A MAXIMAL THEOREM FOR HOLOMORPHIC SEMIGROUPS ON VECTOR-VALUED SPACES

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Abstract. Suppose that $1 < p \leq \infty$, (Ω, μ) is a σ-finite measure space and E is a closed subspace of a Lebesgue–Bochner space $L^p(\Omega; X)$, consisting of functions on Ω that take their values in some complex Banach space X. Suppose also that $-A$ is invertible and generates a bounded holomorphic semigroup $\{T_z\}$ on E. If $0 < \alpha < 1$ and f belongs to the domain of A^α then the maximal function $\sup_z \|T_z f\|_X$, where the supremum is taken over any given sector contained in the sector of holomorphy, belongs to L^p. This extends an earlier result of Blower and Doust [BD].

1. Introduction

Suppose that $\{T_t\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on a Banach space E. In the case that E is a space of functions f from a set Ω to a normed space X, an important tool in the analysis of such a semigroup is the maximal function Mf where

$$Mf(\omega) = \operatorname*{ess-sup}_{t\geq 0} \|T_t f(\omega)\|_X .$$

The classical theorems of Stein [St] and Cowling [Co] apply to symmetric diffusion semigroups on E, where $E = L^p(\Omega)$ and $1 < p < \infty$, and show that in this case $\|Mf\|_p \leq c \|f\|_p$ for all f in $L^p(\Omega)$.

Taggart [Ta] extended this to the vector-valued context where $E = L^p(\Omega; X)$ and X satisfies a geometric condition weak enough to include, for example, many of the classical reflexive function spaces.

Under much weaker hypotheses, Blower and Doust [BD] showed that in the scalar-valued case, if the semigroup $\{T_t\}_{t>0}$ can be extended to a bounded holomorphic semigroup on sector of the complex plane, then Mf lies in $L^p(\Omega)$ at least for f in a large submanifold of $L^p(\Omega)$.

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In this paper we show that result of [BD] may be extended to the vector-valued case where E is a subspace of $L^p(\Omega; X)$ and X is any complex Banach space. Moreover, the result also holds when the supremum used to define the maximal function is taken over sectors contained in the sector of holomorphy of the semigroup (c.f., the results of [Co] and [Ta]). Both these extensions may be easily deduced by modifying the original argument of [BD].

The paper is organised as follows. In Section 2 we introduce some notation and state the main theorem of the paper. As with the result of [BD], the theorem is proved by representing of the semigroup in terms of fractional powers of its generator and obtaining good L^p bounds for parts of this representation. Some of the arguments of [BD] which made use of Yosida approximants to the semigroup's generator have been replaced by direct appeals to the functional calculus for sectorial operators. Salient facts about the functional calculus are presented in Section 3, while the representation of the semigroup and corresponding bounds are established in Section 4. In Section 5 we complete the proof of the theorem. For further discussion on the application of maximal functions and examples of semigroups to which the theorem applies, see [BD, Sections 1 and 4].

2. Notation and theorem

We begin by introducing some notation. Given θ in $[0, \pi)$, let S^0_θ and S_θ denote the open and closed sectors of $\mathbb{C}$ given by

$$S^0_\theta = \big\{\zeta \in \mathbb{C} \setminus \{0\} : |\arg \zeta| < \theta\big\}$$

and

$$S_\theta = \big\{\zeta \in \mathbb{C} \setminus \{0\} : |\arg \zeta| \le \theta\big\} \cup \{0\},$$

where $\arg \zeta$ denotes the principle argument of a nonzero complex number ζ. Note that $S_0 = [0, \infty)$.

Throughout, suppose that X is a complex Banach space and that (Ω, μ) is a positive σ-finite measure space. When $1 \le q \le \infty$, let $L^q(\Omega; X)$ denote the Lebesgue–Bochner space of strongly measurable functions $f : \Omega \to X$ whose norm is given by

$$\|f\|_{L^q(\Omega;X)} = \left(\int_\Omega \|f(\omega)\|_X^q \, \mathrm{d}\mu(\omega)\right)^{1/q}$$

if $q < \infty$ and $\|f\|_{L^q(\Omega;X)} = \text{ess-sup}_{\omega\in\Omega} \|f(\omega)\|_X$ if $q = \infty$. We write $L^q(\Omega)$ for $L^q(\Omega; \mathbb{C})$.

