p-GROUPS WITH MAXIMAL ELEMENTARY ABELIAN SUBGROUPS OF RANK 2

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Abstract. Let p be an odd prime number and G a finite p-group. We prove that if the rank of G is greater than p, then G has no maximal elementary abelian subgroup of rank 2. It follows that if G has rank greater than p, then the poset $\mathcal{E}(G)$ of elementary abelian subgroups of G of rank at least 2 is connected and the torsion-free rank of the group of endotrivial kG-modules is one, for any field k of characteristic p. We also verify the class-breadth conjecture for the p-groups G whose poset $\mathcal{E}(G)$ has more than one component.

1. Introduction

In this article, we prove the following result, which answers a question raised by the second author in [21, § 2]:

Theorem A. Let p be a prime and G be a finite p-group that possesses a maximal elementary abelian subgroup E of order p^2. Then G has rank at most p if p is odd.

In other words, if a finite p-group G for an odd prime p has a maximal elementary abelian subgroup of rank 2, then G has no elementary abelian subgroup of rank $p+1$. (Recall that the rank of an elementary abelian group of order p^n is n, and that the rank of G is the maximum of the ranks of the elementary abelian subgroups of G.)

Surprisingly, groups satisfying the rather narrow hypothesis of Theorem A appear in several different areas of finite group theory. For example, they require a great deal of attention as possible "small" Sylow subgroups in the proof of the Feit-Thompson Odd Order Theorem ([9, pp. 453-454]; [8, pp. 845, 903]) and of many subsequent theorems on classifying simple groups ([10, pp. 67-69]). More recently, they have been important in the study of endotrivial modules in representation theory, as we explain below. Furthermore, in the special case that $C_G(E) = E$, they form part of the family of p-groups of maximal class, by a theorem of M. Suzuki ([15, Satz III.14.23]; [2, Proposition 1.8]).

N. Blackburn studied these groups extensively in [3]. In particular, he noted that for p odd, the centralizer of E in G is a *soft* subgroup of G, as defined by Héthelyi in [13] (and in Section 2 below). These groups were studied further in [14] and [21].

The condition on E in Theorem A suggests that G must be "small". Indeed, it is easy to show that G cannot possess a normal elementary abelian subgroup of rank

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$p+1$ (this is a special case of Lemma 2.4(b) below). But what about non-normal subgroups?

By considering subgroups of the wreath product $C_p \wr C_p$, we see that the rank, r, of a group G satisfying the hypothesis of Theorem A can have any of the values $2, 3, \dots, p$. One may show (Remark 3.3) that for $p = 2$, the values may also be 3 or 4, but not larger. In contrast, Theorem A shows that if p is odd, the only possible values are $2, 3, \dots, p$.

From Theorem A, we obtain an application to representation theory of arbitrary finite groups. The concepts used in the following statement are explained in Section 3.

Corollary B. *Let p be an odd prime and G^* a finite group having p-rank greater than p. For any field k of characteristic p, the group $T(G^*)$ of endotrivial kG^*-modules has torsion-free rank one. More precisely, any endotrivial kG^*-module is isomorphic to a direct summand of a module of the form $\Omega^n(k) \otimes M$, for some integer n and some torsion endotrivial module M.*

As an independent result on p-groups with maximal elementary abelian subgroups of rank 2, we end with the proof of the class-breadth conjecture for them ([7]). We define the poset $\mathcal{E}(G)$ in Section 2.

Proposition C. *Let p be an odd prime and G a finite p-group. Assume that the poset $\mathcal{E}(G)$ has more than one component. Then the class-breadth conjecture holds for G.*

The paper is organized as follows: In Section 2, we set the notation and definitions. We also review the necessary background, and show that if a finite p-group G has a non-normal maximal elementary abelian subgroup of order p^2, then G possesses a unique normal elementary abelian subgroup of order p^2, which is hence characteristic in G. In Section 3, we prove Theorem A and Corollary B. We prove Proposition C in Section 4.

2. Generalities: Results old and new

Henceforth in this paper, p denotes a prime number and G a finite p-group, that is, a finite group of order a power of p.

Definition 2.1.

