Experimental Studies Of The Non-Adiabatic Escape Problem

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Abstract. Noise-induced transitions between coexisting stable states of a periodically driven nonlinear oscillator have been investigated by means of analog experiments and numerical simulations in the non-adiabatic limit for a wide range of oscillator parameters. It is shown that, for over-damped motion, the field-induced corrections to the activation energy can be described quantitatively in terms of the *logarithmic susceptibility* (LS) and that the measured frequency dispersion of the corresponding corrections for a weakly damped nonlinear oscillator is in qualitative agreement with the theoretical prediction. Resonantly directed diffusion is observed in numerical simulations of a weakly damped oscillator. The possibility of extending the LS approach to encompass escape from the basin of attraction of a quasi-attractor is discussed.

INTRODUCTION

Noise-induced escape from a periodically driven state plays an important role in a wide range of phenomena e.g. the earth's ice-age cycle [1] and excitable neurons [2]. External driving can often exert a marked influence on the escape probability $W \propto \exp(-R/kT)$. The corresponding mechanism is readily understood in the case of adiabatic driving, where the system remains in quasi-equilibrium under the instantaneous value of the driving force, and the activation energy R is given by the height of the potential barrier, as modulated slowly by the driving force.

For higher driving frequencies $\omega_f > t_{\rm rel}^{-1}$ (where $t_{\rm rel}$ is a characteristic relaxation time of the system) a complete theoretical analysis is significantly more complicated because the system cannot be considered to remain in quasi-equilibrium. One might perhaps expect that the change in activation energy R would be proportional to A^2, i.e. that the field would give rise to an effective "heating" of the system (see e.g. [3]). Recently, however, it was suggested [4,5] that the field-induced corrections to the activation energy at high frequencies should – quite counterintuitively – be linear in the field amplitude. The coefficient of proportionality was called the *logarithmic susceptibility* (LS).

The LS-theory [4,5] is based on the concept of optimal paths [6] – the most probable paths along which system fluctuates from a steady state to some specified remote state. In thermal equilibrium, optimal paths are time-reversed relaxational paths of the dissipative system [6], and thus are known or can easily be found experimentally. The main idea underlying the LS-theory is that, although in general the fluctuational and relaxational paths lose their time reversal symmetry under periodic perturbation, the field-induced corrections of the equilibrium optimal path are small. The field-induced corrections to the activation energy are therefore linear in the field and, for periodic driving $F(t) = \sum_k F_k \exp(ik\omega t)$, they can be written

$$\delta R = \min_{t_c} \delta R(t_c),\ \delta R(t_c) = \sum_k F_k \tilde{\chi}(k\omega) e^{ik\omega t_c}, \quad \tilde{\chi}(\omega) = -\int_{-\infty}^{\infty} dt\, \dot{q}^{(0)}(t) e^{i\omega t}. \quad (1)$$

Here, $\tilde{\chi}(\omega)$ is the LS for escape and $\dot{q}^{(0)}(t)$ the velocity along the most probable escape path in the absence of driving ($F(t) = 0$). The LS is a fundamental characteristic of the system dynamics in equilibrium and does not depend on the field.

Some important properties of the LS can readily [7] be verified using (1). In particular, the field-induced corrections to the activation energy should be linear in the amplitude of the harmonic driving. The LS dependence on the phase shift between harmonics of the driving force should display singularities due to the acasual character of the LS [4]. (Note that the integral in (1) is taken from $-\infty$ to $+\infty$ because we consider transitions between two stable states of the system.) The frequency dispersion of the LS should be exponentially sharp [4] for strong damping and may exhibit a resonance-like behaviour in weakly damped nonlinear oscillators: in systems with asymmetric spatially periodic potentials, resonantly directed diffusion [5] may be expected.

To test these predictions, we have built analog electronic models [8] of three different types of nonlinear oscillators. We drive them with zero-mean quasi-white Gaussian noise from a shift-register noise generator, digitize the response $q(t)$, and analyze it with a digital data processor. We have also carried out a complementary digital simulation, using a high-speed pseudo-random generator [9].

