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NONLINEAR CONTROL AND ESTIMATION WITH GENERAL PERFORMANCE CRITERIA

by

Xin Wang

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ABSTRACT NONLINEAR CONTROL AND ESTIMATION WITH GENERAL PERFORMANCE CRITERIA

Xin Wang

Marquette University, 2011

This dissertation is concerned with nonlinear systems control and estimation with general performance criteria. The purpose of this work is to propose general design methods to provide systematic and effective design frameworks for nonlinear system control and estimation problems. First, novel State Dependent Linear Matrix Inequality control approach is proposed, which is optimally robust for model uncertainties and resilient against control feedback gain perturbations in achieving general performance criteria to secure quadratic optimality with inherent asymptotic stability property together with quadratic dissipative type of disturbance reduction. By solving a state dependent linear matrix inequality at each time step, the sufficient condition for the control solution can be found which satisfies the general performance criteria. The results of this dissertation unify existing results on nonlinear quadratic regulator, H_{∞} and positive real control. Secondly, an $H_2 - H_{\infty}$ State Dependent Riccati Equation controller is proposed in this dissertation. By solving the generalized State Dependent Riccati Equation, the optimal control solution not only achieves the optimal quadratic regulation performance, but also has the capability of external disturbance reduction. Numerically efficient algorithms are developed to facilitate effective computation. Thirdly, a robust multicriteria optimal fuzzy control of nonlinear systems is proposed. To improve the optimality and robustness, optimal fuzzy control is proposed for nonlinear systems with general performance criteria. The Takagi-Sugeno fuzzy model is used as an effective tool to control nonlinear systems through fuzzy rule models. General performance criteria have been used to design the controller and the relative weighting matrices of these criteria can be achieved by choosing different coefficient matrices. The optimal control can be achieved by solving the LMI at each time step. Lastly, since any type of controller and observer is subject to actuator failures and sensors failures respectively, novel robust and resilient controllers and estimators are also proposed for nonlinear stochastic systems to address these failure problems. The effectiveness of the proposed control and estimation techniques are demonstrated by simulations of nonlinear systems: the inverted pendulum on a cart and the Lorenz chaotic system, respectively.

Dissertation Supervisor: Prof. Edwin E. Yaz, PhD EECE

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List of Abbreviations

ARE	Algebraic Riccati Equation
CKF	Cubature Kalman Filter
EKF	Extended Kalman Filter
HJE	Hamilton Jacobi Equation
НЛ	Hamilton Jacobi Inequality
LQR	Linear Quadratic Regulator
LQG	Linear Quadratic Gaussian
LMI	Linear Matrix Inequality
NLQR	Nonlinear Quadratic Regulator
NLMI	Nonlinear Matrix Inequality
SDLMI	State Dependent Linear Matrix Inequality
SDRE	State Dependent Riccati Equation
UKF	Unscented Kalman Filter

List of Symbols

\forall	for all
E	belongs to
C	subset of
$\cap (\cup)$	and (or) operation of two sets
Ø	an empty set
\overline{M}	the closure of a set M
Σ()	summation
∏()	product
ā	the complex conjugate of scalar <i>a</i>
a	the absolute value of a scalar <i>a</i>
<i>x</i>	the norm of a vector <i>x</i>
ź	the estimated value of a variable z
A^T (x^T)	the transpose of a matrix A (a vector x)
$A > 0 \ (A \ge 0)$	a positive definite matrix A (a positive semi-definite matrix A)
$z \sim (a, \sigma^2)$	the distribution of a variable <i>z</i> with mean of <i>a</i> and variance of σ^2
$f: S_1 \to S_2$	the function f mapping a set S_1 into a set S_2
$\partial f / \partial x$	the Jacobian matrix
\otimes	the Hardamard product
$E\left\{x\right\} = \overline{x}$	mean/expectation of a random variable
$E_{x}\left\{\cdot ight\}$	expectation taken with respect to random vector x
$E\left\{x/y\right\}$	expectation of x conditional on y

CHAPTER 1 INTRODUCTION

1.1 Background

Nonlinear system theory has been the mainstream of dynamical systems and automatic control science due to its extensive industrial applications in the areas of robotics, aerial vehicles, underwater vehicles, spacecraft, automotives, navigation, telecommunications, signal processing, financial engineering, etc. Broadly speaking, nonlinear system theory can be categorized into nonlinear control and nonlinear estimation. Nonlinear control deals with the following question: given the nonlinear system dynamics, can one achieve the goal of influencing the plant output behavior through the control input in order to yield the desired performance? Nonlinear estimation deals with the following question: given the values of the measurement signal, can one estimate the values of those state vectors, which are not directly measurable, as a function of time? In summary, the goal of nonlinear control and estimation is to synthesize controllers and estimators for nonlinear dynamical systems to achieve desired performance objectives in the presence of disturbances, noises and interferences such as parameter perturbations and unknown dynamics.

1.1.1 Overview of Nonlinear Control

The beginning of modern control theory can be traced back to the 1950s, while classical control developed mostly during World War II. Due to the Cold War, two different approaches to modern control developed: one is the Lyapunov Function approach

developed in the Soviet Union, and the other is the Dynamic Programming and Pontryagin Maximum Principle approach developed in the western world [1].

The Lyapunov Function approach design relies on an energy type of control Lyapunov function for a given nonlinear system which always guarantees control stability. Although the construction of the desired Lyapunov function may not be feasible or it may be difficult to find, the Lyapunov Function approach always guarantees the overall global stability of the system, unlike many other control techniques for nonlinear systems.

The Dynamic Programming was first proposed by Bellman results in Hamilton Jacobi Equations. On the other hand, Pontryagin Maximum principle leads to Euler Lagrange Equations. It is well known that the optimal control solutions of nonlinear systems are conventionally characterized in terms of Hamilton Jacobi Equations (HJEs). The solutions of the HJEs provide the necessary and sufficient optimal control conditions for nonlinear systems. Furthermore, when the controlled system is linear time invariant and the performance index is Linear Quadratic Regulator (LQR), the HJE reduces to Algebraic Riccati Equations (AREs). In this sense, the classical H_2 theory, concerning the Linear Quadratic Gaussian (LQG) control theory, has been set up in the 1950s and 1960s, which assumes perfect models and complete statistical knowledge. However, in the 1970s, it had been found that LQG controller can be highly non-robust with respect to system modeling errors.

Originally introduced by Zames in 1980s, H_{∞} control and estimation theory became one of the most significant accomplishments in automatic control theory during the 1980s and 1990s [3]. As for H_{∞} nonlinear control problems, the optimal solutions are equivalent to solving the corresponding Hamilton Jacobi Inequalities (HJIs). Although Hamilton Jacobi Equations and Hamilton Jacobi Inequalities are powerful tools for nonlinear optimal control theory, neither of them are well suited for industrial applications due to the associated computational complexity. HJEs and HJIs, which are first order partial differential equations and inequalities, cannot be solved for more than a few state variables.

Motivated by the success of optimal control methods for linear system, there has been a great deal of research involved in approximation of the solutions of HJEs and HJIs over the last decade. At the same time, other well-known nonlinear control methods have been developed including feedback linearization, adaptive control, gain scheduling, nonlinear predicative control, fuzzy control and sliding mode control. However, all these approaches are limited in their range of applicability, and the use of one particular control technique for a specific system demands tradeoffs between performance, robustness, stability, optimality and computational complexity.

As powerful alternatives to HJE/HJI techniques, the State Dependent Linear Matrix Inequality (SDLMI) and the State Dependent Riccati Equation (SDRE) techniques provide us very effective algorithms for synthesizing nonlinear feedback controls. Both the State Dependent Linear Matrix Inequality and the State Dependent Riccati Equation utilize state dependent representations of nonlinear systems and some of the earliest work can be found in [30, 31, 41, 42, 46].

The purpose behind the SDLMI approach is to convert a nonlinear system control problem into a convex optimization problem which is solved by Linear Matrix Inequality. Recent developments in convex optimization provide very efficient algorithms for solving LMIs. If a solution can be expressed in a LMI form, then there exist optimization algorithms providing efficient global numerical solutions [5]. Therefore, if the LMI is feasible, then the LMI control technique provides asymptotically stable solutions satisfying various design objectives.