The setting for our main result is as follows. Suppose that E is a closed subspace of $L^p(\Omega; X)$, where $1 < p \leq \infty$, and suppose that $\{T_z : z \in S^0_\theta\}$ is a bounded holomorphic semigroup acting on E for some θ where $0 < \theta < \pi/2$. Let $-A$ denote the infinitesimal generator of this semigroup. (See [RS2, Section X.8] or [Da] for definitions of these terms.) When $0 < \alpha < 1$ we can define the fractional powers A^α and $A^{-\alpha}$ of A (see Remark 3.3). These have the property that $D(A) \subseteq D(A^\alpha)$ and $A^{-\alpha}$ is bounded on E whenever A has a bounded inverse (see [Tan, Theorem 2.3.1]). Given f in $D(A^\alpha) \cap D(A^{-\alpha})$, define $\|f\|_{p,\alpha}$ by

$$\|f\|_{p,\alpha} = \|A^\alpha f\|_{L^p(\Omega;X)} + \left\|A^{-\alpha} f\right\|_{L^p(\Omega;X)}.$$

Whenever $0 \leq \theta' < \theta$ and $f \in E$, define the maximal function $M_{\theta'} f$ by

$$M_{\theta'} f = \sup\left\{\|T_z f\|_X : z \in S_{\theta'}\right\}.$$

Theorem 2.1. *Suppose E, A and $\{T_z : z \in S^0_\theta\}$ are as above, that A is invertible and that $0 < \alpha < 1$. If $f \in D(A^\alpha)$ and $0 \leq \theta' < \theta$ then $M_{\theta'} f \in L^p(\Omega)$ and there is a constant $C(A, \alpha, \theta')$ such that*

$$\|M_{\theta'} f\|_{L^p(\Omega)} \leq C(A, \alpha, \theta') \|f\|_{p,\alpha}.$$

Remark 2.2. As in [BD], we note that if $-A$ generates a bounded holomorphic semigroup then, for each positive number s, the operator $-(sI + A)$ is invertible and also generates a holomorphic semigroup. Thus the invertibility hypothesis of Theorem 2.1 does not restrict the usefulness of the result.

Remark 2.3. The constant $C(A, \alpha, \theta')$ is bounded by

$$\frac{C_\eta \sec(\theta' + \eta) \sec(\alpha\pi/2)}{\pi\alpha}$$

for any η such that $\pi/2 - \theta < \eta < \pi/2 - \theta'$ and where C_η is the constant appearing in the resolvent bound (3.1) for A on $L^p(\Omega; X)$. Note that if the semigroup acts on a range of L^p spaces then these quantities may vary with p.

3. Functional calculus calculus for sectorial operators

In this section we summarise for use in Section 4 a few pertinent facts about the holomorphic functional calculus for sectorial operators.

Definition 3.1. Suppose that $0 \leq \vartheta < \pi$ and that E is any Banach space. We say that an operator A in E is *sectorial of type* ϑ if A is closed, $\sigma(A) \subseteq S_\vartheta$ and for each η in (ϑ, π) there exists a constant C_η such that

$$\left\|(\zeta I - A)^{-1}\right\| \leq C_\eta |\zeta|^{-1} \qquad \forall \zeta \in \mathbb{C} \setminus S_\eta. \tag{3.1}$$

We recall the following important characterisation of generators of holomorphic semigroup. Details may be found in [Da] or [RS2].