(1) An *elementary abelian subgroup* of G is an abelian subgroup E of G of exponent at most p. If $|E| = p^n$, the *rank* of E is the integer n. Hence, the rank of G is the maximum of the ranks of the elementary abelian subgroups of G.
(2) An elementary abelian subgroup E of G is *maximal* if E is not properly contained in any larger elementary abelian subgroup of G.
(3) The elementary abelian subgroups of G of rank at least 2 form a poset $\mathcal{E}(G)$ for the order relation given by inclusion.

Now assume that G has rank at least 2. The groups in Theorem A are important because the study of simple groups and of endotrivial modules for a finite group is usually much easier when $\mathcal{E}(G)$ is connected for a Sylow p-subgroup G. In fact, one easily deduces from [10, Lemma 10.21] that $\mathcal{E}(G)$ is connected if and only if G has a unique elementary abelian subgroup of rank 2 or G has no maximal elementary abelian subgroups of rank 2.

Assume p is odd. We refer the reader to [10, § 10], and especially [10, Lemmas 10.11 and 10.21], for a detailed description of the structure of the poset $\mathcal{E}(G)$. In particular, G possesses a normal elementary abelian subgroup E_0 of rank 2 and if G has rank at least 3, then all the elementary abelian subgroups of G of rank 3 or more lie in a common connected component of $\mathcal{E}(G)$, which contains also a normal elementary abelian subgroup of G of rank 2; the other possible connected components are hence isolated vertices, i.e. maximal elementary abelian subgroups of rank 2. By Lemma 10.21 and Corollary 10.22 of [10], $\mathcal{E}(G)$ is connected if the normal rank of G is greater than p, or if the center of G is not cyclic.

We now quote some useful results from [3], [13], [14] and [21]. Let $|G| = p^n$. If G has a non-normal maximal elementary abelian subgroup E of rank 2, then E determines a strictly increasing chain

$$E \leq N_0 < N_1 < \cdots < N_{r-1} < N_r = G \quad \text{with} \quad |N_i : N_{i-1}| = |G/N_{r-1}| = p\ .$$

Here, $N_0 = C_G(E)$ is a *soft subgroup* of G (as defined in [13], i.e., $C_G(N_0) = N_0$ and $|N_G(N_0)/N_0| = p$) of the form $C_{p^{n-r}} \times C_p$, and $N_i = N_G(N_{i-1})$, for all $1 \leq i \leq r$.

Moreover, $|G : N_0| = p^r$, and N_i has nilpotence class $i+1$, for all $0 \leq i \leq r$. A striking fact is that the size of N_0 does not depend on the choice of the non-normal maximal elementary subgroup E. Finally, the centralizer $C_G(E_0)$ of E_0 is a maximal subgroup of G, and its intersection $H = C_G(E_0) \cap N_{r-1}$ is also independent of E and thus is a characteristic subgroup of index p^2 in G.

For the remainder of this paper, we refer the reader to one of the books [2], [11], or [15] for the background material and the statements about regular p-groups that we use.

Remark 2.2. For convenience, we single out the mechanics of the Lazard correspondence, as we will repeatedly use them. We refer the reader to [18, Chap. 10], and in particular to the results stated in 10.11, 10.13 and on p. 124.

A celebrated theorem of M. Lazard shows that we may define operations $+$ and $[\ ,\ ]$ on any finite p-group H of nilpotence class less than p, in order to make H into a Lie ring H_L such that every automorphism of the group H induces an automorphism of the Lie ring H_L, and each element of H in H_L has the same order under $+$ as its order in the group H. Moreover, each subgroup of the group H is a Lie subring of H_L.

The case of interest to us is when H is a subgroup of exponent p of a finite p-group G, say $|H| = p^n$, and $x \in G$ has order p and normalizes H. Then conjugation by x induces an automorphism c_x of order p of the additive group of H_L, which is an elementary abelian group of rank n, and thus a vector space of dimension n over the prime field $\mathbb{F}_p$. By considering the Jordan form of this automorphism, we get the rank of $C_H(x)$ as the number of Jordan blocks of c_x, which is greater than or equal to n/p.

The following result is a consequence of P. Hall's Enumeration Principle (see [22, Theorem IV.4.19 (i)] or [12, Theorem 1.4]). Note that Lemma 2.3 generalizes to finite nilpotent groups, because they are the direct product of their Sylow p-subgroups.