The SDRE control has been emerging as a general design method since the mid-1990s, and provides a systematic and effective design framework for nonlinear systems. Motivated by Linear Quadratic Regulator control using the Algebraic Riccati Equation (ARE), Cloutier *et al.* extended the result to nonlinear quadratic regulator problem by using state dependent coefficient matrices [7, 8]. A discrete-time SDRE method has also been developed in [14]. Due to the computational advantage, stability and effectiveness in control, the SDRE method is of meaningful and practical importance and has already been applied in a wide range of applications, including robotics, guidance and navigation, control of missiles, aircraft, satellite/spacecraft, unmanned aerial vehicles (UAVs), ship systems, autonomous underwater vehicles (AUVs), automotive systems, chemical processes and biomedical systems, etc. A recent survey of the developments in SDRE method can be found in [6].

1.1.2 Overview of Nonlinear Estimation

The problem of least square estimation of stochastic processes was first investigated by Kolmogorov and Wiener, which can be traced back to 1940s. Wiener is regarded as the pioneer for introducing the use of stochastic models and optimization in estimation and control. The assumption of both Kolmogorov and Wiener's work is that extraneous noises, disturbances and interferences are stationary and implicitly Gaussian. Moreover,

both Kolmogorov and Wiener's theory is limited to the input-output description of the system.

In the late 1950s and early 1960s, a breakthrough on linear system estimation theory was made by Rudolph E. Kalman based on the state space approach. Kalman Filter is one of the most widely used methods for linear system state estimation and tracking due to its simplicity, optimality and robustness. Kalman filtering has been proved to be the optimal H_2 solution for linear systems with extraneous noises, disturbances and interferences, either stationary or non-stationary. The theory has been proven and widely used in guidance and navigation of space vehicle to which the theory was first applied. On the other hand, H_{∞} estimation was investigated in 1980s, in order to maximize the estimator's robustness against the external finite energy disturbances with unknown statistical descriptions. That is why a wide range of disturbances can be accommodated.

Traditional Kalman filtering proves to be the optimal estimation solution for linear systems with additive noise; however, it does not work as well for nonlinear systems. Over the past 40 years, the Extended Kalman Filter (EKF), which locally linearizes the nonlinear model so that Kalman Filter theory can be applied, has been the dominant tool for nonlinear state estimation. However, the EKF is also well-known for being difficult to implement, difficult to tune and unstable for severely nonlinear systems.

Another alternative to EKF, the State Dependent Riccati Equation estimator for nonlinear estimation, has been proposed by Cloutier *et al.* SDRE controllers have been widely deployed in recent advanced nonlinear control systems, and have shown to be far more robust than LQR based on standard linearization techniques applied to nonlinear systems. The SDRE estimator has been shown to be a powerful alternative to EKF in some design applications [6, 7].

Recently, researchers have developed the Unscented Kalman Filter (UKF), showing better estimation in most applications than EKF, especially for severely nonlinear models. As pointed out by Simon J. Julier and Jeffery K. Uhlmann [34], UKF is based on the principle that "it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function or transformation".

Other well-known nonlinear estimation methods have also been developed, including Gaussian Filter (GF), Rao-Blackwellised Particle Filter (RBPF), Cubature Kalman Filter (CKF), etc. However, all these approaches are limited in the range of applicability, and the use of one particular estimation technique for a specific system demands tradeoffs between performance, robustness, stability, optimality and computational complexity. These nonlinear estimation techniques have been extensively applied to ranging from GPS navigation to military sensor networks, from autonomous vehicles trajectory planning to wireless communications [77].

1.2 Outline of Dissertation and Summary of Contributions

This dissertation is concerned with problems of nonlinear control and estimation with general performance criteria. As previously mentioned, the goal of nonlinear control and estimation is to synthesize controllers and estimators for nonlinear dynamical system and measurement models to achieve desired general performance criteria in the presence of disturbances, noises and interferences. The purpose of this dissertation research is to propose controller and estimators that are stable, optimal, robust and resilient. The

controller / estimator design algorithms are verified and demonstrated through computer simulations. The logical dependency of each chapter is shown in Fig.1.1, and the contributions of each chapter in this dissertation are summarized below:

The following five chapters deal with optimal and robust nonlinear control.

- Chapter 2 presents the background theories of nonlinear control, which are necessary for the stability analysis and synthesis design of controllers, including Hamilton Jacobi Equation theory, dissipative system analysis and convex optimization theory with Linear Matrix Inequalities. Moreover, the general performance criteria are proposed in Chapter 2 to achieve the mixed performance objectives of Nonlinear Quadratic Regulator, H_∞ and dissipativity, which will be used throughout the dissertation.
- Chapter 3 presents the State Dependent Linear Matrix Inequality control approach. The general performance criteria are applied to design a controller guaranteeing the quadratic sub-optimality with inherent stability property in combination with dissipativity type of disturbance attenuation. By solving the linear matrix inequality at each time, the optimal control solution can be found to satisfy the desired performance objectives. The benchmark under-actuated system, the inverted pendulum on a cart, is used to demonstrate the effectiveness and robustness of the proposed control techniques. Since any type of controller may be subject to actuator failures, the control of nonlinear stochastic systems under actuator failures is also investigated in Section 3.3, which shows significant improvement over the traditional nonlinear control techniques.
- The robust and resilient State Dependent Linear Matrix Inequality (SDLMI)

control of nonlinear systems with general performance criteria is proposed in Chapter 4, which extends the results in Chapter 3. The controller is robust for model uncertainties and resilient for control gain perturbations. The general performance criteria is a generalization of the Nonlinear Quadratic Regulator, H_{∞} , positive realness and sector bounded constraint performance objectives. The results of this chapter unify existing results on nonlinear quadratic regulator, H_{∞} and positive real control. The inverted pendulum on a cart is also used to demonstrate the effectiveness and robustness of the proposed control techniques.

- Chapter 5 presents H₂ H_∞ State Dependent Riccati Equation control. The traditional SDRE method approaches nonlinear quadratic regulator problem. In this chapter, a novel H₂ H_∞ State Dependent Riccati Equation control approach is developed with the purpose of providing a generalized control framework for nonlinear systems. By solving the generalized State Dependent Riccati Equation, the optimal control solution is found to satisfy mixed performance criteria which guarantees quadratic optimality with inherent stability property in combination with H_∞ type of disturbance reduction. The effectiveness of the proposed technique is demonstrated by simulations involving the control of inverted pendulum on a cart.
- Chapter 6 presents robust multi-criteria optimal fuzzy control of nonlinear systems. To improve the optimality and robustness, optimal fuzzy control methods for nonlinear systems with general performance criteria are proposed and evaluated. The Takagi-Sugeno fuzzy model provides an effective tool to control nonlinear systems through the fuzzy rule models. General performance criteria

have been used to design the controller and the relative weighting matrices of these criteria can be achieved by choosing different coefficient matrices. The optimal control can be obtained by solving LMI at each time. The inverted pendulum on a cart problem is used again as an example to demonstrate its effectiveness. The simulation studies show that the proposed method provides a satisfactory alternative to the existing nonlinear control approaches.



Fig.1.1 Dependency of each chapter in this dissertation

The last two chapters deal with optimal and robust nonlinear estimation.

- Chapter 7 briefly goes over the nonlinear estimation techniques, including EKF, UKF and SDRE estimators. Since nonlinear estimators are the dual problems of nonlinear controllers, Chapter 7 provides the background of nonlinear estimation techniques which bridges nonlinear control theory to the novel resilient stochastic nonlinear filtering with sensor failures theory in Chapter 8.
- Chapter 8 presents resilient filtering for nonlinear systems with random failures. Since any type of observer may be subject to the sensor failures, a novel resilient filtering technique for nonlinear systems with random sensor failures is proposed in Chapter 8, which shows significant improvement over the traditional nonlinear estimation techniques.

Chapter 9 concludes the dissertation and summarized the results, followed by a discussion of related future research directions.

1.3 Notation

The following standard notation is used throughout this work:

 \mathfrak{R}_+ stands for the set of non-negative real numbers, C stands for a complex number, \mathfrak{R}^n stands for the n-dimensional Euclidean space. $P(\cdot)$ is the probability of an event. $E(x) = \overline{x}$ is the mean/expectation value of a random variable $x \cdot x \Box(\overline{x}, X)$ denotes a random variable x with arbitrary distribution with mean \overline{x} and covariance $X \cdot \delta_{k-j}$ is the Kronecker delta function; that is, $\delta_{k-j} = 1$ when k = j; and $\delta_{k-j} = 0$ when $k \neq j$. Let A and B be $n \times m$ matrices, the Hadamard product [13] of A and B is denoted by $A \otimes B$, and is defined as $[A \otimes B]_{i,j} = [A]_{ij} [B]_{ij}$ for $1 \le i \le n, 1 \le j \le m$.