Theorem 3.2. *Suppose that E is a Banach space. A closed operator $-A$ in E generates a bounded holomorphic semigroup $\{T_z : z \in S^0_\theta\}$ on E for some θ in $(0, \pi/2)$ if and only if A is densely defined and sectorial of type $\pi/2 - \theta$.*

We now describe a holomorphic functional calculus for sectorial operators. Suppose that $0 < \vartheta < \nu < \pi$. Let ψ denote the complex-valued function defined on $\mathbb{C}$ by

$$\psi(\zeta) = \zeta/(1+\zeta)^2 \qquad \forall \zeta \in \mathbb{C}.$$

Denote by $H(S^0_\nu)$ the space of all holomorphic functions on S^0_ν. Following the notation of [CDMY], we define the following subspaces of $H(S^0_\nu)$:

$$H^\infty(S^0_\nu) = \Big\{ f \in H(S^0_\nu) : \sup_{\zeta \in S^0_\nu} |f(\zeta)| < \infty \Big\},$$
$$\Psi(S^0_\nu) = \Big\{ f \in H(S^0_\nu) : f\psi^{-s} \in H^\infty(S^0_\nu) \text{ for some } s > 0 \Big\}$$

and

$$\mathscr{F}(S^0_\nu) = \Big\{ f \in H(S^0_\nu) : f\psi^{s} \in H^\infty(S^0_\nu) \text{ for some } s > 0 \Big\}.$$

Note that

$$\Psi(S^0_\nu) \subset H^\infty(S^0_\nu) \subset \mathscr{F}(S^0_\nu) \subset H(S^0_\nu).$$

It is well known (see [CDMY, Section 2]) that if A is a one-to-one sectorial operator of type ϑ on a Banach space E with dense domain and dense range, then A has an $\mathscr{F}(S^0_\nu)$ functional calculus. Moreover, if $f \in \Psi(S^0_\nu)$ then $f(A)$, defined by the contour integral

$$f(A) = \frac{1}{2\pi i} \int_\gamma (\zeta I - A)^{-1} f(\zeta)\, \mathrm{d}\zeta, \tag{3.2}$$

is a bounded operator on E. Here the integral converges absolutely in the uniform topology and the contour γ is given by

$$\gamma(t) = \begin{cases} -te^{-i\eta} & \text{if } -\infty < t \le 0 \\ te^{i\eta} & \text{if } 0 < t < \infty, \end{cases}$$

where η is any angle strictly between ϑ and ν. It can be shown that the definition of $f(A)$ is independent of the choice of angle η in this range.

Remark 3.3. The functional calculus defined above allows one to define fractional powers for sectorial operators, and in particular for generators of holomorphic semigroups. If $0 < |\alpha| < 1$ and $\zeta \in S_\nu$, then we define the fractional power ζ^α by

$$\zeta^\alpha = \exp(\alpha \ln|\zeta| + i\alpha \arg \zeta).$$

Note that the function $\zeta \mapsto \zeta^\alpha$ belongs to $\mathscr{F}(S_\nu^0)$. Thus if A has an $\mathscr{F}(S_\nu^0)$ functional calculus then the operator A^α may be defined by $A^\alpha = g(A)$, where $g(\zeta) = \zeta^\alpha$.

4. A representation for the semigroup

Suppose that $t \in \mathbb{R}$, $0 < \alpha < 1$, $\varphi \in \mathbb{R}$, $\zeta \in \mathbb{C}$ and $|\arg(e^{i\varphi}\zeta)| < \pi/2$. By Fourier inversion,

$$\begin{aligned} e^{-|t|e^{i\varphi}\zeta} &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{e^{i\varphi}\zeta}{(e^{i\varphi}\zeta)^2 + u^2}\, e^{itu}\,\mathrm{d}u \\ &= \frac{e^{i\varphi}}{\pi}\left(\int_{|u|<1} e^{itu} F_{u,\varphi}(\zeta)\,\zeta^{-\alpha}\,\mathrm{d}u + \int_{|u|>1} e^{itu} G_{u,\varphi}(\zeta)\,\zeta^{\alpha}\,\mathrm{d}u\right) \end{aligned}$$

where $0 < \alpha < 1$,

$$F_{u,\varphi}(\zeta) = \frac{\zeta^{1+\alpha}}{(e^{i\varphi}\zeta)^2 + u^2} \qquad \text{and} \qquad G_{u,\varphi}(\zeta) = \frac{\zeta^{1-\alpha}}{(e^{i\varphi}\zeta)^2 + u^2}\,.$$

This observation and the $\mathscr{F}(S_\nu^0)$ functional calculus leads to the following lemma.

