Technical appendix to Locally stationary wavelet fields with application to the modelling and analysis of image texture

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Summary. This report is an appendix to the paper Locally stationary wavelet fields with application to the modelling and analysis of image texture, providing proofs for all the major results.

Keywords: random field; local spectrum; local autocovariance; texture classification; texture model; non-decimated wavelets

1 Introduction

References to Proposition 1 and Theorem 1 refer to items in the article Locally stationary wavelet fields with application to the modelling and analysis of image texture by Eckley, Nason and Treloar (2009) (henceforth ENT).

2 Proofs

Proposition 1. *Let $C_{\mathbf{R}}$ be the autocovariance of a LS2W process, $X_{\mathbf{r}}$, and C as in Definition 6 of ENT. Then as $R, S \to \infty$*

$$|C_{\mathbf{R}}(\mathbf{z}, \tau) - C(\mathbf{z}, \tau)| = O\left\{\min(R, S)^{-1}\right\}, \tag{1}$$

uniformly in $\tau \in \mathbb{Z}^2$ and $\mathbf{z} \in (0, 1)^2$.

Proof of Proposition 1

Using the LS2W process representation in equation 5 of ENT,

$$\begin{aligned} C_{\mathbf{R}}(\mathbf{z}, \boldsymbol{\tau}) &= \text{Cov}(X_{[\mathbf{zR}]}, X_{[\mathbf{zR}]+\boldsymbol{\tau}}) \\ &= \mathbb{E}\left((X_{[\mathbf{zR}]} - \mu_{[\mathbf{zR}]})(X_{[\mathbf{zR}]+\boldsymbol{\tau}} - \mu_{[\mathbf{zR}]+\boldsymbol{\tau}})\right). \end{aligned}$$

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However, by Assumption 1, $\mathbb{E}(X_{\mathbf{r}}) = 0$ for all $\mathbf{r}$. Hence,

$$\begin{aligned}
C_{\mathbf{R}}(\mathbf{z}, \boldsymbol{\tau}) &= \mathbb{E}\left(X_{[\mathbf{zR}]}X_{[\mathbf{zR}]+\boldsymbol{\tau}}\right) \\
&= \mathbb{E}\left(\sum_l \sum_j \sum_{\mathbf{u}} w^l_{j,\mathbf{u}} \psi^l_{j,\mathbf{u}}([\mathbf{zR}])\xi^l_{j,\mathbf{u}} \sum_{l_0}\sum_{j_0}\sum_{\mathbf{u}_0} w^{l_0}_{j_0,\mathbf{u}_0}\psi^{l_0}_{j_0,\mathbf{u}_0}([\mathbf{zR}]+\boldsymbol{\tau})\xi^{l_0}_{j_0,\mathbf{u}_0}\right) \\
&= \mathbb{E}\left(\sum_l \sum_j \sum_{\mathbf{u}} \sum_{l_0}\sum_{j_0}\sum_{\mathbf{u}_0} w^l_{j,\mathbf{u}} w^{l_0}_{j_0,\mathbf{u}_0} \psi^l_{j,\mathbf{u}}([\mathbf{zR}])\psi^{l_0}_{j_0,\mathbf{u}_0}([\mathbf{zR}]+\boldsymbol{\tau})\xi^l_{j,\mathbf{u}}\xi^{l_0}_{j_0,\mathbf{u}_0}\right) \\
&= \sum_l \sum_j \sum_{\mathbf{u}} \sum_{l_0}\sum_{j_0}\sum_{\mathbf{u}_0} w^l_{j,\mathbf{u}} w^{l_0}_{j,\mathbf{u}_0} \psi^l_{j,\mathbf{u}}([\mathbf{zR}])\psi^{l_0}_{j_0,\mathbf{u}_0}([\mathbf{zR}]+\boldsymbol{\tau})\mathbb{E}(\xi^l_{j,\mathbf{u}}\xi^{l_0}_{j_0,\mathbf{u}_0})
\end{aligned}$$

for the $w^l_{j,\mathbf{u}}$ and the $\psi^l_{j,\mathbf{u}}$ are deterministic. Moreover, since

$$\mathrm{Cov}(\xi^l_{j,\mathbf{u}}, \xi^{l_0}_{j_0,\mathbf{u}_0}) = \mathbb{E}(\xi^l_{j,\mathbf{u}}\xi^{l_0}_{j_0,\mathbf{u}_0}) = \delta_{j,j_0}\delta_{l,l_0}\delta_{\mathbf{u},\mathbf{u}_0},$$

it follows that

$$\begin{aligned}
C_{\mathbf{R}}(\mathbf{z}, \boldsymbol{\tau}) &= \sum_l \sum_j \sum_{\mathbf{u}} |w^l_{j,\mathbf{u}}|^2 \psi^l_{j,\mathbf{u}}([\mathbf{zR}])\psi^l_{j,\mathbf{u}}([\mathbf{zR}]+\boldsymbol{\tau}), \qquad \text{next let } \mathbf{u} = \mathbf{p} + [\mathbf{zR}] \\
&= \sum_l \sum_j \sum_{\mathbf{u}} |w^l_{j,\mathbf{p}+[\mathbf{zR}]}|^2 \psi^l_{j,\mathbf{p}+[\mathbf{zR}]}([\mathbf{zR}])\psi^l_{j,\mathbf{p}+[\mathbf{zR}]}([\mathbf{zR}]+\boldsymbol{\tau}) \\
&= \sum_l \sum_j \sum_{\mathbf{u}} |w^l_{j,\mathbf{p}+[\mathbf{zR}]}|^2 \psi_{j,\mathbf{p}}(\mathbf{0})\psi^l_{j,\mathbf{p}}(\boldsymbol{\tau}).
\end{aligned}$$

We now derive two limit results which are required to complete this proof.

Limit result 1 Using the definition of the local wavelet spectrum (ENT: Definition 5), it is easily shown that $S^l_j(\mathbf{z}) = |W^l_j(\mathbf{z})|^2$ for all $\mathbf{z} \in (0,1)^2$. Furthermore, Assumption 2 of ENT states that

$$S^l_j(\mathbf{z}) = \lim_{R,S\to\infty} |w^l_{j,[\mathbf{zR}]}|^2 \quad \text{for } \mathbf{z} \in (0,1)^2.$$

By Assumption 2,

$$\sup_{\mathbf{u}} \left| w^l_{j,\mathbf{u}} - W^l_j\left(\frac{\mathbf{u}}{\mathbf{R}}\right)\right| \leq \frac{C^l_j}{\max\{R,S\}}.$$

The triangle inequality implies that

$$\begin{aligned}
\sup_{\mathbf{u}} \left|w^l_{j,\mathbf{u}}\right| - \left|W^l_j\left(\frac{\mathbf{u}}{\mathbf{R}}\right)\right| &\leq \frac{C^l_j}{\max\{R,S\}} \\
\Rightarrow \left|w^l_{j,\mathbf{u}}\right| &= \left|W^l_j\left(\frac{\mathbf{u}}{\mathbf{R}}\right)\right| + O\left(\frac{C^l_j}{\max\{R,S\}}\right) \\
\Rightarrow \left|w^l_{j,\mathbf{u}}\right|^2 &= \left|W^l_j\left(\frac{\mathbf{u}}{\mathbf{R}}\right)\right|^2 + O\left(\frac{C^l_j}{\max\{R,S\}}\right) \quad \text{as } \textstyle\sum_j |W^l_j(\mathbf{z})|^2 < \infty.
\end{aligned}$$

Hence, setting $\mathbf{z} = \mathbf{u}/\mathbf{R}$, we obtain

$$\left| |w^l_{j,\mathbf{u}}|^2 - S^l_j(\mathbf{z}) \right| = O\left(\frac{C^l_j}{\max\{R, S\}} \right). \tag{2}$$

Limit result 2 Recall that the $W^l_j(\mathbf{z})$ are assumed to be Lipschitz continuous functions (with respect to the L_1-norm). Hence,

$$\begin{aligned}
\|W^l_j(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R}) - W^l_j(\mathbf{z})\| &\leq L^l_j \|(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R}) - \mathbf{z}\| \qquad \text{where } \boldsymbol{\tau}/\mathbf{R} = (\tau_1/R, \tau_2/S) \\
\Rightarrow |W^l_j(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R}) - W^l_j(\mathbf{z})| &\leq L^l_j \|\boldsymbol{\tau}/\mathbf{R}\|_1 \\
\Rightarrow |W^l_j(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R})| - |W^l_j(\mathbf{z})| &\leq L^l_j \|\boldsymbol{\tau}/\mathbf{R}\|_1 \qquad \text{by the triangle inequality} \\
\Rightarrow |W^l_j(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R})| &= |W^l_j(\mathbf{z})| + O\left(L^l_j \|\boldsymbol{\tau}/\mathbf{R}\|_1\right) \\
\Rightarrow |W^l_j(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R})|^2 &= |W^l_j(\mathbf{z})|^2 + O\left(L^l_j \|\boldsymbol{\tau}/\mathbf{R}\|_1\right)
\end{aligned}$$

for $\sum_j \sum_l |W^l_j(\mathbf{z})|^2 < \infty$ and the L^l_j are uniformly bounded in (j, l). Hence

$$\begin{aligned}
\left| |W^l_j(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R})|^2 - |W^l_j(\mathbf{z})|^2 \right| &= O\left(L^l_j \left(\frac{|\tau_1|}{R} + \frac{|\tau_2|}{S} \right) \right) \\
\Rightarrow \left| |W^l_j(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R})|^2 - |W^l_j(\mathbf{z})|^2 \right| &= O\left(L^l_j \left(\frac{|\tau_1| + |\tau_2|}{\min\{R, S\}} \right) \right).
\end{aligned}$$