Lemma 2.3. *Let P be a finite p-group and Q a normal subgroup of order p^n of P. For each integer k with $0 \leq k < n$, Q contains a subgroup Q_k of order p^k that is normal in P.*

Proof. We proceed by induction on k. If $k = 0$, the claim trivially holds. Assume $k \geq 1$ and pick a subgroup $Q_1 \leq Q \cap Z(P)$ with $|Q_1| = p$. (Recall that any non-trivial normal subgroup of P intersects $Z(P)$ non-trivially.) Then, $Q_1 \lhd P$. Set $\pi : P \to P/Q_1$ for the natural projection map and write $\overline{K} = \pi(K)$ for the image of a subgroup K of P under π. In $\overline{P}$, we have by induction hypothesis that $\overline{Q}$ contains a normal subgroup $\overline{Q_k}$ of $\overline{P}$ of order p^{k-1}. Therefore, $Q_k = \pi^{-1}(\overline{Q_k})$ is a normal subgroup of P contained in Q and $|Q_k| = p^k$. □

Lemma 2.4. *Suppose that E is a non-normal maximal elementary abelian subgroup of G of rank 2, and H is a subgroup of exponent p in G that is normalized by E. Let $|H| = p^n$. Then:*

(a) *for each positive integer k less than n, H contains a subgroup H_k of order p^k that is normalized by E;*
(b) *$n \leq p$; and*
(c) *if E is not contained in H, then the subgroup H_k in part (a) is unique, for each k.*

Proof. Let $E = \langle z, x \rangle$, with $z \in Z(G)$. Note that H is normal in HE.

Each part of the lemma is vacuous or obvious if $|H| \leq p$ or if $H = E$. So we assume that $|H| \geq p^2$ and that $H \neq E$. Then $C_E(H) = \langle z \rangle$ and

$$C_H(x) = C_H(E) = H \cap E\ . \tag{A}$$

Part (a) is Lemma 2.3 applied to $P = HE$ and $Q = H$. Thus, for each $0 \leq k < n$, the group H contains a subgroup H_k of order p^k that is normalized by E (see also related results in [17, Proposition 0.1]).

For part (b), assume that $n \geq p + 1$. We aim for a contradiction. By (a), we may assume that $n = p + 1$.

Since H has exponent p, it is a regular p-group. Therefore, by a theorem of N. Blackburn ([15, Satz III.14.21]; [2, Theorem 9.5]), H is not a p-group of maximal class.

Suppose first that $x \in H$. Since $|H| > p^2$, we see that

$$\langle x \rangle < C_H(x) = C_H(E) = H \cap E\ .$$

Hence

$$|C_H(x)| = |E| = p^2 \quad , \qquad \text{and} \quad |H : C_H(x)| = |H|/p^2\ .$$

By Suzuki's Theorem mentioned in Section 1, H is a p-group of maximal class, a contradiction. Thus, x lies outside H, and

$$|C_H(x)| = |H \cap E| \leq p\ . \tag{B}$$

Since $|H| = p^{p+1}$ and H does not have maximal class, H has class at most $p-1$. We appeal to Remark 2.2. Explicitly, conjugation by x induces an automorphism of order p of the additive group of the Lie ring H_L, which is an elementary abelian group of rank $p+1$, and thus a vector space of dimension $p+1$ over the prime field $\mathbb{F}_p$. By considering the Jordan form of this automorphism, we see that it has at least two Jordan blocks. Therefore, $|C_H(x)| \geq p^2$. But $|C_H(x)| \leq p$ by (B), a contradiction.

For part (c), we assume that E is not contained in H. By (A), we have $|C_H(x)| = |H \cap E| \leq p$. By part (b), $|H| \leq p^p$. Hence, H has nilpotence class at most $p-1$. As in (b), we apply Lazard's theorem and consider the Jordan form of the automorphism of H_L induced by conjugation by x. As $|C_H(x)| \leq p$, this is a single Jordan block of degree n. Therefore, x preserves a unique k-dimensional subspace of H_L over $\mathbb{F}_p$, which proves (c). $\square$

From these technicalities, we draw the following conclusion.

Proposition 2.5. *Assume that p is odd and that G has rank greater than 2. If G has some non-normal maximal elementary abelian subgroup of rank 2, then G has a unique normal elementary abelian subgroup of rank 2, which is hence a characteristic subgroup of G.*

Proof. Suppose that G contains a normal elementary abelian subgroup F of rank 2 other than the chosen subgroup E_0 which we introduced after Definition 2.1. Let $H = E_0F$. Then F contains Z, and E_0/Z and F/Z are contained in the center of G/Z. Therefore, H has order p^3 and nilpotence class at most 2, and possesses more than one elementary abelian subgroup of order p^2. A review of the groups of order p^3 for odd p (or an application of [11, Theorem 12.4.3], since H is a regular p-group) shows that H has exponent p.