For $x \in \Re^n$, the Euclidean norm is $||x|| = (x^T x)^{1/2}$ where $(\cdot)^T$ represents the transpose. $\Re^{n \times m}$ is the set of $n \times m$ real matrices. $A \in \Re^{m \times n}$ denotes an $m \times n$ matrix with real elements. A^{-1} is the inverse of matrix A, A > 0 (A < 0) represents A is a positive (negative) definite matrix, and I_m is an identity matrix of dimension $m \times m$. $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) denotes the minimum (maximum) eigenvalues of the symmetric matrix A.

Continuous-time case: L_2 is the space of finite dimensional vectors with finite energy: $\int_0^\infty ||x(t)||^2 dt < \infty$ Let L_{2e}^n be the extended space of L_2 space defined by:

$$L_{2e}^n = \{f : f \text{ is a measureable function: } \mathfrak{R}_+ \to \mathfrak{R}^n,$$

with property that $F_T f \in L_2$ for all finite $T \in \mathfrak{R}_+\}$

where $F_T f(t) = \begin{cases} f(t), & 0 \le t \le T \\ 0, & T < t \end{cases}$ is called the truncation function on \mathfrak{R}_+ with values in

 \Re^n . The inner product in this space is defined by $\langle u, v \rangle = \int_0^T u^T v dt$, for $u, v \in L_{2e}^n$.

Discrete-time case: The inner product on \Re^n is defined by $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$. l_2 is the space of infinite sequences of finite dimensional vectors with finite energy: $\sum_{k=0}^{\infty} ||x_k||^2 < \infty$.

The following lemmas are used extensively in derivations in this dissertation:

Lemma 1.1:
$$AB^{T} + BA^{T} \le \alpha AA^{T} + \alpha^{-1}BB^{T}$$

Proof
$$(1.1)$$

This can be proven easily by considering

$$(\alpha^{1/2}A - \alpha^{-1/2}B)(\alpha^{1/2}A - \alpha^{-1/2}B)^T \ge 0$$
 (1.2)

Also, by choosing *A*, *B* matrices as $A = \begin{bmatrix} a^T \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ b^T \end{bmatrix}$, we have

$$\begin{bmatrix} 0 & a^T b \\ b^T a & 0 \end{bmatrix} \leq \begin{bmatrix} \zeta a^T a & 0 \\ 0 & \zeta^{-1} b^T b \end{bmatrix}$$
(1.3)

Lemma 1.2: Denote $X = P^{-1}$ for positive definite matrix P > 0, then the following equality always holds

$$X\dot{P}X = -\dot{X} \tag{1.4}$$

This can be proven easily by considering

$$0 = \frac{d}{dt} (I) = \frac{d}{dt} (PP^{-1}) = \frac{d}{dt} (P) \cdot P^{-1} + P \cdot \frac{d}{dt} (P^{-1})$$
(1.5)

Note that since *P* will be used to describe the energy content for the quadratic energy function $V(x) = x^T P x$, which needs to decrease due to the asymptotic stability requirement, $X = P^{-1}$ matrix will be increasing in time. Therefore, we conclude $X\dot{P}X = -\dot{X} < 0$.

Lemma 1.3: Schur's complement: The matrix inequality $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0$ is equivalent to

 $A - BC^{-1}B^T > 0$ for C > 0, which is also equivalent to $C - B^T A^{-1}B > 0$ for A > 0.

Lemma 1.4: Rayleigh's Inequality, which is $\lambda_{\min}(A) \|x\|^2 \le x^T Ax \le \lambda_{\max}(A) \|x\|^2$. Matrix form of Rayleigh's inequality [15] can also be stated as: for matrices $X = X^T \in \Re^{n \times n}$ and $Y \in \Re^{m \times n}$, the matrix inequality $\lambda_{\min}(X) YY^T \le YXY^T \le \lambda_{\max}(X) YY^T$ holds.

CHAPTER 2 BACKGROUND THEORY FOR NONLINEAR CONTROL

The nonlinear system control solution leads to a Hamilton Jacobi Equation and Hamilton Jacobi Inequality. The classical nonlinear optimal control is reviewed, followed by a brief summary of dissipative system analysis and convex optimization theory. More importantly, we propose the general performance criteria which will be used throughout the dissertation.

2.1 Hamilton Jacobi Equations

The Principle of Optimality, also commonly referred to as Dynamic Programming, was proposed by Bellman in 1952. The Hamilton Jacobi Equation (HJE) is derived using the dynamic programming theory for optimal control [31]. Consider the general nonlinear system dynamics described by

$$\dot{x} = f(x, u, t), x(t_0) \text{ and } t_0 \text{ given}$$
 (2.1)

Here, $x \in \Re^n$ represents the state vector, $u \in \Re^m$ represents the control input, the nonlinear function $f:(\Re^n, \Re^m, \Re) \to \Re^n$ is continuous differentiable in all of its arguments. The optimal control we consider is to determine the admissible control input u(t) to minimize the cost function

$$V(x(t_0), u, t_0) = \psi(x(t_f), t_f) + \int_{t_0}^{t_f} \phi(x(\tau), u(\tau), \tau) d\tau$$
(2.2)

where both $\psi(x(t_f), t_f)$, the penalty at the final state, and $\phi(x(\tau), u(\tau), \tau)$ are continuously differentiable functions. *V* is called the performance index. The optimal cost function, $V^*(x(t), t)$, is the minimal performance index, defined by

$$V^{*}(x(t),t) = \min_{u(t)} V(x(t),u,t).$$
(2.3)

By assuming the optimal cost function $V^*(x(t),t)$ exists and is continuously differentiable with respect to both time and state, the optimal cost must satisfy the following partial differential equation:

$$\frac{\partial V^*}{\partial t} = -\min_{u(t)} \left[\phi(x(t), u(t), t) + \frac{\partial V^*}{\partial x} f(x(t), u(t), t) \right]$$
(2.4)

with the boundary condition

$$V^*\left(x\left(t_f\right), t_f\right) = \psi\left(x\left(t_f\right), t_f\right)$$
(2.5)

(2.4) and (2.5) are referred to as the Hamilton Jacobi Equations or Hamilton Jacobi Bellman Equations. The Hamilton Jacobi Equation represents a sufficient and necessary condition for an optimal cost function to exist.

2.2 The Nonlinear Minimax Problem

In this section, we consider the infinite horizon optimal control problem and the inputaffine nonlinear time invariant system, which is a particular form of (2.1), as follows:

$$\dot{x}(t) = f(x(t)) + g_1(x(t))u(t) + g_2(x(t))w(t), x(0) = x_0$$
(2.6)

where $x \in \Re^n$ represents the state vector, $u \in \Re^m$ represents the control input, $w \in \Re^s$ represents the external noises and disturbances. Both g_1, g_2 are assumed to be continuously differentiable in x [31].

The performance objective is to determine the admissible control input to minimize the cost function under the worst possible noises and disturbances effect:

$$\inf_{u(t)} \sup_{w(t) \in L_{2}[0,\infty)} J(u,w)
J(u,w) = \int_{0}^{\infty} (q(x(t)) + u^{T}(t)u(t) - \gamma^{2}w^{T}(t)w(t)) dt$$
(2.7)

with $q(x(t)) \ge 0$ for all x and q(0) = 0. w(t) is assumed to be L_2 type of disturbance. The problem can be viewed as a zero-sum differential game with minimizing the player u and maximizing the player w. The detailed game theory approach to H_{∞} -optimal control can be found in [3].