Lemma 4.1. *Suppose that A and θ are as in the hypothesis of Theorem 2.1. If $0 < \alpha < 1$, $f \in D(A^\alpha)$ and $z \in S_\theta$ then*

$$T_z f = \frac{e^{i\varphi}}{\pi}\int_{|u|<1} e^{itu} F_{u,\varphi}(A) A^{-\alpha} f\,\mathrm{d}u + \frac{e^{i\varphi}}{\pi}\int_{|u|>1} e^{itu} G_{u,\varphi}(A) A^{\alpha} f\,\mathrm{d}u,$$

where $t = |z|$ and $\varphi = \arg(z)$.

The proofs of both lemmata in this section make frequent use of the following fact. If $|\phi| < \pi/2$ then

$$\sup\left\{\frac{t^2 + u^2}{|(te^{i\phi})^2 + u^2|} : t > 0, u > 0\right\} = \sec\phi.$$

This may be deduced using planar trigonometry. We turn now to the proof of Lemma 4.1.

Proof. Suppose that $z = te^{i\varphi} \in S_\theta^0$ and choose ν such that $\pi/2 - \theta < \nu < \pi/2 - |\varphi|$. By the hypotheses on A and Theorem 3.2, A has an $\mathscr{F}(S_\nu^0)$ functional calculus. If $\zeta \in S_\nu^0$ then

$$|F_{u,\varphi}(\zeta)| \le \sec(\varphi + \nu)\,\frac{|\zeta|^{1+\alpha}}{|\zeta|^2 + u^2}$$

and hence $F_{u,\varphi} \in \Psi(S_\nu^0)$ for all nonzero u in $\mathbb{R}$. In fact, if

$$\tilde{F}_{z,\varepsilon}(\zeta) = \int_{\varepsilon<|u|<1} e^{itu} F_{u,\varphi}(\zeta)\,\mathrm{d}u \quad \text{and} \quad \tilde{F}_z(\zeta) = \int_{|u|<1} e^{itu} F_{u,\varphi}(\zeta)\,\mathrm{d}u$$

whenever $0 < \varepsilon < 1$ then

$$\begin{aligned} |\tilde{F}_{z,\varepsilon}(\zeta)| &\le 2\sec(\varphi+\nu) \int_\varepsilon^1 \frac{|\zeta|^{1+\alpha}}{|\zeta|^2 + u^2}\,\mathrm{d}u \\ &= 2\sec(\varphi+\nu)\,|\zeta|^\alpha \big(\arctan(1/|\zeta|) - \arctan(\varepsilon/|\zeta|)\big). \end{aligned}$$

Hence $\tilde{F}_{z,\varepsilon} \in \Psi(S_\nu^0)$ and $\tilde{F}_{z,\varepsilon}$ converges to $\tilde{F}_z$ uniformly on compact subsets of S_ν as $\varepsilon \to 0^+$. This convergence implies that $\tilde{F}_z$ is holomorphic in S_ν and, combining this with the bounds on $\tilde{F}_{z,\epsilon}$, one concludes that $\tilde{F}_z \in \Psi(S_\nu^0)$.

One can similarly show that $G_{u,\varphi}$ and $\tilde{G}_z$, where

$$\tilde{G}_z(\zeta) = \int_{|u|>1} e^{itu} G_{u,\varphi}(\zeta)\,\mathrm{d}u,$$

both belong to $\Psi(S_\nu^0)$.

Finally, since $T_z f = e^{-zA} f$ for all f in E and

$$e^{-z\zeta} = \frac{e^{i\varphi}}{\pi}\left(\tilde{F}_z(\zeta)\,\zeta^{-\alpha} + \tilde{G}_z(\zeta)\,\zeta^\alpha\right) \qquad \forall \zeta \in S_\nu$$

by Fourier inversion, the lemma follows from the $\mathscr{F}(S_\nu^0)$ functional calculus for A. □