Thus,

$$\left| S^l_j(\mathbf{z}+\boldsymbol{\tau}/\mathbf{R})^2 - S^l_j(\mathbf{z})^2 \right| = O\left(L^l_j \left(\frac{|\tau_1| + |\tau_2|}{\min\{R, S\}} \right) \right). \tag{3}$$

With the above limit results in place, we are now in a position to consider the asymptotic convergence of $C_{\mathbf{R}}(\mathbf{z}, \boldsymbol{\tau})$ to $C(\mathbf{z}, \boldsymbol{\tau})$:

$$\begin{aligned}
|C_{\mathbf{R}}(\mathbf{z}, \boldsymbol{\tau}) - C(\mathbf{z}, \boldsymbol{\tau})| &= \left| \sum_l \sum_j \sum_{\mathbf{u}} |w^l_{j,\mathbf{u}+\mathbf{z}\mathbf{R}}|^2 \psi^l_{j,\mathbf{u}}(\mathbf{0}) \psi^l_{j,\mathbf{u}}(\boldsymbol{\tau}) - \sum_l \sum_j S^l_j(\mathbf{z}) \Psi^l_j(\boldsymbol{\tau}) \right| \\
&= \left| \sum_l \sum_j \sum_{\mathbf{u}} \left(|w^l_{j,\mathbf{u}+\mathbf{z}\mathbf{R}}|^2 - S^l_j\left(\frac{\mathbf{u}}{\mathbf{R}} + \mathbf{z}\right) \right) \psi^l_{j,\mathbf{u}}(\mathbf{0}) \psi^l_{j,\mathbf{u}}(\boldsymbol{\tau}) \right. \\
&\quad \left. + \sum_l \sum_j \sum_{\mathbf{u}} S^l_j\left(\frac{\mathbf{u}}{\mathbf{R}} + \mathbf{z}\right) \psi^l_{j,\mathbf{u}}(\mathbf{0}) \psi^l_{j,\mathbf{u}}(\boldsymbol{\tau}) - \sum_{l,j} S^l_j(\mathbf{z}) \Psi^l_j(\boldsymbol{\tau}) \right| \\
&\leq \left| \sum_l \sum_j \sum_{\mathbf{u}} \left(|w^l_{j,\mathbf{u}+\mathbf{z}\mathbf{R}}|^2 - S^l_j\left(\frac{\mathbf{u}}{\mathbf{R}} + \mathbf{z}\right) \right) \psi^l_{j,\mathbf{u}}(\mathbf{0}) \psi^l_{j,\mathbf{u}}(\boldsymbol{\tau}) \right| \\
&\quad + \left| \sum_l \sum_j \sum_{\mathbf{u}} S^l_j\left(\frac{\mathbf{u}}{\mathbf{R}} + \mathbf{z}\right) \psi^l_{j,\mathbf{u}}(\mathbf{0}) \psi^l_{j,\mathbf{u}}(\boldsymbol{\tau}) - \sum_{l,j} S^l_j(\mathbf{z}) \Psi^l_j(\boldsymbol{\tau}) \right|.
\end{aligned}$$

However $\Psi_j^l(\boldsymbol{\tau}) = \sum_{\mathbf{u}} \psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau})$. Hence, using Limit Result 1

$$
\begin{aligned}
|C_{\mathbf{R}}(\mathbf{z},\boldsymbol{\tau}) - C(\mathbf{z},\boldsymbol{\tau})| \leq & \sum_l \sum_j \sum_{\mathbf{u}} \frac{C_j^l}{\max\{R,S\}} \left|\psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau})\right| \\
& + \left| \sum_l \sum_j \sum_{\mathbf{u}} S_j^l\left(\frac{\mathbf{u}}{\mathbf{R}} + \mathbf{z}\right) \psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau}) \right. \\
& \left. - \sum_l \sum_j S_j^l(\mathbf{z}) \sum_{\mathbf{u}} \psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau}) \right| \\
\leq & \sum_l \sum_j \sum_{\mathbf{u}} \frac{C_j^l}{\max\{R,S\}} \left|\psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau})\right| \\
& + \sum_l \sum_j \sum_{\mathbf{u}} \left| S_j^l\left(\frac{\mathbf{u}}{\mathbf{R}} + \mathbf{z}\right) - S_j^l(\mathbf{z}) \right| \left|\psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau})\right|.
\end{aligned}
$$

Using Limit Result 2, in conjunction with the modelling assumptions made in Definition 2 of ENT that the Lipschitz constants L_j^l and W_j^l are uniformly bounded in j, l and that $\sum_l \sum_j 2^{2j} L_j^l < \infty$ and $\sum_l \sum_j C_j^l < \infty$, we obtain

$$
\begin{aligned}
|C_{\mathbf{R}}(\mathbf{z},\boldsymbol{\tau}) - C(\mathbf{z},\boldsymbol{\tau})| = & \sum_l \sum_j \sum_{\mathbf{u}} \frac{C_j^l}{\max\{R,S\}} \left|\psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau})\right| \\
+ & \sum_j \sum_l \sum_{\mathbf{u}} L_j^l \frac{|u_1| + |u_2|}{\min\{R,S\}} \left|\psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau})\right| \\
\leq & \sum_l \sum_j \sum_{\mathbf{u}} \left[\frac{C_j^l + L_j^l(|u_1| + |u_2|)}{\min\{R,S\}} \right] \left|\psi_{j,\mathbf{u}}^l(\mathbf{0})\psi_{j,\mathbf{u}}^l(\boldsymbol{\tau})\right| \\
+ & O\left(\frac{1}{\min\{R,S\}}\right).
\end{aligned}
$$

□

Theorem 1. *For any compactly supported Daubechies wavelet, the family of discrete 2D autocorrelation wavelets $\{\Psi_\eta\}$ is linearly independent. Hence,*

1. *the operator A is invertible (since all of its eigenvalues are positive) and for each $J \in \mathbb{N}$, the norm $||A_J^{-1}||$ is bounded above.*
2. *the LWS is uniquely defined given the corresponding LS2W process.*

Proof of Theorem 1

The structure of the proof for the theorem is similar to that of the one dimensional case, considered by NvSK, although added care is required when dealing with the zeros of $m_0(\omega)$ and $m_1(\omega)$. This is due to the addition of directionality, $l \in \{h, v, d\}$, as well as scale, j, within the decomposition.

Suppose, by way of contradiction, that there exist two spectral representations of the *same* LS2W process. In other words, assume that there exist $w^{(1)}_{\eta,\mathbf{u}}$ and $w^{(2)}_{\eta,\mathbf{u}}$ such that

$$\left| w^{(i)}_{\eta,\mathbf{u}} - W^{(i)}_{\eta}\left(\frac{\mathbf{u}}{\mathbf{R}}\right) \right| = O\left(\frac{1}{\max\{R,S\}}\right) \quad \text{for } i = 1, 2$$

which also possess the same covariance structure. In other words

$$C(\mathbf{z}, \boldsymbol{\tau}) = \sum_{\eta} S^{(1)}_{\eta}(\mathbf{z})\Psi_{\eta}(\boldsymbol{\tau}) = \sum_{\eta} S^{(2)}_{\eta}(\mathbf{z})\Psi_{\eta}(\boldsymbol{\tau})$$

where C is defined in equation 14 of ENT $\forall \mathbf{z} \in (0,1)^2$, $\forall \boldsymbol{\tau} = \mathbb{Z}^2$ and $S^{i}_{\eta}(\mathbf{z}) = \left| W^{(i)}_{\eta}(\mathbf{z}) \right|^2$ for $i = 1, 2$.

Setting $\Delta_{\eta}(\mathbf{z}) \equiv S^{(1)}_{\eta}(\mathbf{z}) - S^{(2)}_{\eta}(\mathbf{z})$ it therefore follows that to prove this result, we must show that

$$\begin{aligned} 0 &= \sum_{\eta} \Delta_{\eta}(\mathbf{z})\Psi_{\eta}(\boldsymbol{\tau}) \qquad \forall \mathbf{z} \in (0,1)^2,\ \forall \boldsymbol{\tau} \in \mathbb{Z}^2, \\ \Rightarrow \quad \Delta_{\eta}(\mathbf{z}) &= 0 \qquad \forall \eta, \forall \mathbf{z} \in (0,1)^2. \end{aligned}$$

What we actually show is that

$$0 = \sum_{\eta} \tilde{\Delta}_{\eta}(\mathbf{z})\Psi_{\eta}(\boldsymbol{\tau}) \qquad \forall \mathbf{z} \in (0,1)^2, \forall \boldsymbol{\tau} \in \mathbb{Z}^2,$$

implies $\tilde{\Delta}_{\eta}(\mathbf{z}) = 0$, $\forall \eta \geq 1$, $\forall \mathbf{z} \in (0,1)^2$. Here $\tilde{\Delta}_{\eta}(\mathbf{z}) = 2^{-2j(\eta)}\Delta_{\eta}(\mathbf{z})$, where $j(\eta) \equiv \eta - \lfloor \frac{\eta-1}{J} \rfloor J$ for $\eta = 1, \ldots, 3J$. $\lfloor \cdot \rfloor$ denotes the floor function. Thus $j(\eta)$ simply refers to scale.

Before proving the theorem we state the following proposition.

Proposition 2. *Let $\hat{\psi}_j(\omega)$ and $\hat{\phi}_j(\omega)$ be the Fourier transforms of $\{\psi_{j,k}\}$ and $\{\phi_{j,k}\}$ respectively. Then*

1. $\hat{\psi}_j(\omega) = \sum_k \psi_{j,k} e^{-i\omega k} = 2^{j/2} m_1(2^{j-1}\omega) \prod_{\ell=0}^{j-2} m_0(2^{\ell}\omega)$,
2. $\hat{\phi}_j(\omega) = \sum_k \phi_{j,k} e^{-i\omega k} = 2^{j/2} \prod_{\ell=0}^{j-1} m_0(2^{\ell}\omega)$,

where $m_0(\omega)$ and $m_1(\omega)$ are the usual frequency response functions of the low- and high-pass filters of Daubechies compactly supported wavelets.