The Hamilton Jacobi Equation associated with this problem is

$$0 = \min_{u} \max_{w} \left\{ V_{x}(x) \cdot \left[f(x) + g_{1}(x)u + g_{2}(x)w \right] + q(x) + u^{T}(t)u(t) - \gamma^{2}w^{T}(t)w(t) \right\}$$
(2.8)

Based on completion of the square arguments, we have the following equality

$$0 = \min_{u} \max_{w} \left\{ \left\| u + \frac{1}{2} g_{1}^{T} V_{x}^{T} \right\|^{2} - \gamma^{2} \left\| w - \frac{1}{2\gamma^{2}} g_{2}^{T} V_{x}^{T} \right\|^{2} + V_{x} f + \frac{1}{4} V_{x} \left(\frac{1}{\gamma^{2}} g_{2} g_{2}^{T} - g_{1} g_{1}^{T} \right) V_{x}^{T} + q \right\}$$

$$(2.9)$$

Therefore, the minimizing control should be

$$u^* = -\frac{1}{2}g_1^T(x)V_x^T(x)$$
 (2.10)

The worst case noise and disturbance is

$$w^* = \frac{1}{2\gamma^2} g_2^T(x) V_x^T(x)$$
 (2.11)

Substituting (2.10) and (2.11) into (2.8), we obtain the corresponding Hamilton Jacobi Equation:

$$V_{x}(x)f(x) + \frac{1}{4}V_{x}(x)\left(\frac{1}{\gamma^{2}}g_{2}(x)g_{2}^{T}(x) - g_{1}(x)g_{1}^{T}(x)\right)V_{x}^{T}(x) + q(x) = 0 \qquad (2.12)$$

2.2.1 Nonlinear Regulator Problem

One of the special cases of minimax control problem is the nonlinear regulator problem, in which case there is no noise and disturbance effect, i.e. the coefficient matrix $g_2 = 0$. Therefore, the performance index (2.7) becomes:

$$V(x) = \int_0^\infty (q(x(t)) + u^T(t)u(t))dt$$
 (2.13)

The Hamilton Jacobi Equation (2.12) becomes

$$V_{x}(x)f(x) - \frac{1}{4}V_{x}(x)(g_{1}(x)g_{1}^{T}(x))V_{x}^{T}(x) + q(x) = 0$$
(2.14)

Finally, the optimal control input can be obtained from (2.10)

$$u^* = -\frac{1}{2}g_1^T(x)V_x^T(x)$$
 (2.15)

It is also noteworthy that the solution to the nonlinear regulator problem corresponds to that of the minimax problem with $\gamma \rightarrow \infty$.

2.2.2 Nonlinear H_{∞} Control Problem

If we further assume $q(x) = h^{T}(x)h(x)$ for some continuous function h(x) with h(0) = 0, and take $z = \begin{bmatrix} h(x) \\ u \end{bmatrix}$ as our performance output, the performance objective

becomes
$$\min_{u(t)} \max_{w(t) \in L_2[0,\infty)} \int_0^\infty \left(z^T(t) z(t) - \gamma^2 w^T(t) w(t) \right) dt$$
(2.16)

which is equivalent to $\int_0^\infty \|z(t)\|^2 dt \le \gamma^2 \int_0^\infty \|w(t)\|^2 dt + K(x)$

where $0 \le K(x) \le \infty$ is a constant, with K(0) = 0. We define the pre-Hamiltonian

$$K_{\gamma} = V_{x}(x)(f(x) + g_{1}(x)u + g_{2}(x)w) + ||z||^{2} - \gamma^{2} ||w||^{2}.$$
(2.17)

The optimal solutions can be obtained based on completion the squares of arguments as before. Here, we find the solutions based on the following conditions:

$$\frac{\partial K_{\gamma}}{\partial w} = 0, \frac{\partial K_{\gamma}}{\partial u} = 0$$
(2.18)

we have

$$u^* = -\frac{1}{2}g_1^T V_x^T$$
 (2.19)

$$w^* = \frac{1}{2\gamma^2} g_2^T V_x^T$$
 (2.20)

which has the saddle point property

$$K_{\gamma}(x, V_{x}, u^{*}, w) \leq K_{\gamma}(x, V_{x}, u^{*}, w^{*}) \leq K_{\gamma}(x, V_{x}, u, w^{*})$$
(2.21)

This leads to the Hamiltonian $H_{\gamma}(x, V_x)$ given as $K_{\gamma}(x, V_x, u^*, w^*)$, i.e.

$$H_{\gamma} = V_{x}(x)f(x) + \frac{1}{4}V_{x}(x)\left(\frac{1}{\gamma^{2}}g_{2}(x)g_{2}^{T}(x) - g_{1}(x)g_{1}^{T}(x)\right)V_{x}^{T}(x) + h^{T}(x)h(x)$$
(2.22)

Immediately following from differential game theory [72, 73], the solution of the state feedback H_{∞} suboptimal control centers around finding solutions to the Hamilton Jacobi Inequality, $H_{\gamma}(x, V_x) \le 0$, i.e.

$$H_{\gamma} = V_{x}(x)f(x) + \frac{1}{4}V_{x}(x)\left(\frac{1}{\gamma^{2}}g_{2}(x)g_{2}^{T}(x) - g_{1}(x)g_{1}^{T}(x)\right)V_{x}^{T}(x) + h^{T}(x)h(x) \le 0$$
(2.23)

2.2.3 State Dependent Riccati Equation Control

The nonlinear quadratic game problem becomes the nonlinear quadratic regulator (NLQR) control problem when there exists a weight on the control action in the performance index

$$V(x) = \min_{u(t)} \int_0^\infty \left(x^T(t) Q x(t) + u^T(t) R u(t) \right) dt$$
 (2.24)

In general, weighting matrices Q, R can be functions of the state variable.

Following the previous procedure, we have the Hamilton Jacobi Equation

$$V_{x}(x)f(x) - \frac{1}{4}V_{x}(x)g_{1}R^{-1}g_{1}^{T}V_{x}^{T}(x) + x^{T}Qx = 0$$
(2.25)

and the optimal control feedback gain:

$$u^* = -\frac{1}{2}R^{-1}g_1^T V_x^T$$
(2.26)

When we have a system in linear form:

$$\dot{x} = A(x)x + B(x)u \tag{2.27}$$

The energy function, $V(x) = x^T P(x)x$, for some P(x), is a continuous positive definite matrix-valued function. The HJE (2.25) becomes a State Dependent Algebraic Riccati Equation (SDRE) as:

$$A^{T}(x)P(x) + P(x)A(x) - P(x)B(x)R^{-1}(x)B^{T}(x)P(x) + Q(x) = 0$$
 (2.28)

The optimal feedback control (2.26) becomes

$$u^* = -R^{-1}(x)B^T(x)P(x)x$$
 (2.29)

which is also known as the State Dependent Riccati Equation (SDRE) control approach for nonlinear systems. SDRE control has been investigated and applied in aerospace and industrial applications in the past decade with growing popularity, due to its effectiveness and robustness for synthesizing nonlinear feedback control [6]. We further advance the SDRE approach in Chapter 5, in which $H_2 - H_{\infty}$ SDRE control approach will be developed.

2.3 Dissipative Systems Analysis and Control

Dissipative theory provides a framework for the design and analysis of system control using an input-output description based on energy related considerations. The application of dissipative theory to linear systems has also received considerable attention over the past two decades. The concept of dissipative system was first introduced in [93], and further generalized in [22]-[25], playing an important role in systems, circuits and controls. The theory of dissipative systems generalizes the basic tools for control design including the passivity theorem, bounded real lemma, Kalman-Yakubovich lemma and circle criterion. Dissipativity performance includes H_{∞} performance, passivity, positive realness and sector bounded constraint as special cases. Research addressing the problems of H_{∞} and positive real control systems can be found in [12, 13, 20, 54, 62]. Control of uncertain linear systems with L_2 -bounded structured uncertainty satisfying H_{∞} and passivity criteria have been tackled in [37, 50]. More recent developments involving the quadratic dissipative control for linear systems problem has been tackled in [64, 94]. This section covers the essential dissipative theory to this work.

2.3.1 Passive Systems

Consider the input-output relations of a system with input u(t) and output y(t) satisfying $H: u \to H(u) = y$, assuming that $H: L_{2,e} \to L_{2,e}$. The system is passive if there is a constant $\beta \ge 0$ such that the following property holds

$$\int_0^T y^T(t)u(t)dt \ge \beta$$
(2.30)

for all functions u(t) and all $T \ge 0$. If in addition, there are constants $\delta \ge 0$ and $\varepsilon \ge 0$ such that $\int_0^T y^T(t)u(t)dt \ge \beta + \delta \int_0^T u^T(t)u(t)dt + \varepsilon \int_0^T y^T(t)y(t)dt \qquad (2.31)$

for all functions u(t) and all $T \ge 0$, then the system is input strictly passive if $\delta > 0$, output strictly passive if $\varepsilon > 0$, and very strict passive if $\delta > 0, \varepsilon > 0$.