For the proof of Theorem 2.1 we require good bounds for $F_{u,\varphi}(A)$ and $G_{u,\varphi}(A)$, which will be expressed in terms of two operators introduced below. If $f \in L^p(\Omega; X)$, u is a nonzero real number and $\pi/2 - \theta < \eta < \pi$, then we define the scalar-valued functions $\Gamma_{u,\eta} f$ and $\Delta_{u,\eta} f$ by

$$\Gamma_{u,\eta} f = \frac{1}{2\pi} \int_0^\infty F_{u,0}(t)\left(\left\|(te^{i\eta}I - A)^{-1} f\right\|_X + \left\|(te^{-i\eta}I - A)^{-1} f\right\|_X\right)\mathrm{d}t$$

and

$$\Delta_{u,\eta} f = \frac{1}{2\pi} \int_0^\infty G_{u,0}(t)\left(\left\|(te^{i\eta}I - A)^{-1} f\right\|_X + \left\|(te^{-i\eta}I - A)^{-1} f\right\|_X\right)\mathrm{d}t,$$

where convergence of both improper integrals takes place in $L^p(\Omega)$.

Lemma 4.2. *Suppose that* $0 < \theta' < \pi/2 - \eta < \theta$, $f \in L^p(\mu; X)$ *and* u *is a nonzero real number. Then*

$$\|(F_{u,\varphi}(A)f)(\omega)\|_X \leq \sec(\theta' + \eta)(\Gamma_{u,\eta} f)(\omega)$$

and

$$\|(G_{u,\varphi}(A)f)(\omega)\|_X \leq \sec(\theta' + \eta)(\Delta_{u,\eta} f)(\omega)$$

for μ*-almost every* ω *in* Ω *and for all* φ *in* $[-\theta', \theta']$*. Moreover,*

$$\|\Gamma_{u,\eta} f\|_{L^p(\Omega)} \leq \frac{C_\eta}{2} \sec(\alpha\pi/2)|u|^{-1+\alpha}$$

and

$$\|\Delta_{u,\eta} f\|_{L^p(\Omega)} \leq \frac{C_\eta}{2} \sec(\alpha\pi/2)|u|^{-1-\alpha},$$

where C_η *is the constant appearing in the resolvent bound (3.1) for* A*.*

Proof. If ν is chosen such that $\pi/2 - \theta < \eta < \nu < \pi/2 - \theta'$ then $F_{u,\varphi} \in \Psi(S^0_\nu)$. Hence operator $F_{u,\varphi}(A)$ has an integral representation of the form (3.2). Now if $\{g_\varepsilon\}$ is a convergent net in $L^p(\Omega; X)$, then

$$\left\|\lim_\varepsilon g_\varepsilon\right\|_X = \lim_\varepsilon \|g_\varepsilon\|_X\,,$$

where convergence on the left is in $L^p(\Omega; X)$ while convergence on the right in is $L^p(\Omega)$. So we may move the X-norm through the improper integral representing $F_{u,\varphi}(A)f$ to obtain

$$\begin{aligned}
\|F_{u,\varphi}(A)f\|_X &\leq \frac{1}{2\pi}\int_0^\infty |F_{u,\varphi}(te^{i\eta})| \left\|(te^{i\eta}I - A)^{-1} f\right\|_X \,\mathrm{d}t \\
&\quad + \frac{1}{2\pi}\int_0^\infty |F_{u,\varphi}(te^{-i\eta})| \left\|(te^{-i\eta}I - A)^{-1} f\right\|_X \,\mathrm{d}t \\
&\leq \sec(\varphi + \eta)\Gamma_{u,\eta} f \\
&\leq \sec(\theta' + \eta)\Gamma_{u,\eta} f.
\end{aligned}$$

By resolvent bounds for A, we also have

$$\begin{aligned}
\|\Gamma_{u,\eta} f\|_{L^p(\Omega)} &\leq \frac{C_\eta}{\pi} \|f\|_{L^p(\Omega)} \int_0^\infty \frac{t^{1-\alpha}}{t^2 + u^2} \frac{\mathrm{d}t}{t} \\
&\leq \frac{C_\eta}{2} \sec(\alpha\pi/2)|u|^{1-\alpha} \|f\|_{L^p(\Omega)}\,.
\end{aligned}$$

The bounds for $\|G_{u,\varphi}(A)f\|_X$ and $\|\Delta_{u,\eta} f\|_{L^p(\Omega)}$ are verified in a similar fashion. □

5. Proof of the maximal theorem

Assume the setting and hypotheses of Theorem 2.1.