Proof of Proposition 2

Part (i) was shown in NvSK, part (ii) can be shown similarly: both are simple consequences of the scaling relations between wavelets and father wavelets.

□

To start, recall that the operator $A = (A_{\eta,\nu})_{\eta,\nu\geq 1}$ is defined by $A_{\eta,\nu} = \sum_{\boldsymbol{\tau}} \Psi_\eta(\boldsymbol{\tau})\Psi_\nu(\boldsymbol{\tau})$. However, by Parseval's relation

$$A_{\eta,\nu} = \sum_{\boldsymbol{\tau}} \Psi_\eta(\boldsymbol{\tau})\Psi_\nu(\boldsymbol{\tau}) = \left(\frac{1}{2\pi}\right)^2 \int\int \widehat{\Psi}_\eta(\boldsymbol{\omega})\widehat{\Psi}_\nu(\boldsymbol{\omega})\, d\boldsymbol{\omega}, \tag{4}$$

where $\widehat{\Psi}_\eta(\boldsymbol{\omega})$ takes one of the following forms:

$$\begin{aligned}
|\widehat{\Psi}^v_j(\boldsymbol{\omega})|^2 &= 2^{2j}|m_1(2^{j-1}\omega_1)|^2|m_0(2^{j-1}\omega_2)|^2 \textstyle\prod_{p=0}^{j-2} |m_0(2^p w_1)m_0(2^p w_2)|^2 \\
|\widehat{\Psi}^h_j(\boldsymbol{\omega})|^2 &= 2^{2j}|m_0(2^{j-1}\omega_1)|^2|m_1(2^{j-1}\omega_2)|^2 \textstyle\prod_{p=0}^{j-2} |m_0(2^p w_1)m_0(2^p w_2)|^2 \\
|\widehat{\Psi}^d_j(\boldsymbol{\omega})|^2 &= 2^{2j}|m_1(2^{j-1}\omega_1)|^2|m_1(2^{j-1}\omega_2)|^2 \textstyle\prod_{p=0}^{j-2} |m_0(2^p w_1)m_0(2^p w_2)|^2
\end{aligned}$$

The above follows as a consequence of the Fourier properties of discrete father wavelet filters and discrete wavelets (see for example Lemma 3.1 of Eckley (2001), the separability of the 2D wavelets and the result that $\widehat{\Psi}^l_j(\omega) = \left|\widehat{\psi}^l_j(\omega)\right|^2$. Thus, $0 = \sum_\eta \tilde{\Delta}(\mathbf{z})\Psi_\eta(\boldsymbol{\tau})$ implies that

$$\Rightarrow 0 = \sum_\eta \tilde{\Delta}_\eta(\mathbf{z})\Psi_\eta(\boldsymbol{\tau}) \sum_\nu \tilde{\Delta}_\nu \Psi_\nu(\boldsymbol{\tau}), \quad \forall \mathbf{z} \in (0,1)^2, \forall \boldsymbol{\tau} \in \mathbb{Z}^2.$$

Hence $0 = \sum_\eta \sum_\nu \tilde{\Delta}_\eta(\mathbf{z})\tilde{\Delta}_\nu(\mathbf{z}) \sum_{\boldsymbol{\tau}} \Psi_\eta(\boldsymbol{\tau})\Psi_\nu(\boldsymbol{\tau})$.

Applying Parseval's relation, (4), we obtain

$$\begin{aligned}
0 &= \sum_\eta \sum_\nu \tilde{\Delta}_\eta(\mathbf{z})\tilde{\Delta}_\nu(\mathbf{z}) \left(\frac{1}{2\pi}\right)^2 \int\int \widehat{\Psi}_\eta(\boldsymbol{\omega})\widehat{\Psi}_\nu(\boldsymbol{\omega})\, d\boldsymbol{\omega} \\
&= \int\int d\boldsymbol{\omega} \left(\sum_\eta \tilde{\Delta}_\eta(\mathbf{z})\widehat{\Psi}_\eta(\boldsymbol{\omega})\right)^2.
\end{aligned} \tag{5}$$

By Definition 4 of ENT, $S_\eta(\mathbf{z})$ is positive, hence $|S_\eta(z)| = S_\eta(z)$. Furthermore, it is easily shown that $\sum_\eta S_\eta(\mathbf{z}) < \infty$ (see Eckley (Property 3.1, 2001) for details), uniformly in $\mathbf{z}$. Thus, $\sum_\eta |\Delta_\eta(\mathbf{z})| < \infty$ and hence $\sum_\eta 2^{2j(\eta)}|\tilde{\Delta}_\eta(\mathbf{z})| < \infty$. We can infer that $\sum_\eta \tilde{\Delta}_\eta(\mathbf{z})\widehat{\Psi}_\eta(\boldsymbol{\omega})$ is a continuous function for $\boldsymbol{\omega} \in [-\pi,\pi]^2$ because $2^{-2j(\eta)}\widehat{\Psi}_\eta(\boldsymbol{\omega})$

is continuous in this domain (it is simply a trigonometric polynomial in two variables, uniformly bounded above by 1). Hence, (5) if and only if

$$0 = \sum_{\eta} \tilde{\Delta}_{\eta}(\mathbf{z})\widehat{\Psi}_{\eta}(\boldsymbol{\omega}), \quad \forall \boldsymbol{\omega} \in [-\pi, \pi]^2, \forall \mathbf{z} \in (0,1)^2.$$

All that remains now is to demonstrate the pointwise implication of $\tilde{\Delta}_{\eta}(\mathbf{z}) = 0 \; \forall \eta \geq 1, \forall \mathbf{z} \in (0,1)^2$. To achieve this, we use continuity arguments and the insertion of the zeros of $|m_0(2^l\omega)|^2$ and $|m_1(2^l\omega)|^2$.

We start by fixing $\mathbf{z} \in (0,1)^2$ and set $\tilde{\Delta}_{\eta} = \tilde{\Delta}_{\eta}(\mathbf{z})$ at this fixed point $\mathbf{z}$. Then,

$$\begin{aligned}
0 &= \sum_{\eta} \tilde{\Delta}_{\eta}\widehat{\Psi}_{\eta}(\boldsymbol{\omega}) \\
&= \sum_{\eta=1}^{J} \tilde{\Delta}_{\eta}\widehat{\Psi}_{\eta}(\boldsymbol{\omega}) + \sum_{\eta=J+1}^{2J} \tilde{\Delta}_{\eta}\widehat{\Psi}_{\eta}(\boldsymbol{\omega}) + \sum_{\eta=2J+1}^{3J} \tilde{\Delta}_{\eta}\widehat{\Psi}_{\eta}(\boldsymbol{\omega}) \\
&= \sum_{\eta=1}^{J} \tilde{\Delta}_{\eta} 2^{2j} |m_1(2^{j-1}\omega_1)|^2 |m_0(2^{j-1}\omega_2)|^2 \prod_{l=0}^{j-2} |m_0(2^l w_1) m_0(2^l w_2)|^2 \\
&\quad + \sum_{\eta=J+1}^{2J} \tilde{\Delta}_{\eta} 2^{2j} |m_0(2^{j-1}\omega_1)|^2 |m_1(2^{j-1}\omega_2)|^2 \prod_{l=0}^{j-2} |m_0(2^l w_1) m_0(2^l w_2)|^2 \\
&\quad + \sum_{\eta=2J+1}^{3J} \tilde{\Delta}_{\eta} 2^{2j} |m_1(2^{j-1}\omega_1)|^2 |m_1(2^{j-1}\omega_2)|^2 \prod_{l=0}^{j-2} |m_0(2^l w_1) m_0(2^l w_2)|^2 .
\end{aligned} \tag{6}$$

From Daubechies (1992, Chapter 5) we know that m_0 is a 2π-periodic function which is such that $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$ and,

$$|m_0(\pi)|^2 = 0. \tag{7}$$

Thus, $|m_0(0)|^2 = 1$. Recall also that $|m_1(\omega)|^2 = 1 - |m_0(\omega)|^2$ for Daubechies compactly supported wavelets.

To show that $\tilde{\Delta}_1, \tilde{\Delta}_{J+1}$ and $\tilde{\Delta}_{2J+1}$ are all zero, consider the following: Let $\omega_1 = \pi$ and ω_2 vary. Then by the construction of $\widehat{\Psi}_{\eta}(\omega_1, \omega_2)$ and using (7) it follows that $\widehat{\Psi}_{\eta}(\pi, \omega_2) = 0$ for $\eta = 2, 3, \ldots, J, J+1, \ldots, 2J, 2J+2, \ldots, 3J$. However since $|m_1(\pi)|^2 = 1$, equation 6 simplifies to

$$\begin{aligned}
0 &= \tilde{\Delta}_1 4 |m_1(\pi)|^2 |m_0(\omega_2)|^2 + \tilde{\Delta}_{2J+1} 4 |m_1(\pi)|^2 |m_1(\omega_2)|^2 \\
&= \tilde{\Delta}_1 |m_0(\omega_2)|^2 + \tilde{\Delta}_{2J+1} |m_1(\omega_2)|^2, \quad \forall \omega_2 \in [-\pi, \pi]
\end{aligned} \tag{8}$$

Now suppose, without loss of generality, that $\omega_2 = 0$. Then $|m_1(0)|^2 = 1 - |m_0(0)|^2 = 0$. Hence, $0 = \tilde{\Delta}_1 |m_0(0)|^2 + \tilde{\Delta}_{2J+1} |m_1(0)|^2 = \tilde{\Delta}_1 |m_0(0)|^2$. In other words, $\tilde{\Delta}_1 = 0$.