As E/Z is not normal in G/Z and since H/Z is central in G/Z, we see that E is not contained in H. By Lemma 2.4, E normalizes only one subgroup of order p^2 in H. But E normalizes E_0 and F, a contradiction. $\square$

Remark 2.6. We refer the reader to [21, Corollary 2.3] for the case when G has only normal elementary abelian subgroups of rank 2.

3. Proof of the main result

For convenience, we appeal to some additional standard notation. For any finite p-group G of nilpotence class c, we write

$$1 = Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots \leq Z_c(G) = G \,,$$

$$\text{with} \quad Z_1(G) = Z(G) \text{ , and } \quad Z_i(G)/Z_{i-1}(G) = Z\big(G/Z_{i-1}(G)\big) \text{ , } \forall\, 1 \leq i \leq c \,,$$

for the subgroups in the upper central series of G. Recall that the least positive integer c with $Z_c(G) = G$ is the nilpotence class of G. For subgroups H, K of G, the subgroup $\langle H^K \rangle$ of G is generated by the K-conjugates of H. In particular, $\langle H^G \rangle$ is the normal closure of H in G. Also, $\Omega_d(H)$ is the subgroup $\langle x \in H \mid x^{p^d} = 1 \rangle$ of G generated by the elements of order at most p^d, for any integer $d \geq 1$.

The following lemma is equivalent to [12, Theorem 2.49, (i)].

Lemma 3.1. *Suppose that N is a normal subgroup of G and k is an integer, $k \geq 0$.*
(a) *If $N \cap Z_k(G) = N \cap Z_{k+1}(G)$, then $N \leq Z_k(G)$.*
(b) *If $|N| = p^k$, then $N \leq Z_k(G)$.*

Proof. For part (a), let $M = N \cap Z_k(G)$ and $\overline{G} = G/M$, and let $\overline{X} = XM/M$ for every subgroup X of G. Then, $\overline{N} \lhd \overline{G}$, and since $M \leq Z_k(G)$, the definition of the upper central series gives

$$Z(\overline{G}) \leq Z_{k+1}(G)/M \quad \text{and so} \quad \overline{N} \cap Z(\overline{G}) \leq (N \cap Z_{k+1}(G))/M = 1 \ .$$

Hence, $\overline{N} = 1$.

Part (b) follows from (a). Indeed, let $N \lhd G$. Assume that $N \not\leq Z_k(G)$. Then

$$1 = N \cap Z_0(G) < N \cap Z_1(G) < \cdots < N \cap Z_{k+1}(G) \ ,$$

is a strictly increasing chain of subgroups of G. Thus, we must have $|N| > p^k$. $\square$

In view of [10, Proposition 10.17] (or [17, Theorem]), if $p = 3$ and G has rank at least 4, then G has normal rank 4. Consequently, Theorem A holds for $p = 3$, as also observed in [21]. So, we may in addition suppose that $p \geq 5$ from now on. Thus, Theorem A follows from our next result.

Theorem 3.2. *Let p be a prime greater than 3, and assume that G has order p^n. If G has a non-normal maximal elementary abelian subgroup of rank 2, then G has rank at most p.*

Proof. We assume that G has rank greater than p and work toward a contradiction.

Let E be a non-normal maximal elementary abelian subgroup of rank 2 in G. By hypothesis, $p \geq 5$ and G contains an elementary abelian subgroup A of rank $p+1$.

By [1, Theorem D], we may choose A to be normal in its normal closure, $\langle A^G \rangle$, in G. Let $N = \langle A^G \rangle$.

Since $A \lhd N$, Lemma 2.3 says that A contains a normal subgroup B of N having order p^{p-1}.

Let $M = \Omega_1(Z_{p-1}(N))$. Then $M \lhd G$ and $B \leq M$ by Lemma 3.1. Since $Z_{p-1}(N)$ has class at most $p-1$, it is a regular p-group. Therefore, M has exponent p because it is a regular p-group generated by elements of order p. Since $M \lhd G$, Lemma 2.4 yields that $|M| \leq p^p$. Hence, $|M : B| \leq p$.

