

Now D is elementary, by [13, Theorem 2.5], and so splits over each of its ideals, by Lemma 2.3 of [12]. This yields that $D = Z\dot{+}E$ for some subalgebra E of D, whence $A_i\dot{+}E \cong \bar{L}_i$ has the form given in Corollary 6.4. Assertions (iii) and (iv) now follow.

Now suppose that conditions (i)-(iv) are satisfied. Adopting the same notation as above we have that L_i/Z is an A-algebra, by (iv), and that L_i/A_i is an A-algebra, by (ii). It follows that L_i is an A-algebra, by Lemma 2.2. As this is true for each $1 \leq i \leq n$ repeated use of Lemma 2.2 yields that L is an A-algebra.

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