To show that $\tilde{\Delta}_{2J+1}$ is zero, reconsider (8):

$$\begin{aligned}
0 &= \tilde{\Delta}_1|m_1(\pi)|^2|m_0(\omega_2)|^2 + \tilde{\Delta}_{2J+1}|m_1(\pi)|^2|m_1(\omega_2)|^2 \\
&= \tilde{\Delta}_{2J+1}|m_1(\pi)|^2|m_1(\omega_2)|^2, \qquad \text{as } \tilde{\Delta}_1 \text{ is zero} \\
&= \tilde{\Delta}_{2J+1}|m_1(\omega_2)|^2 \quad \forall\omega_2 \in [-\pi,\pi].
\end{aligned}$$

Setting $\omega_2 = \pi$, we obtain, $0 = \tilde{\Delta}_{2J+1}|m_1(\pi)|^2$,

$$\implies \tilde{\Delta}_{2J+1} = 0. \tag{9}$$

To conclude this part of the proof, it remains to show that $\tilde{\Delta}_{J+1} = 0$. To this end, reconsider (6) setting $\omega_2 = \pi$ and letting ω_1 vary. Then, as $|m_0(\pi)|^2 = 0$, it follows that $\widehat{\Psi}_\eta(\omega_1,\pi) = 0$ for all η except $\eta = J+1$ and $2J+1$. However, we have already shown that $\tilde{\Delta}_{2J+1} = 0$. Thus (6) simplifies to $0 = \tilde{\Delta}_{J+1}|m_0(\omega_1)|^2$ $\forall\omega_1 \in [-\pi,\pi]$. Setting $\omega_1 = 0$ ($\Rightarrow |m_0(w_1)|^2 = 1$), we find that $\tilde{\Delta}_{J+1} = 0$.

We have therefore shown that $\tilde{\Delta}_1, \tilde{\Delta}_{J+1}$ and $\tilde{\Delta}_{2J+1} = 0$. Thus (6) simplifies to

$$\begin{aligned}
0 = |m_0(w_1)m_0(w_2)|^2 \Bigg\{ &\sum_{\eta=2}^{J} \tilde{\Delta}_\eta 2^{2j}|m_1(2^{j-1}\omega_1)|^2|m_0(2^{j-1}\omega_2)|^2 \prod_{l=1}^{j-2}|m_0(2^l w_1)m_0(2^l w_2)|^2 \\
&+ \sum_{\eta=J+2}^{2J} \tilde{\Delta}_\eta 2^{2j}|m_0(2^{j-1}\omega_1)|^2|m_1(2^{j-1}\omega_2)|^2 \prod_{l=1}^{j-2}|m_0(2^l w_1)m_0(2^l w_2)|^2 \\
&+ \sum_{\eta=2J+2}^{3J} \tilde{\Delta}_\eta 2^{2j}|m_1(2^{j-1}\omega_1)|^2|m_1(2^{j-1}\omega_2)|^2 \prod_{l=1}^{j-2}|m_0(2^l w_1)m_0(2^l w_2)|^2 \Bigg\}.
\end{aligned} \tag{10}$$

As $|m_0(\omega)|^2$ and $|m_1(\omega)|^2$ are analytic and $m_0(\omega), m_1(\omega)$, as trigonometric polynomials, have finitely many zeros, it follows that the (continuous) function in the braces must vanish identically. Setting $\omega_1 = \pi/2$ and letting ω_2 vary, we find that $|m_0(2w_1)|^2 = |m_0(\pi)|^2 = 0$ and $|m_1(2\omega_1)|^2 = 1$. Hence (10) reduces to

$$\begin{aligned}
0 &= \tilde{\Delta}_2 2^4|m_1(\pi)|^2|m_0(2\omega_2)|^2 + \tilde{\Delta}_{2J+2}2^4|m_1(\pi)|^2|m_1(2\omega_2)|^2 \\
0 &= \tilde{\Delta}_2|m_0(2\omega_2)|^2 + \tilde{\Delta}_{2J+2}|m_1(2\omega_2)|^2 \quad \forall\omega_2 \in [-\pi,\pi].
\end{aligned} \tag{11}$$

Without loss of generality, let $\omega_2 = 0$. Then as $|m_1(0)|^2 = 0$, the above simplifies to $\tilde{\Delta}_2 = 0$. Thus the expression in (11), where ω_2 can take *any* value, simplifies to $0 = \tilde{\Delta}_{2J+2}|m_1(2\omega_2)|^2$. Setting $\omega_2 = \pi/2$, we obtain $0 = \tilde{\Delta}_{2J+2}|m_1(\pi)|^2 = \tilde{\Delta}_{2J+2}$.

Finally to show that $\tilde{\Delta}_{J+2} = 0$, reconsider (10), this time allowing ω_1 to vary and setting $\omega_2 = \pi/2$.The expression reduces to

$$\begin{aligned}
0 &= \tilde{\Delta}_{J+2}2^4|m_0(2\omega_1)|^2|m_1(\pi)|^2 + \tilde{\Delta}_{2J+2}2^4|m_1(2\omega_1)|^2|m_1(\pi)|^2 \quad \text{but } \tilde{\Delta}_{2J+2} = 0, \\
&= \tilde{\Delta}_{J+2}|m_)(2\omega_1)|^2 \qquad \forall\omega_1 \in [-\pi,\pi].
\end{aligned}$$

Setting $\omega_1 = 0$ it follows that

$$\tilde{\Delta}_{J+2} = 0.$$

Continuing with this scheme for $j(\eta) = 3, 4, 5, \ldots$ leads to the result that

$$\tilde{\Delta}_\eta(\mathbf{z}) = 0 \quad \forall \eta, \forall \mathbf{z} \in (0,1)^2.$$

Hence the LWS are uniquely defined given the corresponding LS2W process. Furthermore, since we have shown that $0 = \sum_\eta \tilde{\Delta}_\eta(\mathbf{z})\Psi_\eta(\boldsymbol{\tau})$ if, and only if $\tilde{\Delta}_j(\mathbf{z}) = 0$, we have that $\{\Psi_\eta(\boldsymbol{\tau})\}_{\eta=1}^{\infty}$ are linearly independent. Moreover, since A is the Inner Product (or Gram) matrix of the Ψ_η, A is clearly symmetric and also positive definite. Consequently the eigenvalues of A are positive.

□

Corollary 1. *The inverse formula of equation 14 of ENT is*

$$S_j^l(\mathbf{z}) = \sum_{\eta_1} A^{-1}_{\eta,\eta_1} \sum_{\tau} C(\mathbf{z},\tau)\Psi_{\eta_1}(\tau). \tag{12}$$

Proof of Corollary 1

This proof is identical to that of the one-dimensional case considered by Nason *et al.* (Proposition 2, 2000). Consider,

$$\sum_{\eta_1} A^{-1}_{\eta,\eta_1} \sum_{\boldsymbol{\tau}} C(\mathbf{z},\boldsymbol{\tau})\Psi_\eta(\boldsymbol{\tau})$$

By definition, $C(\mathbf{z},\boldsymbol{\tau}) = \sum_\nu S_\nu \Psi_\nu(\boldsymbol{\tau})$. Hence

$$\begin{aligned}
\sum_{\eta_1} A^{-1}_{\eta,\eta_1} \sum_{\boldsymbol{\tau}} C(\mathbf{z},\boldsymbol{\tau})\Psi_\eta(\boldsymbol{\tau}) &= \sum_{\eta_1} A^{-1}_{\eta,\eta_1} \sum_{\boldsymbol{\tau}} \left\{ \sum_\nu S_\nu(\mathbf{z})\Psi_\nu(\boldsymbol{\tau}) \right\} \Psi_\eta(\boldsymbol{\tau}) \\
&= \sum_{\eta_1} A^{-1}_{\eta,\eta_1} \sum_\nu S_\nu(\mathbf{z}) \sum_{\boldsymbol{\tau}} \Psi_\nu(\boldsymbol{\tau})\Psi_{\eta_1}(\boldsymbol{\tau}).
\end{aligned}$$

The order of the summations may be changed above for $\sum_\eta S_\eta(\mathbf{z}) < \infty \; \forall \mathbf{z}$ whilst the sum over $\boldsymbol{\tau}$ is finite. By definition $\sum_{\boldsymbol{\tau}} \Psi_\nu(\boldsymbol{\tau})\Psi_{\eta_1}(\boldsymbol{\tau}) = A_{\nu,\eta_1} = A_{\nu,\eta_1}$. Hence,

$$\begin{aligned}
\sum_{\eta_1} A^{-1}_{\eta,\eta_1} \sum_{\boldsymbol{\tau}} C(\mathbf{z},\boldsymbol{\tau})\Psi_\eta(\boldsymbol{\tau}) &= \sum_{\eta_1} A^{-1}_{\eta,\eta_1} \sum_\nu S_\nu(\mathbf{z}) A_{\nu,\eta_1} \\
&= \sum_\nu S_\nu(\mathbf{z}) \sum_{\eta_1} A^{-1}_{\eta,\eta_1} A_{\eta_1,\nu} \\
&= \sum_\nu S_\nu(\mathbf{z})\delta_{\eta,\nu} \\
&= S_\eta(\mathbf{z}).
\end{aligned}$$

□

Theorem 2. *Let* $\mathbf{z} = (z_1, z_2)$, $\mathbf{R} = (R, S)$ *and* $[\mathbf{zR}] = ([\mathbf{z}_1 R], [\mathbf{z}_2 S])$ *where* $R = 2^J, S = 2^K$ *for some* $J, K \in \mathbb{N}$. *Further, assume that the* $\{\xi_{\eta,\mathbf{r}}\}$ *are Gaussian. Then,*

$$\mathbb{E}\left(I_{\eta,[\mathbf{zR}]}\right) = \sum_{\eta_1} A_{\eta\eta_1} S_{\eta_1}(\mathbf{z}) + O\left(\frac{1}{\min\{R,S\}}\right). \tag{13}$$