Let $Y = \Omega_1(Z_2(N))$ and $W = \Omega_1(Z(N))$. Then

$$W \leq Y \leq M \quad \text{and} \quad W, Y \lhd G \ .$$

Assume first that $Y \leq A$. Then $C_G(Y) \lhd G$ and $A \leq C_G(Y)$. Therefore, $N = \langle A^G \rangle \leq C_G(Y)$, and $Y \leq Z(N)$. More generally, observe similarly that any normal abelian subgroup of G that is contained in any conjugate of A is necessarily contained in $Z(N)$. Then

$$A \cap Z_2(N) = A \cap Y = A \cap Z(N) \ ,$$

and $A \leq Z(N)$, by Lemma 3.1. But then,

$$A \leq \Omega_1(Z_{p-1}(N)) = M \quad \text{and} \quad p^{p+1} = |A| \leq |M| \leq p^p \ ,$$

a contradiction. Thus, Y is not contained in A. Therefore, $B < BY \leq M$.

Since $|M : B| \leq p$, we have $M = BY$. Moreover, $Y/W \leq Z(N/W)$. Therefore, M/W is centralized by AW/W. As $M \lhd G$, it follows that M/W is centralized by $\langle A^G \rangle W/W$, i.e., by N/W. Therefore, $M \leq Z_2(N)$. But now,

$$A \cap Z_3(N) \leq A \cap Z_{p-1}(N) = A \cap M = A \cap Z_2(N) .$$

So $A \cap Z_3(N) = A \cap Z_2(N)$. By Lemma 3.1, $A \leq Z_2(N)$. Hence, $A \leq M$, and we obtain a contradiction as in the previous paragraph. □

Now we obtain our main result.

Theorem A. Let p be an odd prime and let G be a finite p-group. If G has rank at least $p+1$, then G has no maximal elementary abelian subgroup of order p^2.

Theorem A contrasts sharply with the situation for $p = 2$.

Remark 3.3. Suppose G is a 2-group possessing a maximal elementary abelian subgroup of rank 2. By Lemma 2.4, G has no normal elementary abelian subgroup of rank 3. Therefore, by a theorem of Anne MacWilliams Patterson [20], every subgroup of G is generated by 4 or fewer elements. Hence, G has rank at most 4.

Examples in [10, p. 68] show that G may have rank 3. Here, we give an example of rank 4.

Let $\mathbb{F}$ be the finite field of order 4. For each a, b, c in $\mathbb{F}$, let $M(a, b, c)$ be the 3×3 matrix over $\mathbb{F}$ given by

$$M(a, b, c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Let U be the set of all matrices $M(a, b, c)$. Then U is a group under multiplication and is a Sylow 2-subgroup of $\mathrm{GL}_3(4)$.

For each a in $\mathbb{F}$, let $\bar{a} = a^2$; thus, we obtain the unique non-trivial field automorphism of $\mathbb{F}$. Let t be the mapping on U given by

$$M(a, b, c)^t = M(\bar{b}, \bar{a}, \bar{a}\bar{b} + \bar{c}) .$$

Then t is an automorphism of order two of U (and comes from a unitary automorphism of order two of $\mathrm{GL}_3(4)$). Let G be the semi-direct product of U by $\langle t \rangle$.

Note that $C_U(t)$ is the group of all matrices of the form $M(a, \bar{a}, c)$ such that $c + \bar{c} = a\bar{a}$. This group is a quaternion group of order 8, and $C_G(t) = C_U(t) \times \langle t \rangle$. This shows that G possesses a maximal elementary abelian subgroup of rank 2, namely, $Z(C_U(t)) \times \langle t \rangle$. However, it is easy to see that U, and hence G, possess an elementary abelian subgroup of rank 4. Therefore, G has rank 4.

We end this section with a consequence of Theorem A concerning some important finitely generated representations of an arbitrary finite group G^* over a field k of characteristic p. (Hence, for the remainder of this section, we let G^* denote an arbitrary finite group.) The relationship between the group of endotrivial kG^*-modules $T(G^*)$ and the result stated in Theorem A is that the torsion-free rank of the group $T(G^*)$ equals the number of conjugacy classes of connected components of the poset $\mathcal{E}(G^*)$ ([5, § 3]). By [4] and [21], this number is at most 5 if $p = 2$ and at most $p + 1$ if p is odd. In the particular case that $T(G^*)$ has torsion-free rank 1, the description of $T(G^*)$ is much easier, according to the results and notation of [5] (explained below). Indeed, in this case, any endotrivial kG^*-module is isomorphic to a direct summand of a module of the form

$$\Omega^n(k) \otimes M$$

for some integer n and some torsion endotrivial kG^*-module M. Hence, Theorem A provides a criterion for this to happen which only depends on the p-rank of G^*.