Assume that for a given system, there is a continuous energy function $V(t) \ge 0$ such that

$$V(T) - V(0) \leq \int_0^T y^T(t) u(t) dt \qquad (2.32)$$

for all functions u(t), for all $T \ge 0$ and all V(0). Then the system is passive. Assume, in addition there are constants $\delta \ge 0$ and $\varepsilon \ge 0$ such that

$$V(T) - V(0) \leq \int_0^T y^T(t)u(t)dt - \delta \int_0^T u^T(t)u(t)dt - \varepsilon \int_0^T y^T(t)y(t)dt \qquad (2.33)$$

for all functions u(t) and all $T \ge 0$, then the system is input strictly passive if $\delta > 0$, output strictly passive if $\varepsilon > 0$, and very strict passive if $\delta > 0, \varepsilon > 0$. 2.3.2 Bounded Real and Positive Real for Linear Systems

A transfer function g(s) of a linear time-invariant single input single output system is said to be bounded real [36], if

- 1. g(s) is analytic in $\operatorname{Re}(s) > 0$.
- 2. g(s) is real for real and positive s.
- 3. $|g(s)| \le 1$ for all $\operatorname{Re}(s) > 0$.

This definition can be extended to a matrix function, G(s), of a linear time-invariant multiple input multiple output system as follows:

A transfer matrix, $G(s) \in C^{m \times m}$, is bounded real if all elements of G(s) are analytic for $\operatorname{Re}(s) \ge 0$ and the H_{∞} norm satisfies $||G(s)||_{\infty} \le 1$ where we recall that $||G(s)||_{\infty} = \sup_{\omega \in \Re} \sigma_{\max} (G(j\omega))$. Equivalently, the second condition can be replaced by $I_m - G^T(j\omega)G(j\omega) \ge 0$ for all $\omega \in \Re$. Strict bounded realness can be achieved by changing the inequality from greater than (≥ 0) to simply greater (> 0).

A transfer function h(s) is said to be positive real if

- 1. h(s) is analytic in $\operatorname{Re}(s) > 0$.
- 2. h(s) is real for real and positive s.
- 3. $\operatorname{Re}[h(s)] \ge 0$ for all $\operatorname{Re}(s) > 0$.

When the definition of positive realness extends to a multivariable system, the transfer matrix, $H(s) \in C^{m \times m}$, is positive real if we have

1. H(s) has no pole in $\operatorname{Re}(s) > 0$.

- 2. H(s) is real for all positive real s.
- 3. $H(s) + H^*(s) \ge 0$ for all $\operatorname{Re}(s) > 0$.

The relation between passivity and positive realness can be stated as follows: A system with transfer function, h(s), is passive if and only if the transfer function h(s) is positive real.

2.3.3 Kalman-Yakubovich-Popov Lemma

The Kalman-Yakubovich-Popov Lemma (KYP), also known as Positive Real Lemma, is considered to be one of the milestones in control and estimation theory, due to its extensive applications in stability, dissipativity, passivity, optimal control, adaptive control and stochastic control. The importance of the KYP Lemma is that it establishes equivalence between the conditions in the frequency domain for a system to be positive real and the input-output relationship of that system in the time domain. The KYP Lemma presents the conditions on the matrices describing the state space representation of the system.

Lemma 2.1: Consider a linear time invariant system described by the following state space representation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$
(2.34)

with initial condition $x(0) = x_0$. The Positive Real Lemma can be stated as follows [36]:

Let the system (2.34) be controllable and observable. The system transfer function $H(s) = C(sI_n - A)^{-1}B + D$ with $A \in \Re^{n \times n}, B \in \Re^{n \times m}, C \in \Re^{m \times n}, D \in \Re^{m \times m}$ is Positive Real
with $H(s) \in \mathbb{R}^{m \times m}$, $s \in C$, if and only if there exists matrices $P = P^T > 0$, $P \in \mathbb{R}^{n \times n}$, $L \in \mathbb{R}^{n \times m}$ and $W \in \mathbb{R}^{m \times m}$ such that

$$\begin{cases} PA + A^{T}P = -LL^{T} \\ PB - C^{T} = -LW \\ D + D^{T} = W^{T}W \end{cases}$$
(2.35)

Notice that the set of equations can also be written as

$$\begin{bmatrix} -PA - A^T P & C^T - PB \\ * & D + D^T \end{bmatrix} = \begin{bmatrix} L \\ W^T \end{bmatrix} \begin{bmatrix} L^T & W \end{bmatrix} \ge 0$$
(2.36)

Proof

Only the sufficiency portion will be presented in this section; the necessity part of the proof involves the Youla factorization. Let equations (2.35) be satisfied, then [36]

$$H(s) + H^{T}(\overline{s}) = D^{T} + D + B^{T}(\overline{s}I_{n} - A^{T})^{-1}C^{T} + C(sI_{n} - A)^{-1}B + W^{T}W$$

+ $B^{T}[(\overline{s}I_{n} - A^{T})^{-1}P + P(sI_{n} - A)^{-1}]B + B^{T}(\overline{s}I_{n} - A^{T})^{-1}LW + W^{T}L^{T}(sI_{n} - A)^{-1}B$
= $W^{T}W + B^{T}(\overline{s}I_{n} - A^{T})^{-1}[P(s + \overline{s}) - PA - A^{T}P](sI_{n} - A)^{-1}B + B^{T}(\overline{s}I_{n} - A^{T})^{-1}LW + W^{T}L^{T}(sI_{n} - A)^{-1}B$
= $\left(W^{T}W + B^{T}(\overline{s}I_{n} - A^{T})^{-1}LW + W^{T}L^{T}(sI_{n} - A)^{-1}B\right)$
+ $B^{T}(\overline{s}I_{n} - A^{T})^{-1}LL^{T}(sI_{n} - A)^{-1}B$
= $\left[W^{T} + B^{T}(\overline{s}I_{n} - A^{T})^{-1}LL^{T}(sI_{n} - A)^{-1}B\right] + B^{T}(\overline{s}I_{n} - A^{T})^{-1}P(sI_{n} - A)^{-1}B(s + \overline{s})$
= $\left[W^{T} + B^{T}(\overline{s}I_{n} - A^{T})^{-1}L\right]\left[W + L^{T}(sI_{n} - A)^{-1}B\right] + B^{T}(\overline{s}I_{n} - A^{T})^{-1}P(sI_{n} - A)^{-1}B(s + \overline{s})$
(2.37)

which is positive semidefinite for all $\operatorname{Re}(s) > 0$.

2.3.4 Dissipative Systems

The system Σ is said to be dissipative if there exists an energy storage function $V(x) \ge 0$ such that the following dissipation inequality holds [36]:

$$V(x(T)) \le V(x(0)) + \int_0^T \omega(u(t), y(t)) dt$$
(2.38)

along all possible trajectories of Σ starting at x(0), for all $T \ge 0$.

Based on different supply rates $\omega(u(t), y(t))$, we have the general supply rate:

$$\omega(u(t), y(t)) = y^{T}Qy + u^{T}Ru + y^{T}Su$$
(2.39)

with $Q = Q^T$, $R = R^T$.

If $Q = 0, R = -\varepsilon I_m, \varepsilon > 0, S = I_m$, the system is said to be input strictly passive, i.e.

$$\int_{0}^{T} y^{T}(t) u(t) dt \ge \beta + \varepsilon \int_{0}^{T} u^{T}(t) u(t) dt .$$
(2.40)

If $R = 0, Q = -\delta I_m, \delta > 0, S = I_m$, the system is said to be output strictly passive, i.e.

$$\int_0^T y^T(t)u(t)dt \ge \beta + \delta \int_0^T y^T(t)y(t)dt.$$
(2.41)

If $Q = -\delta I_m$, $R = -\varepsilon I_m$, $\delta > 0$, $\varepsilon > 0$, $S = I_m$, the system is said to be very strictly passive,

i.e.
$$\int_0^T y^T(t)u(t)dt \ge \beta + \delta \int_0^T y^T(t)y(t)dt + \varepsilon \int_0^T u^T(t)u(t)dt. \qquad (2.42)$$

2.4 General Performance Criteria

Based on the performance criteria found in the literature given above, in this section, we now discuss general performance criteria for nonlinear control design, which yields a mixed Nonlinear Quadratic Regular (NLQR) in combination with H_{∞} or dissipativity performance index. The commonly used system performance criteria, including bounded realness, positive realness, sector boundedness and quadratic cost criterion, become special cases of the general performance criteria. The general performance criteria facilitate the controller and estimator design to allow for better tradeoffs between performance, robustness, stability, optimality and computational cost.