Proof of Theorem 2

Let p=$[\mathbf{zR}]$. By definition,

$$\begin{aligned}\mathbb{E}(I^l_{j,\mathbf{p}}) &= \mathbb{E}\left[(d^l_{j,\mathbf{p}})^2\right] \\ &= \mathbb{E}\left[\left(\sum_{\mathbf{r}} X_{\mathbf{r}} \psi^l_{j,\mathbf{p}}(\mathbf{r})\right)^2\right].\end{aligned}$$

As $\{X_{\mathbf{r}}\}$ is assumed to be a LS2W process, we obtain

$$\begin{aligned}\mathbb{E}(I^l_{j,\mathbf{p}}) &= \mathbb{E}\left[\left(\sum_{\mathbf{r}}\left\{\sum_{l,j,\mathbf{u}} w^l_{j,\mathbf{u}}\psi^l_{j,\mathbf{u}}(\mathbf{r})\xi^l_{j,\mathbf{u}}\right\}\psi^l_{j,\mathbf{p}}(\mathbf{r})\right)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{\mathbf{r}_1}\sum_{l_1,j_1,\mathbf{u}_1} w^{l_1}_{j_1,\mathbf{u}_1}\psi^l_{j_1,\mathbf{u}_1}(\mathbf{r}_1)\xi^{l_1}_{j_1,\mathbf{u}_1}\psi^l_{j,\mathbf{p}}(\mathbf{r}_1)\right.\right. \\ &\qquad \left.\left.\sum_{\mathbf{r}_2}\sum_{l_2,j_2,\mathbf{u}_2} w^{l_2}_{j_2,\mathbf{u}_2}\psi^l_{j_2,\mathbf{u}_2}(\mathbf{r}_2)\xi^{l_2}_{j_2,\mathbf{u}_2}\psi^l_{j,\mathbf{p}}(\mathbf{r}_2)\right)\right] \\ &= \sum_{\mathbf{r}_1,\mathbf{r}_2}\sum_{l_1,l_2}\sum_{j_1,j_2}\sum_{\mathbf{u}_1,\mathbf{u}_2} w^{l_1}_{j_1,\mathbf{u}_1}w^{l_2}_{j_2,\mathbf{u}_2}\psi^l_{j_1,\mathbf{u}_1}(\mathbf{r}_1)\psi^l_{j_2,\mathbf{u}_2}(\mathbf{r}_2)\psi^l_{j,\mathbf{p}}(\mathbf{r}_1)\psi^l_{j,\mathbf{p}}(\mathbf{r}_2)\mathbb{E}(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}).\end{aligned}$$

By the orthonormality of the increment sequence and Assumption 1, it follows that

$$\begin{aligned}\mathrm{Cov}(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}) &= \mathbb{E}(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}) \\ &= \delta_{j_1,j_2}\delta_{l_1,l_2}\delta_{\mathbf{u}_1,\mathbf{u}_2}.\end{aligned}$$

Hence,

$$\mathbb{E}(I^l_{j,\mathbf{p}}) = \sum_{l_1,j_1,\mathbf{u}}(w^{l_1}_{j_1,\mathbf{u}})^2\sum_{\mathbf{r}_1}\psi^{l_1}_{j_1,\mathbf{u}}(\mathbf{r}_1)\psi^l_{j,\mathbf{p}}(\mathbf{r}_1)\sum_{\mathbf{r}_2}\psi^{l_1}_{j_1,\mathbf{u}}(\mathbf{r}_2)\psi^l_{j,\mathbf{p}}(\mathbf{r}_2). \tag{14}$$

Upon making the substitution $\mathbf{u} = \mathbf{x} + \mathbf{p}$ we obtain:

$$\begin{aligned}\mathbb{E}(I^l_{j,\mathbf{p}}) &= \sum_{l_1,j_1,\mathbf{x}}(w^{l_1}_{j_1,\mathbf{x}+\mathbf{p}})^2\left\{\sum_{\mathbf{r}}\psi^{l_1}_{j_1,\mathbf{x}+\mathbf{p}}(\mathbf{r})\psi^l_{j,\mathbf{p}}(\mathbf{r})\right\}^2 \\ &= \sum_{l_1,j_1,\mathbf{x}}(w^{l_1}_{j_1,\mathbf{x}+\mathbf{p}})^2\left\{\sum_{\mathbf{r}}\psi^{l_1}_{j_1,\mathbf{x}+\mathbf{p}-\mathbf{r}}\psi^l_{j,\mathbf{p}-\mathbf{r}}\right\}^2.\end{aligned} \tag{15}$$

As the sum over $\mathbf{x}$ ranges over $\{\mathbf{x} = (x_1, x_2) : x_1.x_2 \in \mathbb{Z}\}$, it follows that $\mathbf{p}$ in the final summation of equation (15) becomes redundant. Hence,

$$\mathbb{E}(I^l_{j,\mathbf{p}}) = \sum_{l_1}\sum_{j_1}\sum_{\mathbf{x}}(w^{l_1}_{j_1,\mathbf{x}+\mathbf{p}})^2 \left\{\sum_{\mathbf{r}} \psi^{l_1}_{j_1,\mathbf{x}-\mathbf{r}}\psi^{l}_{j,-\mathbf{r}}\right\}^2.$$

It is easily shown that

$$\left||w^l_{j,[\mathbf{z}\mathbf{R}]+\mathbf{x}}|^2 - S^l_j\left(\mathbf{z}+\frac{\mathbf{x}}{\mathbf{R}}\right)\right| \leq \frac{C^l_j}{\max\{R,S\}}.$$

See the proof of Proposition 1 for further details. Hence

$$|w^l_{j,[\mathbf{z}\mathbf{R}]+\mathbf{x}}|^2 - S^l_j\left(\mathbf{z}+\frac{\mathbf{x}}{\mathbf{R}}\right) \leq \frac{C^l_j}{\max\{R,S\}}.$$

In other words,

$$|w^{l_1}_{j_1,\mathbf{x}+\mathbf{p}}|^2 = S^{l_1}_{j_1}\left(\frac{\mathbf{x}+\mathbf{p}}{\mathbf{R}}\right) + O\left(\frac{C^{l_1}_{j_1}}{\max\{R,S\}}\right).$$

Thus,

$$\begin{aligned}\mathbb{E}(I^l_{j,\mathbf{p}}) &= \sum_{l_1}\sum_{j_1}\sum_{\mathbf{x}}\left(S^{l_1}_{j_1}\left(\frac{\mathbf{x}+\mathbf{p}}{\mathbf{R}}\right) + O\left(\frac{C^{l_1}_{j_1}}{\max\{R,S\}}\right)\right)\left\{\sum_{\mathbf{r}} \psi^{l_1}_{j_1,\mathbf{x}-\mathbf{r}}\psi^{l}_{j,-\mathbf{r}}\right\}^2 \\ &= \sum_{l_1}\sum_{j_1}\sum_{\mathbf{x}} S^{l_1}_{j_1}\left(\frac{\mathbf{x}+\mathbf{p}}{\mathbf{R}}\right)\left\{\sum_{\mathbf{r}} \psi^{l_1}_{j_1,\mathbf{x}-\mathbf{r}}\psi^{l}_{j,-\mathbf{r}}\right\}^2 + O\left(\frac{1}{\max\{R,S\}}\right).\end{aligned}$$

Aside: The remainder term can be brought out because

1. the number of terms in the wavelet product $\left\{\sum_{\mathbf{r}} \psi^{l_1}_{j_1,\mathbf{x}-\mathbf{r}}\psi^{l}_{j,-\mathbf{r}}\right\}$ is finite and bounded as a function of $\mathbf{x}$ due to j being fixed and the fact that discrete wavelets have compact support.
2. and as $\sum_l \sum_j C^l_j < \infty$.

Moreover, as we show in the proof of Proposition 1, if we set $\mathbf{z} = (z_1, z_2)$ and $\tau = (\tau_1, \tau_2)$, then

$$\left|S^l_j(z_1+\tau_1/R, z_2+\tau_2/S) - S^l_j(z_1,z_2)\right| = O\left(L^l_j\left(\frac{|\tau_1|}{R}+\frac{|\tau_2|}{S}\right)\right).$$

Thus,

$$S^l_j(z_1+\tau_1/R, z_2+\tau_2/S) = S^l_j(z_1,z_2) + O\left(L^l_j\frac{|\tau_1|+|\tau_2|}{\min\{R,S\}}\right). \qquad (16)$$

Incorporating this Lipschitz property of the $\{S_j^l\}$, (16), we obtain

$$
\begin{aligned}
\mathbb{E}(I_{j,\mathbf{p}}^l) &= \sum_{l_1}\sum_{j_1}\sum_{\mathbf{x}}\left(S_{j_1}^{l_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)+O\left(\frac{L_j^l\|\mathbf{x}\|_1}{\min\{R,S\}}\right)\right)\left\{\sum_{\mathbf{r}}\psi_{j_1,\mathbf{x}-\mathbf{r}}^{l_1}\psi_{j,-\mathbf{r}}^l\right\}^2 \\
&\qquad +O\left(\frac{1}{\max\{R,S\}}\right) \\
&\quad \sum_{l_1}\sum_{j_1}\sum_{\mathbf{x}}\left\{S_{j_1}^{l_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)\left\{\sum_{\mathbf{r}}\psi_{j_1,\mathbf{x}-\mathbf{r}}^{l_1}\psi_{j,-\mathbf{r}}^l\right\}^2\right\}+O\left(\frac{1}{\min\{R,S\}}\right),
\end{aligned}
$$

again due to $\left\{\sum_{\mathbf{r}}\psi_{j_1,\mathbf{x}-\mathbf{r}}^{l_1}\psi_{j,-\mathbf{r}}^l\right\}$ being finite and the summability of the Lipschitz constants L_j^l.