THE STRONG DAMPING LIMIT

We consider first the fluctuations of an overdamped Brownian particle driven by a periodic (but not necessarily sinusoidal) force $F(t)$ and white Gaussian noise $\xi(t)$,

$$\dot{q} = K(q,t) + \xi(t), \quad K(q,t) = -U'(q) + F(t), \quad (2)$$

where $\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t')$ and the noise intensity $D = kT$, and we consider the double well Duffing potential $U(q) = -q^2/2 + q^4/4$, as used in diverse scientific contexts (see e.g. [10] and references therein).

We have measured the field-dependent correction to the period-averaged escape rate from the stable state, $\bar{W} = 1/\langle t \rangle$. It is given by $\exp(-\delta R/D)$ for $|\delta R| \gg D$ [4,5,11]. For a sinusoidal driving force, the correction to the activation energy of escape (1) is $\delta R = -2|F_1 \tilde{\chi}(\omega)|$. The major observation is that R is indeed *linear* in the force amplitude (see Fig. 1a). The slope yields the absolute value of the LS. Its frequency dependence, a fundamental characteristic of the original equilibrium system, is also in good agreement (inset of Fig. 1a) with the theoretical predictions.

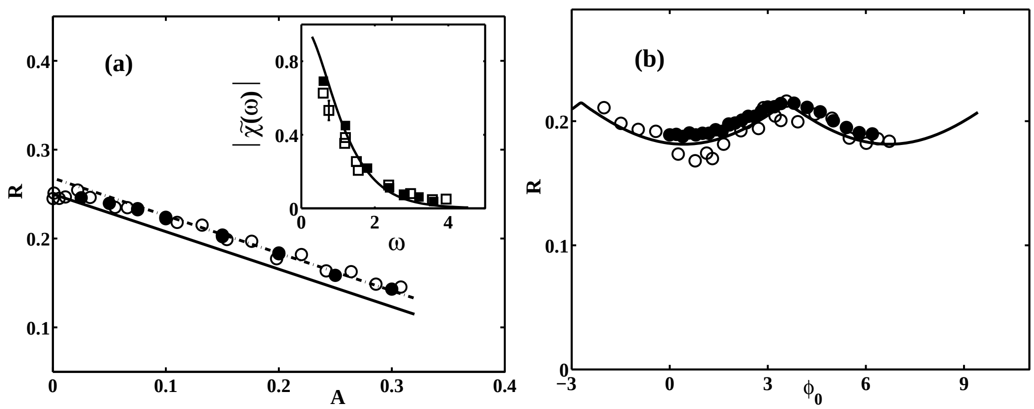

FIGURE 1. (a) The dependence of the activation energy R on the amplitude A of the harmonic driving force $F(t) = A\cos(1.2t)$, as determined by electronic experiment (open circles), numerical simulations (filled circles) and analytic calculation (solid line) based on (1); the dash-dot line is a guide to the eye. Inset: the absolute value of the LS of the system $|\tilde{\chi}(\omega)|$ (1) measured (open and filled squares for experiment and numerical simulation, respectively) and calculated (full curve) as a function of frequency ω. (b) The activation barrier R as a function of phase difference ϕ_0 for a biharmonic driving force. Calculations based on (1) (full curve) are compared with data from electronic (filled circles) and numerical (open circles) experiments.

The results demonstrate that the variation of the activation energy with field can be well described in terms of the LS for a wide range of parameter values. We note however a small deviation from the theory at small amplitudes of the external drive, and the consequent systematic shift of the experimental and numerical points above the theory (solid line). It is attributable to the finite noise intensities used in the experiment, and the fact that the $D \to 0$ limit of the theory breaks down for amplitudes F of the external driving for which $|\chi F|/D$ is small; it can be accounted for by an extension [11] of the theory taking account of changes in the prefactor.

To verify experimentally the acasual character of the LS we seek the *switching* between escape paths predicted [4] for the biharmonic field $F(t) = 0.1\cos(1.2t) + 0.3\cos(2.4t + \phi_0)$. For a field of this kind, $\delta R(t_c)$ (1) may have two local minima; but the activation energy will correspond to the *absolute* minimum. Thus, as ϕ_0 is changed, the escape path will switch from being determined by one local minimum to being determined by the other. The activation energy will consequently display a singularity as a function of ϕ_0, as clearly demonstrated in Fig. 1(b).