Suppose that $f \in D(A^\alpha)$ and define $v : \Omega \times S_{\theta'} \to X$ by

$$v(\omega, z) = T_z f(\omega) \qquad \forall\, (\omega, z) \in \Omega \times S_{\theta'} .$$

Note that $M_{\theta'} f \in L^p(\Omega)$ if and only if $v \in L^p(\Omega; L^\infty(S_{\theta'}; X))$, where

$$\|v\|_{L^p(L^\infty)} = \left(\int_\Omega \operatorname*{ess-sup}_{z \in S_{\theta'}} \|v(\omega, z)\|_X^p \; \mathrm{d}\mu(\omega) \right)^{1/p}$$

and where we have written $L^p(L^\infty)$ for $L^p(\Omega; L^\infty(S_{\theta'}; X))$.

Our aim now is to embed $L^p(L^\infty)$ inside the dual of a suitable Banach space Z and to then show that

$$\|v\|_{L^p(L^\infty)} = \sup \left\{ |\langle g, v \rangle| \;:\; \|g\|_Z \leq 1 \right\} \tag{5.1}$$

is finite.

Each operator T_z of the semigroup acts on the closed subspace E of $L^p(\Omega; X)$. Thus in particular $v(\omega, z)$ can be considered as an element of X^{**} for each ω and z. Writing Y for X^*, we note that the standard duality theory for Lebesgue-Bochner spaces (see [DU, Chapter IV]) says that $L^\infty(S_{\theta'}; Y^*) \subseteq L^1(S_{\theta'}; Y)^*$ isometrically, and so

$$L^p(\Omega; L^\infty(S_{\theta'}; Y^*)) \subseteq L^p(\Omega; L^1(S_{\theta'}; Y)^*).$$

But if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$L^p(\Omega; L^1(S_{\theta'}; Y)^*) \subseteq L^q(\Omega; L^1(S_{\theta'}; Y))^*$$

isometrically, and hence

$$L^p(\Omega; L^\infty(S_{\theta'}; X)) \subseteq L^p(\Omega; L^\infty(S_{\theta'}; X^{**})) \subseteq L^q(\Omega; L^1(S_{\theta'}; Y))^*. \tag{5.2}$$

As above we shall write $L^q(L^1)$ for $L^q(\Omega; L^1(S_{\theta'}; Y))$. From (5.1) and (5.2) it follows that

$$\begin{aligned}
&\|v\|_{L^p(L^\infty)} \\
&\quad = \sup \left\{ \left| \int_\Omega \langle g(\omega, \cdot), v(\omega, \cdot) \rangle \, d\mu(\omega) \right| \;:\; \|g\|_{L^q(L^1)} \leq 1 \right\}. \\
&\quad = \sup \left\{ \left| \int_\Omega \int_{S_{\theta'}} \langle g(\omega, z), v(\omega, z) \rangle_{\langle Y, X \rangle} \, \mathrm{d}z \, \mathrm{d}\mu(\omega) \right| \;:\; \|g\|_{L^q(L^1)} \leq 1 \right\}.
\end{aligned}$$

Suppose then that $g \in L^q(\Omega; L^1(S_{\theta'}; Y))$ and $\|g\|_{L^q(L^1)} \le 1$. Writing z as $te^{i\varphi}$ and using Lemma 4.1, we find that

$$\begin{aligned}\langle g, v\rangle &= \int_\Omega \int_{S_{\theta'}} \langle g(\omega, z), T_z f(\omega)\rangle_{\langle Y,X\rangle} \,\mathrm{d}z\,\mathrm{d}\mu(\omega)\\ &= \frac{1}{\pi}\int_{|u|<1}\int_\Omega\int_{S_{\theta'}} \langle g(\omega,z), e^{i\varphi}e^{itu}F_{u,\varphi}(A)A^{-\alpha}f(\omega)\rangle_{\langle Y,X\rangle}\,\mathrm{d}z\,\mathrm{d}\mu(\omega)\,\mathrm{d}u\\ &\quad + \frac{1}{\pi}\int_{|u|>1}\int_\Omega\int_{S_{\theta'}} \langle g(\omega,z), e^{i\varphi}e^{itu}G_{u,\varphi}(A)A^{\alpha}f(\omega)\rangle_{\langle Y,X\rangle}\,\mathrm{d}z\,\mathrm{d}\mu(\omega)\,\mathrm{d}u,\end{aligned}$$