Expanding the squared wavelet product term yields

$$
\mathbb{E}(I_{j,\mathbf{p}}^l)=\sum_{l_1}\sum_{j_1}\sum_{\mathbf{x}}S_{j_1}^{l_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)\left\{\sum_{\mathbf{r}_1}\psi_{j_1,\mathbf{x}-\mathbf{r}_1}^{l_1}\psi_{j,-\mathbf{r}_1}^l\sum_{\mathbf{r}_2}\psi_{j_1,\mathbf{x}-\mathbf{r}_2}^{l_1}\psi_{j,-\mathbf{r}_2}^l\right\}+O\left(\frac{1}{\min\{R,S\}}\right).
$$

Upon making the substitution $\mathbf{s}=\mathbf{r}_2-\mathbf{r}_1$ we obtain

$$
\begin{aligned}
\mathbb{E}(I_{j,\mathbf{p}}^l) &= \sum_{l_1}\sum_{j_1}\sum_{\mathbf{x}}S_{j_1}^{l_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)\left\{\sum_{\mathbf{r}_1}\psi_{j_1,\mathbf{x}-\mathbf{r}_1}^{l_1}\psi_{j,-\mathbf{r}_1}^l\sum_{\mathbf{s}}\psi_{j_1,\mathbf{x}-\mathbf{s}-\mathbf{r}_1}^{l_1}\psi_{j,-\mathbf{s}-\mathbf{r}_1}^l\right\} \\
&\qquad +O\left(\frac{1}{\min\{R,S\}}\right) \\
&= \sum_{l_1}\sum_{j_1}S_{j_1}^{l_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)\sum_{\mathbf{r}_1}\sum_{\mathbf{s}}\psi_{j,-\mathbf{r}_1}^l\psi_{j,-\mathbf{s}-\mathbf{r}_1}^l\sum_{\mathbf{x}}\psi_{j_1,\mathbf{x}-\mathbf{r}_1}^{l_1}\psi_{j_1,\mathbf{x}-\mathbf{r}_1-\mathbf{s}}^{l_1} \\
&\qquad +O\left(\frac{1}{\min\{R,S\}}\right).
\end{aligned}
$$

By recognition, this last summation is simply the discrete a.c. wavelet, $\Psi_{j_1}^{l_1}(\mathbf{s})$. Thus,

$$
\begin{aligned}
\mathbb{E}(I_{j,\mathbf{p}}^l) &= \sum_{l_1}\sum_{j_1}S_{j_1}^{l_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)\sum_{\mathbf{s}}\Psi_{j_1}^{l_1}(\mathbf{s})\sum_{-\mathbf{r}_1}\psi_{j,\mathbf{r}_1}^l\psi_{j,-\mathbf{s}-\mathbf{r}_1}^l+O\left(\frac{1}{\min\{R,S\}}\right) \\
&= \sum_{l_1}\sum_{j_1}S_{j_1}^{l_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)\sum_{\mathbf{s}}\Psi_{j_1}^{l_1}(\mathbf{s})\Psi_j^l(\mathbf{s})+O\left(\frac{1}{\min\{R,S\}}\right). \qquad (17)
\end{aligned}
$$

Setting $\eta=(j,l)$ and $\eta_1=(j_1,l_1)$, and recalling that $\sum_{\mathbf{s}}\Psi_{\eta_1}(\mathbf{s})\Psi_\eta(\mathbf{s})=A_{\eta,\eta_1}$, equation (17) reduces to:

$$
\mathbb{E}(I_{\eta,\mathbf{p}})=\sum_{\eta_1}A_{\eta,\eta_1}S_{\eta_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)+O\left(\frac{1}{\min\{R,S\}}\right),
$$

as required.

□

Theorem 3. *Assume that the $\{\xi_{\eta,\mathbf{r}}\}$ are again Gaussian. Then the covariance between $I^{l_1}_{j_1,\mathbf{p}}$ and $I^{l_2}_{j_2,\mathbf{q}}$ may be expressed as follows:*

$$Cov(I^{l_1}_{j_1,\mathbf{p}} I^{l_2}_{j_2,\mathbf{q}}) = 2\left\{\sum_{l_0}\sum_{j_0}\sum_{\mathbf{u}_0}(w^{l_0}_{j_0,\mathbf{u}_0})^2 \alpha^{l_1,l_0}_{j_1,j_0}(\mathbf{p},\mathbf{u}_0)\alpha^{l_2,l_0}_{j_2,j_0}(\mathbf{q},\mathbf{u}_0)\right\}^2.$$

Thus the correlation between these quantities decreases with increasing distance between location $\mathbf{p}$ at scale-direction (j_1, l_1) and the location $\mathbf{q}$ at (j_2, l_2). In particular, when $j_1 = j_2$, the covariance is zero when $\|\mathbf{p} - \mathbf{q}\|$ exceeds the overlap of the corresponding wavelets support. Moreover

$$\begin{aligned}
\mathrm{Var}(I^l_{j,\mathbf{p}}) &= 2\mathbb{E}(I^l_{j,\mathbf{p}})^2 \\
&= 2\left(\sum_{\eta_1} A_{\eta\eta_1} S_{\eta_1}([\mathbf{p}/\mathbf{R}])\right)^2 + O\left(\frac{2^{j(\eta)}}{\min(R,S)}\right),
\end{aligned} \tag{18}$$

where $j(\eta) \equiv \eta - \lfloor \frac{\eta-1}{J} \rfloor J$ simply denotes the scale element of $\eta(j,l)$.

Proof of Theorem 3

Variance: The variance of a wavelet periodogram,

$$\begin{aligned}
\mathrm{Var}(I^l_{j,\mathbf{p}}) &= \mathrm{Var}\left((d^l_{j,\mathbf{p}})^2\right) \\
&= \mathbb{E}\left((d^l_{j,\mathbf{p}})^4\right) - \mathbb{E}\left((d^l_{j,\mathbf{p}})^2\right)^2.
\end{aligned}$$

We already know the asymptotic form of $\mathbb{E}\left((d^l_{j,\mathbf{p}})^2\right)$. We therefore focus on

$$\begin{aligned}
\mathbb{E}\left((d^l_{j,\mathbf{p}})^4\right) &= \mathbb{E}\left(\left(\sum_{\mathbf{r}} X_{\mathbf{r}}\psi^l_{j,\mathbf{p}}(\mathbf{r})\right)^4\right) \\
&= \mathbb{E}\left(\left(\sum_{\mathbf{r}}\sum_{l_1}\sum_{j_1}\sum_{\mathbf{u}_1} w^{l_1}_{j_1,\mathbf{u}_1}\psi^{l_1}_{j_1,\mathbf{u}_1}(\mathbf{r})\xi^{l_1}_{j_1,\mathbf{u}_1}\psi^l_{j,\mathbf{p}}(\mathbf{r})\right)^4\right) \\
&= \mathbb{E}\left(\prod_{i=1}^4\sum_{\mathbf{r}_i}\sum_{l_i}\sum_{j_i}\sum_{\mathbf{u}_i} w^{l_i}_{j_i,\mathbf{u}_i}\psi^{l_i}_{j_i,\mathbf{u}_i}(\mathbf{r}_i)\xi^{l_i}_{j_i,\mathbf{u}_i}\psi^l_{j,\mathbf{p}}(\mathbf{r}_i)\right) \\
&= \prod_{i=1}^4\sum_{\mathbf{r}_i}\sum_{l_i}\sum_{j_i}\sum_{\mathbf{u}_i}\mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}\xi^{l_3}_{j_3,\mathbf{u}_3}\xi^{l_4}_{j_4,\mathbf{u}_4}\right) w^{l_i}_{j_i,\mathbf{u}_i}\psi^{l_i}_{j_i,\mathbf{u}_i}(\mathbf{r}_i)\psi^l_{j,\mathbf{p}}(\mathbf{r}_i).
\end{aligned}$$

Consider the term $\mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}\xi^{l_3}_{j_3,\mathbf{u}_3}\xi^{l_4}_{j_4,\mathbf{u}_4}\right)$. Using a result due to (Isserlis, 1918),

$$\begin{aligned}\mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}\xi^{l_3}_{j_3,\mathbf{u}_3}\xi^{l_4}_{j_4,\mathbf{u}_4}\right) &= \mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}\right)\mathbb{E}\left(\xi^{l_3}_{j_3,\mathbf{u}_3}\xi^{l_4}_{j_4,\mathbf{u}_4}\right) \\ &\quad +\mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_3}_{j_3,\mathbf{u}_3}\right)\mathbb{E}\left(\xi^{l_2}_{j_2,\mathbf{u}_2}\xi^{l_3}_{j_4,\mathbf{u}_3}\right) \\ &\quad +\mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_4}_{j_4,\mathbf{u}_4}\right)\mathbb{E}\left(\xi^{l_2}_{j_2,\mathbf{u}_2}\xi^{l_3}_{j_3,\mathbf{u}_3}\right) + \kappa_4\end{aligned}$$

where κ_4 is the fourth order cumulant of the distribution of $\left\{\xi^{l_1}_{j_1,\mathbf{u}_1},\xi^{l_2}_{j_2,\mathbf{u}_2},\xi^{l_3}_{j_3,\mathbf{u}_3},\xi^{l_4}_{j_4,\mathbf{u}_4}\right\}$. Moreover when $\{\xi^{l}_{j,\mathbf{u}}\}$ is Gaussian, as in this case, $\kappa_4 \equiv 0$. (See Priestley (Section 5.3, 1981) for further details.)