2.4.1 Continuous-Time General Performance Criteria

Before proposing continuous time general performance criteria, let us introduce the following quadratic energy supply function $E: L_{2e}^r \times L_{2e}^p \times \mathfrak{R}^+ \to \mathfrak{R}$ associated with the system equations, defined by [22]-[25]:

$$E(z, w, T) = \langle z, Mz \rangle_T + 2 \langle z, Sw \rangle_T + \langle w, Nw \rangle_T$$
(2.43)

where $M \in \Re^{r \times r}$, $S \in \Re^{r \times p}$, $N \in \Re^{p \times p}$ are the chosen weighing matrices. Next, from the definition of dissipativity, we have:

Given matrices $M \in \Re^{r \times r}$, $S \in \Re^{r \times p}$, $N \in \Re^{p \times p}$ with M, N symmetric, the system with energy function (2.43) is said to be (M, S, N)-dissipative if for some real function $\beta(\cdot)$ with $\beta(0) = 0$,

$$E(z, w, T) + \beta(x_0) \ge 0, \forall w \in L_{2e}, \forall T \ge 0$$

$$(2.44)$$

Furthermore, if for some scalar $\alpha > 0$,

$$E(z, w, T) + \beta(x_0) \ge \alpha \langle w, w \rangle_T, \forall w \in L_{2e}, \forall T \ge 0$$
(2.45)

The system is said to be strictly (M, S, N)-dissipative.

The following theorem considers general performance criteria to provide us a general framework for control design. The special cases of the general performance criteria are discussed in Remark 2.1 and Remark 2.2.

Theorem 2.1: Consider the quadratic function $V = x^T P x > 0$, matrices $M \in \Re^{r \times r}$, $S \in \Re^{r \times p}$, $N \in \Re^{p \times p}$ with M, N symmetric, $Q \in \Re^{n \times n}, Q > 0, R \in \Re^{m \times m}, R > 0$ with Q, Rsymmetric, the system control will achieve mixed nonlinear quadratic regulator and dissipative performance if the following condition holds:

$$\dot{V} + x^T Q x + u^T R u - \left(z^T M z + 2z^T S w + w^T N w\right) \le 0, \forall T \ge 0$$
(2.46)

Proof

By integrating (2.46) from 0 to *T*, we have

$$\int_{0}^{T} \left(z^{T} M z + 2 z^{T} S w + w^{T} N w \right) dt \ge \int_{0}^{T} x^{T} Q x dt + \int_{0}^{T} u^{T} R u dt + V \left(x(T) \right) - V \left(x(0) \right), \forall T \ge 0$$
(2.47)

Let
$$\beta(x_0) = V(x(0)), V(x) = x^T P x, V(x(T)) \ge 0,$$
 (2.48)

implies $\int_0^T (z^T M z + 2z^T S w + w^T N w) dt + \beta(x_0) \ge 0, \forall T \ge 0$, which is the condition of (M, S, N)-dissipative.

By adding the terms $x^TQx + u^TRu$, we include the nonlinear quadratic regulator control performance into the original (M, S, N)-dissipative criteria.

Remark 2.1: Notice that both H_{∞} and passivity are special cases of (M, S, N)-dissipativity. The special cases are summarized as follows:

Case 1: $M = -I, S = 0, N = \gamma^2 I$, the strict (M, S, N)-dissipativity reduces to H_{∞} design [13]. The overall control design satisfies mixed NLQR- H_{∞} performance.

Case 2: M = 0, S = I, N = 0, the strict (M, S, N)-dissipativity reduces to strict positive realness [62]. The overall control design satisfies mixed NLQR-strict positive realness performance.

Case 3: $M = -\theta I, S = (1 - \theta)I, N = \theta \gamma^2 I$, the strict (M, S, N)-dissipativity reduces to mixed H_{∞} and positive real performance design, when $\theta \in (0,1)$. The overall control design satisfies mixed NLQR- H_{∞} -positive real performance.

Case 4:
$$M = -I, S = \frac{1}{2} (K_1 + K_2)^T, N = -\frac{1}{2} (K_1^T K_2 + K_2^T K_1)$$
, where K_1 and K_2 are

constant matrices of appropriate dimensions, the strict (M, S, N)-dissipativity reduces to a sector-bounded constraint [18]. The overall control design satisfies mixed NLQR-sector bounded constraint performance.

Remark 2.2: If the coefficient matrices M, S, N are scalars, we denote

$$M = -\alpha, S = \beta/2, N = -\gamma \tag{2.49}$$

The general performance criteria (2.46) becomes

$$\dot{V} + x^T Q x + u^T R u + \alpha \cdot z^T z - \beta \cdot z^T w + \gamma \cdot w^T w \le 0$$
(2.50)

with Q > 0, R > 0 functions of x.

Note that upon integration over time from 0 to T, (2.50) yields

$$V(T) + \int_0^T \left[x^T Q x + u^T R u \right] dt + \int_0^T \left[\alpha \cdot z^T z - \beta \cdot z^T w + \gamma \cdot w^T w \right] dt \le V(0)$$
(2.51)

for all T > 0.

By properly specifying the value of the weighing matrices Q, R, C, D and α, β, γ , general performance criteria can be used in nonlinear control design, which yields a mixed Nonlinear Quadratic Regular (NLQR) in combination with H_{∞} or passivity performance index. For example, if we take $\alpha = 1, \beta = 0, \gamma < 0$, (2.50) and (2.51) yield

$$\dot{V} + x^{T}Qx + u^{T}Ru + z^{T}z + \gamma \cdot w^{T}w \leq 0$$

$$V(T) + \int_{0}^{T} \left[x^{T}Qx + u^{T}Ru + z^{T}z \right] dt \leq V(0) - \gamma \int_{0}^{T} w^{T}w \cdot dt$$
(2.52)

which is a mixed suboptimal NLQR- H_{∞} design. The possible mixed performance criteria which can be used in this framework with different design parameters α, β, γ are given in Tab.2.1.

α	β	γ	Performance criteria
1	0	<0	Suboptimal NLQR– H_{∞} Design
0	1	0	NLQR-Passivity Design
0	1	>0	NLQR-Input Strict Passivity Design
>0	1	0	NLQR-Output Strict Passivity Design
>0	1	>0	NLQR-Very Strict Passivity

Tab.2.1.Various performance criteria in a general framework

2.4.2 Discrete-Time General Performance Criteria

Before proposing discrete time general performance criteria, let us introduce the following quadratic energy supply function, E, associated with the system equations, defined in [22]-[25]:

$$E(z_k, w_k) = \langle z_k, M z_k \rangle + 2 \langle z_k, S w_k \rangle + \langle w_k, N w_k \rangle$$
(2.53)

where $M \in \Re^{r \times r}$, $S \in \Re^{r \times p}$, $N \in \Re^{p \times p}$ are the chosen weighing matrices. Next, from the definition of dissipativity, we have:

Given matrices $M \in \Re^{r \times r}$, $S \in \Re^{r \times p}$, $N \in \Re^{p \times p}$ with M, N symmetric, the system with energy function (2.53) is said to be (M, S, N)-dissipative if for some real function $\beta(\cdot)$ with $\beta(0) = 0$,

$$E(z_k, w_k) + \beta(x_0) \ge 0, \forall w \in l_2, \forall k \ge 0.$$

$$(2.54)$$

Furthermore, if for some scalar $\alpha > 0$,

$$E(z_k, w_k) + \beta(x_0) \ge \alpha \langle w_k, w_k \rangle, \forall w \in l_2, \forall k \ge 0,$$
(2.55)

the system is said to be strictly (M, S, N)-dissipative.

The following theorem is the discrete counter part of Theorem 2.2. The special cases of the general performance criteria are discussed in Remark 2.3 and Remark 2.4.