WEAK DAMPING

In the limit of small damping we have considered the following model

$$\ddot{q} + 2\Gamma\dot{q} + U'(q) = F(t) + \xi(t), \quad \langle \xi(t)\xi(t')\rangle = 4\Gamma kT\delta(t-t'). \tag{3}$$

It was predicted earlier [5] that for this model the high-frequency field $F(t)$ may resonantly decrease the activation energy of escape from one of the potential wells. Moreover, for systems with spatially periodic asymmetric potentials the effect is different in opposite directions, which gives rise to resonantly directed diffusion.

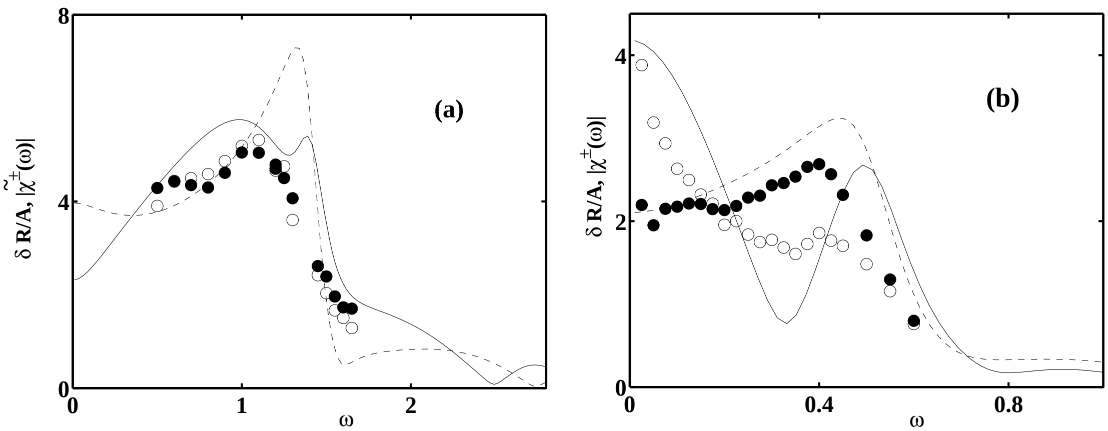

FIGURE 2. (a) The LS $|\tilde{\chi}^{\pm}(\omega)|$ of the escape rates to the right + (solid curve) and to the left − (dashed curve) compared with corrections to the activation energy scaled by the field amplitude $\delta R/A$ obtained in the numerical simulations (open circles, to the right; and filled circles, to the left) for two sets of parameters. (a) The temperature T varied from 0.1 to 0.4 (i.e. $\Gamma \leq kT$), $U(q) = \sin q + 0.3\sin(2q+0.4)$, $F(t) = 0.3\sin(\omega t)$; (b) T varied from 0.01 to 0.04 (i.e. $\Gamma > kT$), $U(q) = 0.1\sin q + 0.049\sin(2q+0.2)$, $F(t) = 0.03\sin(\Omega t)$. The damping was $\Gamma = 0.1$ in both cases.

We have investigated this effect by means of numerical simulations. The two cases shown in Fig. 2 should be carefully distinguished. In Fig. 2(a) results are shown for the set of parameters taken from [5]. To be able to observe fluctuational transitions, the temperature had to be set within the range 0.1–0.4, so that $\Gamma \leq kT$. In this case there is no difference between the measured transition rates to the right and to the left. Resonantly directed diffusion – as indicated by the different frequency-dependences and crossings of the LSs for transitions to the right and left – can occur only for $\Gamma \gg kT$, as in Fig. 2(b)), i.e. if the change of energy per oscillation is larger than kT and the concept of the optimal path is meaningful. Agreement with theory is no more than semi-quantitative, but we note that the approximate theory is only valid within a relatively narrow parameter range that varies with the frequency of the driving field.