Using this quadravariate decomposition, the expression of $\mathbb{E}\left((d^l_{j,\mathbf{p}})^4\right)$ simplifies to

$$\begin{aligned}\mathbb{E}\left((d^l_{j,\mathbf{p}})^4\right) &= \prod_{i=1}^{4}\sum_{\mathbf{r}_i}\sum_{l_i}\sum_{j_i}\sum_{\mathbf{u}_i} w^{l_i}_{j_i,\mathbf{u}_i}\psi^{l_i}_{j_i,\mathbf{u}_i}(\mathbf{r}_i)\psi^{l}_{j,\mathbf{p}}(\mathbf{r}_i)\left\{\mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}\right)\mathbb{E}\left(\xi^{l_3}_{j_3,\mathbf{u}_3}\xi^{l_4}_{j_4,\mathbf{u}_4}\right)\right. \\ &\quad \left. + \mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_3}_{j_3,\mathbf{u}_3}\right)\mathbb{E}\left(\xi^{l_2}_{j_2,\mathbf{u}_2}\xi^{l_3}_{j_4,\mathbf{u}_3}\right) + \mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_4}_{j_4,\mathbf{u}_4}\right)\mathbb{E}\left(\xi^{l_2}_{j_2,\mathbf{u}_2}\xi^{l_3}_{j_3,\mathbf{u}_3}\right)\right\} \\ &= I_1 + I_2 + I_3,\end{aligned}$$

where, for example,

$$I_1 = \prod_{i=1}^{4}\sum_{\mathbf{r}_i}\sum_{l_i}\sum_{j_i}\sum_{\mathbf{u}_i}\mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}\right)\mathbb{E}\left(\xi^{l_3}_{j_3,\mathbf{u}_3}\xi^{l_4}_{j_4,\mathbf{u}_4}\right)w^{l_i}_{j_i,\mathbf{u}_i}\psi^{l_i}_{j_i,\mathbf{u}_i}(\mathbf{r}_i)\psi^{l}_{j,\mathbf{p}}(\mathbf{r}_i). \quad (19)$$

By construction

$$\begin{aligned}\mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1}\xi^{l_2}_{j_2,\mathbf{u}_2}\right) &= \mathrm{Cov}(\xi^{l_1}_{j_1,\mathbf{u}_1},\xi^{l_2}_{j_2,\mathbf{u}_2}) \\ &= \delta_{j_1,j_2}\delta_{\mathbf{u}_1,\mathbf{u}_2}\delta_{l_1,l_2}.\end{aligned}$$

Hence (19) simplifies as follows:

$$\begin{aligned}
I_1 &= \prod_{i=1}^{2} \sum_{\mathbf{r}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w^{l_i}_{j_i,\mathbf{u}_i} \psi^{l_i}_{j_i,\mathbf{u}_i}(\mathbf{r}_i) \psi^{l}_{j,\mathbf{p}}(\mathbf{r}_i) \mathbb{E}\left(\xi^{l_1}_{j_1,\mathbf{u}_1} \xi^{l_2}_{j_2,\mathbf{u}_2}\right) \\
&\quad \times \prod_{i=3}^{4} \sum_{\mathbf{r}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w^{l_i}_{j_i,\mathbf{u}_i} \psi^{l_i}_{j_i,\mathbf{u}_i}(\mathbf{r}_i) \psi^{l}_{j,\mathbf{p}}(\mathbf{r}_i) \mathbb{E}\left(\xi^{l_3}_{j_3,\mathbf{u}_3} \xi^{l_4}_{j_4,\mathbf{u}_4}\right) \\
&= \sum_{l_1} \sum_{j_1} \sum_{\mathbf{u}_1} (w^{l_1}_{j_1,\mathbf{u}_1})^2 \sum_{\mathbf{r}_1} \psi^{l_1}_{j_1,\mathbf{u}_1}(\mathbf{r}_1) \psi^{l}_{j,\mathbf{p}}(\mathbf{r}_1) \sum_{\mathbf{r}_2} \psi^{l_1}_{j_1,\mathbf{u}_1}(\mathbf{r}_2) \psi^{l}_{j,\mathbf{p}}(\mathbf{r}_2) \\
&\quad \sum_{l_3} \sum_{j_3} \sum_{\mathbf{u}_3} (w^{l_3}_{j_3,\mathbf{u}_3})^2 \sum_{\mathbf{r}_3} \psi^{l_3}_{j_3,\mathbf{u}_3}(\mathbf{r}_3) \psi^{l}_{j,\mathbf{p}}(\mathbf{r}_3) \sum_{\mathbf{r}_4} \psi^{l_3}_{j_3,\mathbf{u}_3}(\mathbf{r}_4) \psi^{l}_{j,\mathbf{p}}(\mathbf{r}_4) \\
&= \left[\sum_{l_1} \sum_{j_1} \sum_{\mathbf{u}_1} (w^{l_1}_{j_1,\mathbf{u}_1})^2 \sum_{\mathbf{r}_1} \psi^{l_1}_{j_1,\mathbf{u}_1}(\mathbf{r}_1) \psi^{l}_{j,\mathbf{p}}(\mathbf{r}_1) \sum_{\mathbf{r}_2} \psi^{l_1}_{j_1,\mathbf{u}_1}(\mathbf{r}_2) \psi^{l}_{j,\mathbf{p}}(\mathbf{r}_2) \right]^2 \\
&= \mathbb{E}(I^{l}_{j,\mathbf{p}})^2 \quad \text{(by recognition from formula (14))} \\
&= I_2 \quad \text{and} \quad I_3.
\end{aligned}$$

Thus, (changing to $\eta(j,l)$ notation)

$$\begin{aligned}
\mathrm{Var}(I_{\eta,\mathbf{p}}) &= 3\mathbb{E}(I_{\eta,\mathbf{p}})^2 - \mathbb{E}(I_{\eta,\mathbf{p}})^2 \\
&= 2\mathbb{E}(I_{\eta,\mathbf{p}})^2.
\end{aligned}$$

However, from Theorem 2, we know that

$$\mathbb{E}(I_{\eta,\mathbf{p}}) = \sum_{\eta_1} S_{\eta_1}\left(\frac{\mathbf{p}}{\mathbf{R}}\right) A_{\eta,\eta_1} + O\left(\frac{1}{\min\{R,S\}}\right).$$

Hence,

$$\begin{aligned}
\mathrm{Var}(I_{\eta,\mathbf{p}}) &= 2\mathbb{E}(I_{\eta,\mathbf{p}})^2 \\
&= 2\left\{\sum_{\eta_1} A_{\eta,\eta_1} S_{\eta}\left(\frac{\mathbf{p}}{\mathbf{R}}\right) + O\left(\frac{1}{\min\{R,S\}}\right)\right\}^2.
\end{aligned}$$

From the work of Nason *et al./* (2000) it is known that $\Psi^{l}_{j}(\boldsymbol{\tau}) = O(1)$, uniformly in $\boldsymbol{\tau}$. Hence it follows that

$$A_{\eta(j,l),\eta(j_1,l_1)} = \sum_{\boldsymbol{\tau}} \Psi^{l}_{j}(\boldsymbol{\tau}) \Psi^{l_1}_{j_1}(\boldsymbol{\tau}) = O(2^{2j(\eta)}).$$

Thus, as η is fixed

$$\mathrm{Var}(I_{\eta,\mathbf{p}}) = 2\left\{\sum_{\eta_1} A_{\eta,\eta_1} S_{\eta}\left(\frac{\mathbf{p}}{\mathbf{R}}\right)\right\}^2 + O\left(\frac{2^{2j(\eta)}}{\min\{R,S\}}\right).$$

□

Covariance:

$$
\begin{aligned}
\mathrm{Cov}(I_{j_a,\mathbf{p}}^{l_a}, I_{j_b,\mathbf{q}}^{l_b}) &= \mathrm{Cov}\left(\left(d_{j_a,\mathbf{p}}^{l_a}\right)^2, \left(d_{j_b,\mathbf{q}}^{l_b}\right)^2\right) \\
&= \mathbb{E}\left(\left(d_{j_a,\mathbf{p}}^{l_a}\right)^2 \left(d_{j_b,\mathbf{q}}^{l_b}\right)^2\right) - \mathbb{E}\left(\left(d_{j_a,\mathbf{p}}^{l_a}\right)^2\right)\mathbb{E}(\left(d_{j_b,\mathbf{p}}^{l_b}\right)^2\right)
\end{aligned}
$$

We already know the form of $\mathbb{E}\left(\left(d_{j,\mathbf{p}}^{l}\right)^2\right)$. Hence we focus on the term

$$
\begin{aligned}
\mathbb{E}\left(\left(d_{j_a,\mathbf{p}}^{l_a}\right)^2 \left(d_{j_b,\mathbf{q}}^{l_b}\right)^2\right) &= \mathbb{E}\left(\left(\sum_{\mathbf{r}} X_{\mathbf{r}} \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r})\right)^2 \left(\sum_{\mathbf{s}} X_{\mathbf{s}} \psi_{j_b,\mathbf{p}}^{l_s}(\mathbf{s})\right)^2\right) \\
&= \mathbb{E}\left(\left(\sum_{\mathbf{r}} \sum_{l_1} \sum_{j_1} \sum_{\mathbf{u}_1} w_{j_1,\mathbf{u}_1}^{l_1} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{r}) \xi_{j_1,\mathbf{u}_1}^{l_1} \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r})\right)^2\right. \\
&\qquad \left. \times \left(\sum_{\mathbf{s}} \sum_{l_2} \sum_{j_2} \sum_{\mathbf{u}_2} w_{j_2,\mathbf{u}_2}^{l_2} \psi_{j_2,\mathbf{u}_2}^{l_2}(\mathbf{s}) \xi_{j_2,\mathbf{u}_2}^{l_2} \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s})\right)^2\right) \\
&= \prod_{i=1}^{2} \sum_{\mathbf{r}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w_{j_i,\mathbf{u}_i}^{l_i} \psi_{j_i,\mathbf{u}_i}^{l_i}(\mathbf{r}_i) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_i) \\
&\qquad \prod_{i=3}^{4} \sum_{\mathbf{s}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w_{j_i,\mathbf{u}_i}^{l_i} \psi_{j_i,\mathbf{u}_i}^{l_i}(\mathbf{s}_i) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s}_i) \\
&\qquad \mathbb{E}\left(\xi_{j_1,\mathbf{u}_1}^{l_1} \xi_{j_2,\mathbf{u}_2}^{l_2} \xi_{j_3,\mathbf{u}_3}^{l_3} \xi_{j_4,\mathbf{u}_4}^{l_4}\right)
\end{aligned}
$$