Theorem 2.2: Consider the quadratic function $V_k = x_k^T P_k x_k > 0$, matrices $M \in \Re^{r \times r}$, $S \in \Re^{r \times p}$, $N \in \Re^{p \times p}$ with M, N symmetric, $Q \in \Re^{n \times n}, Q > 0, R \in \Re^{m \times m}, R > 0$ with Q, Rsymmetric, the system control will achieve mixed nonlinear quadratic regulator and dissipative performance if the following condition holds:

$$V_{k+1} - V_k + x_k^T Q x_k + u_k^T R u_k - \left(z_k^T M z_k + 2 z_k^T S w_k + w_k^T N w_k \right) \le 0, \forall k \ge 0$$
(2.56)

Proof

Note that upon summation over *k*, we have

$$\sum_{k=0}^{N-1} \left[z_k^T M z_k + 2 z_k^T S w_k + w_k^T N w_k \right] \ge \sum_{k=0}^{N-1} \left[x_k^T Q x_k + u_k^T R u_k \right] + V_N - V_0$$
(2.57)

Let $\beta(x_0) = V_0$, $V_k(x) = x_k^T P_k x_k$, $V_N \ge 0$, (2.57) implies

$$\sum_{k=0}^{N-1} \left(z_k^T M z_k + 2 z_k^T S w_k + w_k^T N w_k \right) + \beta \left(x_0 \right) \ge 0$$
(2.58)

which is the condition of (M, S, N)-dissipative.

By adding the terms $x_k^T Q x_k + u_k^T R u_k$, we include the nonlinear quadratic regulator control performance within the original (M, S, N)-dissipative criteria.

Remark 2.3: Notice that both H_{∞} and passivity are special cases of (M, S, N)-dissipativity. The special cases are summarized as follows:

Case 1: $M = -I, S = 0, N = \gamma^2 I$, the strict (M, S, N)-dissipativity reduces H_{∞} design [13]. The overall control design satisfies mixed NLQR- H_{∞} performance.

Case 2: M = 0, S = I, N = 0, the strict (M, S, N)-dissipativity reduces to strict positive

realness [62]. The overall control design satisfies mixed NLQR-strict positive realness performance.

Case 3: $M = -\theta I, S = (1 - \theta)I, N = \theta \gamma^2 I$, the strict (M, S, N)-dissipativity reduces to mixed H_{∞} and positive real performance design, when $\theta \in (0,1)$. The overall control design satisfies mixed NLQR- H_{∞} -positive real performance.

Case 4:
$$M = -I, S = \frac{1}{2} (K_1 + K_2)^T, N = -\frac{1}{2} (K_1^T K_2 + K_2^T K_1)^T$$
, where K_1 and K_2 are

constant matrices of appropriate dimensions, the strict (M, S, N)-dissipativity reduces to a sector-bounded constraint [18]. The overall control design satisfies mixed NLQR-sector bounded constraint performance.

Remark 2.4: If the coefficient matrices M, S, N are scalers, we denote

$$M = -\alpha, S = \beta/2, N = -\gamma \tag{2.59}$$

The general performance criteria (2.56) becomes

$$V_{k+1} - V_k + x_k^T Q_k x_k + u_k^T R_k u_k + \alpha \cdot z_k^T z_k - \beta \cdot z_k^T w_k + \gamma \cdot w_k^T w_k \le 0$$

$$(2.60)$$

with $Q_k > 0, R_k > 0$ being functions of x_k .

Note that upon summation over k, (2.60) yields

$$V_{N} + \sum_{k=0}^{N-1} \left[x_{k}^{T} Q_{k} x_{k} + u_{k}^{T} R_{k} u_{k} + \alpha \cdot z_{k}^{T} z_{k} - \beta \cdot z_{k}^{T} w_{k} + \gamma \cdot w_{k}^{T} w_{k} \right] \leq V_{0}$$

$$(2.61)$$

By properly specifying the value of the weighing matrices Q_k, R_k, C_k, D_k and α, β, γ , general performance criteria can be used in nonlinear control design, which yields a mixed Nonlinear Quadratic Regulator (NLQR) in combination with H_{∞} or passivity performance index. For example, if we take $\alpha = 1, \beta = 0, \gamma < 0$, (2.60) and (2.61) yields

$$V_{k+1} - V_{k} + x_{k}^{T} Q_{k} x_{k} + u_{k}^{T} R_{k} u_{k} + z_{k}^{T} z_{k} + \gamma \cdot w_{k}^{T} w_{k} \leq 0$$

$$V_{N} + \sum_{k=0}^{N-1} \left[x_{k}^{T} Q_{k} x_{k} + u_{k}^{T} R_{k} u_{k} + z_{k}^{T} z_{k} \right] \leq V_{0} - \gamma \cdot \sum_{k=0}^{N-1} w_{k}^{T} w_{k}$$
(2.62)

which is mixed suboptimal NLQR- H_{∞} design.

The possible performance criteria which can be used in this framework with different design parameters α, β, γ are summarized in Tab.2.2. Note also that for $\alpha = 0, \beta = 0, \gamma = 0$, it follows from (2.60) that $\sum_{k=0}^{\infty} ||x_k||^2 < \infty$ and therefore, the controlled system is exponentially asymptotically stable for all of the criteria given in the Tab.2.2.

Tab.2.2. Various performance criteria in a general framework

α	β	γ	Performance criteria
1	0	<0	Suboptimal NLQR– H_{∞} Design
0	1	0	NLQR-Passivity Design
0	1	>0	NLQR-Input Strict Passivity Design
>0	1	0	NLQR-Output Strict Passivity Design
>0	1	>0	NLQR-Very Strict Passivity

2.5 Convex Optimization and Linear Matrix Inequality

A Linear Matrix Inequality (LMI) has the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0$$
(2.63)

where $x \in \Re^m$ (with x_i as the i^{th} entry of the vector) is the unknown variable and the symmetric matrices $F_i = F_i^T \in \Re^{n \times n}$, i = 0, ..., m are given. The inequality F(x) > 0means $x^T F(x) x > 0$ for all $x \in \Re^n$. When the matrices, F_i 's, are diagonal, the LMI F(x) > 0 is just a set of linear inequalities. Nonlinear convex inequalities are converted into LMI using Schur's complements; the basic idea is given as follows [5]:

The LMI:
$$\begin{bmatrix} Q(x) & S(x) \\ * & R(x) \end{bmatrix} > 0$$
 where $Q(x) = Q^T(x), R(x) = R^T(x)$ and $S(x)$

depend affinely on x, is equivalent to

$$R(x) > 0, Q(x) - S(x)R^{-1}(x)S^{T}(x) > 0.$$
(2.64)

$$Q(x) > 0, R(x) - S^{T}(x)Q^{-1}(x)S(x) > 0.$$
 (2.65)

Note that symbol "*" denotes the term necessary to make the whole matrix symmetric.

2.6 Summary

Chapter 2 briefly introduces the background theory of nonlinear dynamic system analysis and controls, including Hamilton Jacobi Equation and Hamilton Jacobi Inequality, Dissipative theory and Convex Optimization with Linear Matrix Inequalities. Moreover, general performance criteria have been introduced in Chapter 2, which will be used throughout the dissertation.

CHAPTER 3 STATE DEPENDENT LINEAR MATRIX INEQUALITY CONTROL APPROACH TO NONLINEAR SYSTEMS

This chapter presents novel State Dependent Linear Matrix Inequality (SDLMI) control designs for both continuous time and discrete time nonlinear dynamical systems with general performance criteria. By solving the State Dependent Linear Matrix Inequality at each time step, the optimal control solutions can be found to satisfy mixed performance criteria guaranteeing quadratic optimality with inherent stability property in combination with H_{∞} or a passivity type of disturbance reduction. The effectiveness of the proposed technique is demonstrated by simulations involving the benchmark underactuated system: the inverted pendulum on a cart [78, 83, 87].

3.1 The State Dependent LMI Control of Continuous Time Nonlinear Systems with General Performance Criteria

In this section, we discuss the nonlinear state feedback control problem for continuous time nonlinear control systems using the State Dependent Linear Matrix Inequality approach. We characterize the solution of the nonlinear continuous time control system with State Dependent LMIs, which are essentially equivalent to the classical Hamilton Jacobi Inequalities. General performance criteria are used to design the controller in order to guarantee quadratic optimality with inherent stability property in combination with H_{∞} or a passivity type of disturbance attenuation [78].