where the use of Fubini's theorem is justified by estimates in Section 4. The modulus of the first of these terms may be estimated using Hölder's inequality and Lemma 4.2 so that

$$\begin{aligned}&\left|\int_{|u|<1}\int_\Omega\int_{S_{\theta'}} \langle g(\omega,z), e^{i\varphi}e^{itu}F_{u,\varphi}(A)A^{-\alpha}f(\omega)\rangle_{\langle Y,X\rangle}\,\mathrm{d}z\,\mathrm{d}\mu(\omega)\,\mathrm{d}u\right|\\ &\quad\le \int_{-1}^{1}\int_\Omega\int_{S_{\theta'}} \|g(\omega,z)\|_Y \left\|F_{u,\varphi}(A)A^{-\alpha}f(\omega)\right\|_X \,\mathrm{d}z\,\mathrm{d}\mu(\omega)\,\mathrm{d}u\\ &\quad\le \sec(\theta'+\eta)\int_{-1}^{1}\int_\Omega\int_{S_{\theta'}} \|g(\omega,z)\|_Y\,(\Gamma_{u,\eta}A^{-\alpha}f)(\omega)\,\mathrm{d}z\,\mathrm{d}\mu(\omega)\,\mathrm{d}u\\ &\quad\le \sec(\theta'+\eta)\int_{-1}^{1}\int_\Omega \|g(\omega,\,\cdot\,)\|_{L^1(S_{\theta'};Y)}\,(\Gamma_{u,\eta}A^{-\alpha}f)(\omega)\,\mathrm{d}\mu(\omega)\,\mathrm{d}u\\ &\quad\le \sec(\theta'+\eta)\int_{-1}^{1} \|g\|_{L^q(L^1)}\left\|\Gamma_{u,\eta}A^{-\alpha}f\right\|_{L^p(\Omega;X)}\,\mathrm{d}u\\ &\quad\le \frac{C_\eta}{2}\sec(\theta'+\eta)\sec(\alpha\pi/2)\left\|A^{-\alpha}f\right\|_{L^p(\Omega;X)}\int_{-1}^{1}|u|^{-1+\alpha}\,\mathrm{d}u\\ &\quad\le C_\eta\sec(\theta'+\eta)\sec(\alpha\pi/2)\,\alpha^{-1}\left\|A^{-\alpha}f\right\|_{L^p(\Omega;X)},\end{aligned}$$

where $\theta' < \pi/2 - \eta < \theta$. A similar calculation shows that

$$\begin{aligned}&\left|\int_{|u|>1}\int_\Omega\int_{S_{\theta'}} \langle g(\omega,z), e^{i\varphi}e^{itu}G_{u,\varphi}(A)A^{\alpha}f(\omega)\rangle_{\langle Y,X\rangle}\,\mathrm{d}z\,\mathrm{d}\mu(\omega)\,\mathrm{d}u\right|\\ &\quad\le C_\eta\sec(\theta'+\eta)\sec(\alpha\pi/2)\,\alpha^{-1}\|A^{\alpha}f\|_{L^p(\Omega;X)}.\end{aligned}$$

It follows therefore that

$$|\langle g, v\rangle| \le \frac{C_\eta\sec(\theta'+\eta)\sec(\alpha\pi/2)}{\pi\alpha}\|f\|_{p,\alpha}$$

and hence

$$\|M_{\theta'}f\|_{L^p(\Omega)} \le \frac{C_\eta\sec(\theta'+\eta)\sec(\alpha\pi/2)}{\pi\alpha}\|f\|_{p,\alpha}$$

as required.

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