Using Isserlis' theorem, together with the fact that the fourth order joint cumulant of Gaussian random variables is zero, we can expand the above expression as follows:

$$
\begin{aligned}
\mathbb{E}\left(\left(d_{j_a,\mathbf{p}}^{l_a}\right)^2 \left(d_{j_b,\mathbf{q}}^{l_b}\right)^2\right) &= \prod_{i=1}^{2} \sum_{\mathbf{r}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w_{j_i,\mathbf{u}_i}^{l_i} \psi_{j_i,\mathbf{u}_i}^{l_i}(\mathbf{r}_i) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_i) \\
&\qquad \prod_{i=3}^{4} \sum_{\mathbf{s}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w_{j_i,\mathbf{u}_i}^{l_i} \psi_{j_i,\mathbf{u}_i}^{l_i}(\mathbf{s}_i) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s}_i) \\
&\qquad \left\{\mathbb{E}\left(\xi_{j_1,\mathbf{u}_1}^{l_1} \xi_{j_2,\mathbf{u}_2}^{l_2}\right) \mathbb{E}\left(\xi_{j_3,\mathbf{u}_3}^{l_3} \xi_{j_4,\mathbf{u}_4}^{l_4}\right) + \mathbb{E}\left(\xi_{j_1,\mathbf{u}_1}^{l_1} \xi_{j_3,\mathbf{u}_3}^{l_3}\right) \mathbb{E}\left(\xi_{j_2,\mathbf{u}_2}^{l_2} \xi_{j_4,\mathbf{u}_3}^{l_3}\right)\right. \\
&\qquad \qquad \left. + \mathbb{E}\left(\xi_{j_1,\mathbf{u}_1}^{l_1} \xi_{j_4,\mathbf{u}_4}^{l_4}\right) \mathbb{E}\left(\xi_{j_2,\mathbf{u}_2}^{l_2} \xi_{j_3,\mathbf{u}_3}^{l_3}\right)\right\} \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Now recall that by construction $\mathbb{E}(\xi_{j_1,\mathbf{u}_1}^{l_1} \xi_{j_2,\mathbf{u}_2}^{l_2}) = \delta_{j_1,j_2} \delta_{\mathbf{u}_1,\mathbf{u}_2} \delta_{l_1,l_2}$. It therefore follows that:

$$\begin{aligned}
I_1 &= \prod_{i=1}^{2} \sum_{\mathbf{r}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w_{j_i,\mathbf{u}_i}^{l_i} \psi_{j_i,\mathbf{u}_i}^{l_i}(\mathbf{r}_i) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_i) \\
&\quad \prod_{i=3}^{4} \sum_{\mathbf{s}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w_{j_i,\mathbf{u}_i}^{l_i} \psi_{j_i,\mathbf{u}_i}^{l_i}(\mathbf{s}_i) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s}_i) \\
&\quad \mathbb{E}\left(\xi_{j_1,\mathbf{u}_1}^{l_1} \xi_{j_2,\mathbf{u}_2}^{l_2}\right) \mathbb{E}\left(\xi_{j_3,\mathbf{u}_3}^{l_3} \xi_{j_4,\mathbf{u}_4}^{l_4}\right) \\
&= \sum_{l_1} \sum_{j_1} \sum_{\mathbf{u}_1} (w_{j_1,\mathbf{u}_1}^{l_1})^2 \sum_{\mathbf{r}_1} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{r}_1) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_1) \sum_{\mathbf{r}_2} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{r}_2) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_2) \\
&\quad \sum_{l_3} \sum_{j_3} \sum_{\mathbf{u}_3} (w_{j_3,\mathbf{u}_3}^{l_3})^2 \sum_{\mathbf{s}_3} \psi_{j_3,\mathbf{u}_3}^{l_3}(\mathbf{s}_3) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s}_3) \sum_{\mathbf{s}_4} \psi_{j_3,\mathbf{u}_3}^{l_3}(\mathbf{s}_4) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s}_4).
\end{aligned}$$

However, recall from equation (14) that

$$\mathbb{E}(I_{j_a,\mathbf{p}}^{l_a}) = \sum_{l_1} \sum_{j_1} \sum_{\mathbf{u}_1} (w_{j_1,\mathbf{u}_1}^{l_1})^2 \sum_{\mathbf{r}_1} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{r}_1) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_1) \sum_{\mathbf{r}_2} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{r}_2) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_2)$$

Hence, $I_1 = \mathbb{E}(I_{j_a,\mathbf{p}}^{l_a}) \mathbb{E}(I_{j_b,\mathbf{q}}^{l_b})$. Furthermore,

$$\begin{aligned}
I_2 &= \prod_{i=1}^{2} \sum_{\mathbf{r}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w_{j_i,\mathbf{u}_i}^{l_i} \psi_{j_i,\mathbf{u}_i}^{l_i}(\mathbf{r}_i) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_i) \\
&\quad \prod_{i=3}^{4} \sum_{\mathbf{s}_i} \sum_{l_i} \sum_{j_i} \sum_{\mathbf{u}_i} w_{j_i,\mathbf{u}_i}^{l_i} \psi_{j_i,\mathbf{u}_i}^{l_i}(\mathbf{s}_i) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s}_i) \\
&\quad \mathbb{E}\left(\xi_{j_1,\mathbf{u}_1}^{l_1} \xi_{j_3,\mathbf{u}_3}^{l_3}\right) \mathbb{E}\left(\xi_{j_2,\mathbf{u}_2}^{l_2} \xi_{j_4,\mathbf{u}_4}^{l_4}\right) \\
&= \sum_{l_1} \sum_{j_1} \sum_{\mathbf{u}_1} (w_{j_1,\mathbf{u}_1}^{l_1})^2 \sum_{\mathbf{r}_1} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{r}_1) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_1) \sum_{\mathbf{s}_3} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{s}_3) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s}_3) \\
&\quad \sum_{l_2} \sum_{j_2} \sum_{\mathbf{u}_2} (w_{j_2,\mathbf{u}_2}^{l_2})^2 \sum_{\mathbf{r}_2} \psi_{j_2,\mathbf{u}_2}^{l_2}(\mathbf{r}_2) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_2) \sum_{\mathbf{s}_4} \psi_{j_2,\mathbf{u}_2}^{l_2}(\mathbf{s}_4) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{s}_4) \\
&= \left[\sum_{l_1} \sum_{j_1} \sum_{\mathbf{u}_1} (w_{j_1,\mathbf{u}_1}^{l_1})^2 \sum_{\mathbf{r}_1} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{r}_1) \psi_{j_a,\mathbf{p}}^{l_a}(\mathbf{r}_1) \sum_{\mathbf{r}_2} \psi_{j_1,\mathbf{u}_1}^{l_1}(\mathbf{r}_2) \psi_{j_b,\mathbf{q}}^{l_b}(\mathbf{r}_2) \right]^2.
\end{aligned}$$

Finally, it is easily shown that $I_3 = I_2$.

Drawing our expressions for I_1, I_2 and I_3 together we find that,

$$\begin{aligned}
\mathrm{Cov}(I^{l_a}_{j_a,\mathbf{p}}, I^{l_b}_{j_b,\mathbf{q}}) &= \mathbb{E}(I^{l_a}_{j_a,\mathbf{p}} I^{l_b}_{j_b,\mathbf{q}}) - \mathbb{E}(I^{l_a}_{j_a,\mathbf{p}})\mathbb{E}(I^{l_b}_{j_b,\mathbf{q}}) \\
&= I_1 + I_2 + I_3 - \mathbb{E}(I^{l_a}_{j_a,\mathbf{p}})\mathbb{E}(I^{l_b}_{j_b,\mathbf{q}}) \\
&= 2\left[\sum_{l_1}\sum_{j_1}\sum_{\mathbf{u}_1}(w^{l_1}_{j_1,\mathbf{u}_1})^2 \sum_{\mathbf{r}_1}\psi^{l_1}_{j_1,\mathbf{u}_1}(\mathbf{r}_1)\psi^{l_a}_{j_a,\mathbf{p}}(\mathbf{r}_1)\sum_{\mathbf{r}_2}\psi^{l_1}_{j_1,\mathbf{u}_1}(\mathbf{r}_2)\psi^{l_b}_{j_b,\mathbf{q}}(\mathbf{r}_2)\right]^2 \\
&= 2\left[\sum_{l_1}\sum_{j_1}\sum_{\mathbf{u}_1}(w^{l_1}_{j_1,\mathbf{u}_1})^2 \alpha^{l_1,l_a}_{j_1,j_a}(\mathbf{u}_1,\mathbf{p})\alpha^{l_1,l_b}_{j_1,j_b}(\mathbf{u}_1,\mathbf{q})\right]^2 .
\end{aligned}$$

□

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