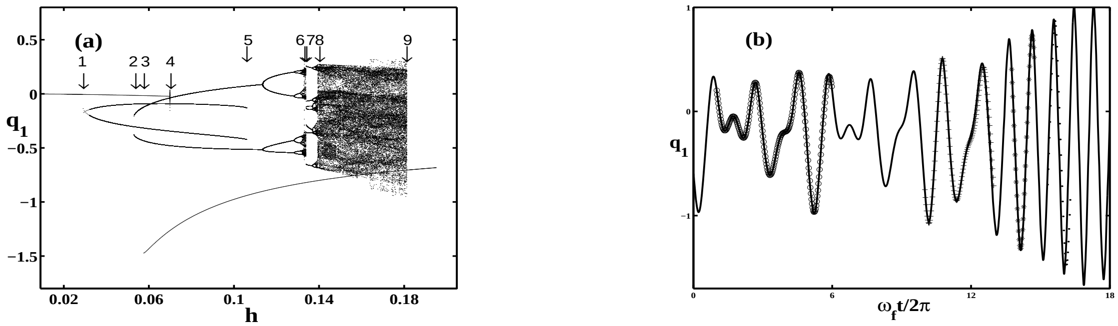

FIGURE 3. (a) A bifurcation diagram in Poincare cross-section obtained for (4) with $\omega_f t = 0(\mathrm{mod}2\pi), \omega_f = 1.005$ shows values of q_1. Arrows 1 and 4 indicate the region of hysteresis for the period 2 resonance. The region of coexistence of the two resonances of period 2 lies between arrows 2 and 5; that of the large stable limit cycle of period 1 lies between arrows 3 and 9. Arrows 6–9 show the boundaries of the chaotic states. (b) The optimal escape path from the quasi-attractor to the stable limit cycle, found in the numerical simulations by use of the prehistory probability distribution, is shown by the solid line. Single periods of the unstable saddle cycles of period 5, 3 and 1 are shown by open circles, pluses and asterisks respectively; the stable limit cycle is shown by dots.

NOISY ESCAPE FROM A QUASI-ATTRACTOR

We have also investigated the stochastic dynamics of a periodically driven nonlinear oscillator with equation of motion in the form

$$\ddot{\mathbf{q}} + 2\Gamma\dot{\mathbf{q}} + \omega_0^2\mathbf{q} + \beta\mathbf{q}^2 + \gamma\mathbf{q}^3 = h\cos(\omega_f t))\} + \xi(t),$$
$$<\xi(t)>= 0, \quad <\xi(t)\xi(0)>= 4kT\Gamma\delta(t), \quad \Gamma \ll \omega_f, \quad \frac{9}{10} < \frac{\beta^2}{\gamma\omega_0^2} < 4. \qquad (4)$$

For these parameters the potential is monostable and the dependence of the energy of oscillations on the frequency is non-monotonic. It was shown earlier [12] that chaos appears in (4) at relatively small driving amplitudes, $h \approx 0.1$.

A bifurcation diagram is shown in the Fig. 3a for one set of parameters: the chaotic state appears as the result of period-doubling bifurcations, and thus corresponds to a quasi-attractor. The working point was chosen in the region of coexistence of the stable limit cycle and quasi-attractor: $h = 0.13, \omega_f = 0.95$.

A statistical analysis of the measured fluctuational trajectories [8] and the corresponding realisations of the random force reveals the following scenario of escape from the basin of attraction of the quasi-attractor (see Fig. 3b). The system comes first to the unstable limit cycle of period 5 embedded in the quasi-attractor, and then slides down the unstable manifold of this cycle. At this moment the optimal force (deterministically related to the optimal path via the equations of motion (4)) switches on and the system is driven by noise to the period 3 unstable limit cycle, which is not part of the quasi-attractor and can be considered as its boundary. Next, the optimal force drives the system from the period 3 limit cycle to the boundary of the basin of attraction of the quasi-attractor. Thus the problem of escape from a quasi-attractor can be considered in terms of fluctuational transitions between limit cycles of low period and their corresponding logarithmic susceptibilities. Given that the LS-theory can be extended to include the case of fluctuational transitions between limit cycles (M.I. Dykman and V.N. Smelyanskiy, private communication) we may hope to apply it to estimate the probability of noise-induced escape from the basin of attraction of a quasi-attractor.

CONCLUSIONS

In conclusion, we have shown experimentally that thermally activated escape under nonequilibrium conditions can be understood in terms of the logarithmic susceptibility, and that the latter is a physically observable quantity. We have verified experimentally the predicted [4] non-analytic behaviour of the activation energy as a function of the field parameters for biharmonic driving and the exponentially sharp frequency dispersion of the LS in the limit of strong damping. Resonantly directed diffusion was observed in numerical simulations.

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