For completeness, we explain the above concepts. We let k denote both a chosen field of characteristic p and the 1-dimensional trivial kG^*-module. The modules $\Omega^n(k)$ are the *syzygies* of k. These are defined inductively as follows: Let $P_* \twoheadrightarrow k$ be a minimal projective resolution of k. Then, $\Omega^0(k) = k$ and for $n > 0$,

$$\Omega^n(k) = \ker\left(P_{n-1} \twoheadrightarrow \Omega^{n-1}(k)\right) .$$

For $n < 0$, we set $\Omega^n(k) = \Omega^{-n}(k)^*$, the k-linear dual of $\Omega^{-n}(k)$. Also, M is a torsion endotrivial module if there is a positive integer m and a projective kG^*-module F such that $M^{\otimes m} \cong k \oplus F$. For additional background material on endotrivial modules, we refer the reader to [6] and [5].

Now, to obtain Corollary B, we also recall that for an arbitrary finite group G^* and prime number p, the *p-rank* of G^* is the rank of a Sylow p-subgroup S_p of G^*. Note that the poset $\mathcal{E}(G^*)$ has at most as many conjugacy classes of components as the poset $\mathcal{E}(S_p)$, and $\mathcal{E}(G^*)$ is non-empty whenever $\mathcal{E}(S_p)$ is non-empty. Therefore, if $\mathcal{E}(S_p)$ is connected, then the components of $\mathcal{E}(G^*)$ form a single conjugacy class. This proves:

Corollary B. Let p be an odd prime and G^* a finite group having p-rank greater than p. For any field k of characteristic p, the group $T(G^*)$ of endotrivial kG^*-modules has torsion-free rank one. More precisely, any endotrivial kG^*-module is isomorphic to a direct summand of a module of the form $\Omega^n(k) \otimes M$, for some integer n and some torsion endotrivial module M.

4. The class-breadth conjecture

We end this note with the class-breadth conjecture for the finite p-groups G whose poset $\mathcal{E}(G)$ has more than one component.

Let G be a finite p-group. For x in G, the *breadth* $b(x)$ of x is given by $p^{b(x)} = |G : C_G(x)|$. In particular, $b(x) = 0$ if and only if x lies in $Z(G)$. The *breadth* $b(G)$ of G is the maximum of $b(x)$ as x ranges over G.

Let $c(G)$ denote the nilpotence class of G. The *class-breadth conjecture* (also known as the Breadth Conjecture) states that the inequality

$$c(G) \leq b(G) + 1$$

always holds. Although counterexamples have been found for $p = 2$, none is known for p odd. For background and recent results about the class-breadth conjecture, we refer the reader to [19] and [7]. In particular, several cases are known to be true, and moreover, the bound is optimal, in the sense that there are groups for which the equality $c(G) = b(G) + 1$ holds. The finite abelian p-groups and those of maximal nilpotence class are such instances, and [19] presents further cases.

Proposition C. Let p be an odd prime and G a finite p-group. Assume that the poset $\mathcal{E}(G)$ has more than one component. Then the class-breadth conjecture holds for G.

Proof. Write $c = c(G)$ for the nilpotence class of G. Let $E = \langle x, z\rangle$ be a maximal elementary abelian subgroup of G, with $z \in Z(G)$. By [3, Theorem], we obtain the equalities $C_G(E) = \langle x\rangle \times Z(N_G(E))$, with $Z(N_G(E))$ cyclic, and

$$|G : C_G(E)| = |G : C_G(x)| = p^{c-1}.$$

Hence $c = b(x) + 1$. Since $b(G) \geq b(x)$, the class-breadth conjecture $c \leq b(G) + 1$ holds for G. □

Remark 4.1. Observe that a similar proof shows that the class-breadth conjecture holds for any finite p-group G having some soft subgroup A such that, for every proper subgroup H of G containing A, the nilpotence class of $N_G(H)$ is one more than the nilpotence class of H.

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