3.1.1 System Model and Performance Index

Consider the input-affine continuous-time nonlinear system represented by the following differential equation: $\dot{x} = F(x, u, w) = A(x) \cdot x + B(x) \cdot u + F(x) \cdot w$ (3.1) where $x \in \Re^n$ state variable of the dynamical system $u \in \Re^m$ applied input $w \in \Re^q$ L_2 type of disturbance A, B, F known coefficient matrices of appropriate dimensions, which can be functions of x

The performance output $z \in \Re^p$ is given by

$$z = C(x) \cdot x + D(x) \cdot w \tag{3.2}$$

where C, D are in general, state dependent coefficient matrices of appropriate dimensions. However, in the following, *x*-dependence will not be shown for notational simplicity.

It is assumed that state feedback is available. If the state variables are not available from the measurement, nonlinear estimators can be set up to obtain the state estimates. The state feedback control input is given by

$$u = K(x)x \tag{3.3}$$

The optimal control problem we consider is to determine an admissible control u to satisfy the performance objective (2.50). The general performance criteria is given as

$$\dot{V} + x^T Q x + u^T R u + \alpha \cdot z^T z - \beta \cdot z^T w + \gamma \cdot w^T w \le 0$$
(2.50)

with Q > 0, R > 0 functions of x.

3.1.2 Continuous Time State Dependent LMI Control

Theorem 3.1: Given the system model (3.1), performance output (3.2), control equation (3.3) and performance index (2.50), if there exist matrices $S = P^{-1} > 0$ and Y for all t > 0, such that the following state dependent LMI holds:

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} \\ * & \Xi_{22} & 0 & 0 & 0 \\ * & * & I & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} \ge 0$$
(3.4)

where

$$\Xi_{11} = -\left(SA^{T} + Y^{T}B^{T} + AS + BY\right)$$

$$\Xi_{12} = -\left(F + \alpha \cdot SC^{T}D - 0.5 \cdot \beta SC^{T}\right)$$

$$\Xi_{13} = SQ^{T/2}$$

$$\Xi_{14} = Y^{T}R^{T/2}$$

$$\Xi_{15} = \alpha^{1/2}SC^{T}$$

$$\Xi_{22} = -\alpha \cdot D^{T}D + 0.5 \cdot \beta\left(D + D^{T}\right) - \gamma I$$
(3.5)

then the performance criteria inequality (2.50) is satisfied. The nonlinear feedback gain of

controller is given by
$$K = Y \cdot P$$
 (3.6)

Proof

By applying system model (3.1), performance output (3.2) and state feedback input (3.3), the performance criteria (2.50) becomes

$$(A \cdot x + B \cdot u + F \cdot w)^{T} Px + x^{T} P(A \cdot x + B \cdot u + F \cdot w) + x^{T} Qx + u^{T} Ru + \alpha \cdot (C \cdot x + D \cdot w)^{T} (C \cdot x + D \cdot w) - \beta \cdot (C \cdot x + D \cdot w)^{T} w + \gamma \cdot w^{T} w \le 0$$
(3.7)

Equivalently, we have

$$\begin{bmatrix} x^T & w^T \end{bmatrix} \Psi \begin{bmatrix} x & w \end{bmatrix}^T = \begin{bmatrix} x^T & w^T \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \le 0$$
(3.8)

where

$$\Psi_{11} = (A + BK)^{T} P + P(A + BK) + Q + K^{T}RK + \alpha \cdot C^{T}C$$

$$\Psi_{12} = PF + \alpha \cdot C^{T}D - 0.5 \cdot \beta C^{T}$$

$$\Psi_{22} = \alpha \cdot D^{T}D - 0.5 \cdot \beta (D + D^{T}) + \gamma I$$
(3.9)

Therefore, (3.8) is equivalent to matrix $\Psi \le 0$. Let us denote the notation $S = P^{-1}$, then (3.3) yields $Y = K \cdot P^{-1} = K \cdot S$.

By pre-multiplying and post multiplying the matrix Ψ with block $diag\{S, I\}$, the following inequality holds

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix} \le 0 \tag{3.10}$$

where

$$\Theta_{11} = SA^{T} + Y^{T}B^{T} + AS + BY + SQS + Y^{T}RY + \alpha \cdot SC^{T}CS$$

$$\Theta_{12} = F + \alpha \cdot SC^{T}D - 0.5 \cdot \beta SC^{T}$$

$$\Theta_{22} = \alpha \cdot D^{T}D - 0.5 \cdot \beta (D + D^{T}) + \gamma I$$
(3.11)

Equivalently, we have

$$\begin{bmatrix} SA^{T} + Y^{T}B^{T} + AS + BY & F + \alpha \cdot SC^{T}D - 0.5 \cdot \beta SC^{T} \\ * & \alpha \cdot D^{T}D - 0.5 \cdot \beta (D + D^{T}) + \gamma I \end{bmatrix}$$

$$+ \begin{bmatrix} SQS + Y^{T}RY + \alpha \cdot SC^{T}CS & 0 \\ 0 & 0 \end{bmatrix} \leq 0$$
(3.12)

Since

$$\begin{bmatrix} SQS + Y^{T}RY + \alpha \cdot SC^{T}CS & 0\\ 0 & 0 \end{bmatrix} = \begin{bmatrix} SQ^{T/2} & Y^{T}R^{T/2} & \alpha^{1/2}SC^{T}\\ 0 & 0 & 0 \end{bmatrix} \cdot I^{-1} \cdot \begin{bmatrix} Q^{1/2}S & 0\\ R^{1/2}Y & 0\\ \alpha^{1/2}CS & 0 \end{bmatrix}$$
(3.13)

By applying Schur complement, the LMI solution is obtained

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} \\ * & \Xi_{22} & 0 & 0 & 0 \\ * & * & I & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} \ge 0$$
(3.14)

where

$$\Xi_{11} = -\left(SA^{T} + Y^{T}B^{T} + AS + BY\right)$$

$$\Xi_{12} = -\left(F + \alpha \cdot SC^{T}D - 0.5 \cdot \beta SC^{T}\right)$$

$$\Xi_{13} = SQ^{T/2}$$

$$\Xi_{14} = Y^{T}R^{T/2}$$

$$\Xi_{15} = \alpha^{1/2}SC^{T}$$

$$\Xi_{22} = -\alpha \cdot D^{T}D + 0.5 \cdot \beta\left(D + D^{T}\right) - \gamma I$$
(3.15)

Hence if the LMI (3.14) holds, performance criteria inequality (2.50) is satisfied.

3.1.3 Application to the Inverted Pendulum on a Cart

The inverted pendulum on a cart system stabilization is a classical control problem and has been used widely as a benchmark for testing control algorithms. The pendulum mass is above the pivot point which is mounted on a horizontally moving cart. It is used in this work to demonstrate the effectiveness and robustness of the State Dependent LMI control approach.

Fig.3.1 shows the physical representation of the inverted pendulum on a cart system. A beam attached to a cart can rotate freely in the vertical 2-dimensional plane. The angle of the beam with respect to the vertical direction is denoted angle θ . The cart can only move in the 1-dimensional track, with position *x*. The external force *F*, the control input acting on the cart, is used to stabilize this highly nonlinear system while

satisfying general performance criteria. The control objective is to find the State Dependent LMI control to set the cart position *x*, the velocity of the cart \dot{x} , the angle of the beam θ and the angular velocity $\dot{\theta}$ all to zero.



Fig.3.1. Inverted pendulum system diagram

Traditional nonlinear control techniques assume that θ is a very small angle, so that $\cos(\theta) \approx 1, \sin(\theta) \approx 0$, and linearize the system equation around its equilibrium point afterwards, then apply the traditional linear system control technique. Other nonlinear control methods might be applicable as well. However, it can be shown that the control is not guaranteed to be optimal or stable. In this dissertation, we will not resort to the usual linearization approach. That is why a detailed account of the system modeling is provided. Using the Euler-Lagrange Equation technique, the complete equations of motion for the inverted pendulum on a cart are found to be

$$\begin{cases} (M+m)\ddot{x} + b\dot{x} + mL\ddot{\theta}\cos(\theta) - mL\dot{\theta}^{2}\sin(\theta) = F \\ (I+mL^{2})\ddot{\theta} + mgL\sin(\theta) + mL\ddot{x}\cos(\theta) = 0 \end{cases}$$
(3.16)

where

- M mass of the cart
- *m* mass of the pendulum

b friction coefficient between cart and ground

L length to the pendulum center of mass, length of the pendulum equals 2L

 $I = \frac{1}{3}m(2L)^2$ inertia of the pendulum

F external force, input of the system

When the following definitions are made, $x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta}$ and

$$\Delta_{1} = I + mL^{2} - \frac{m^{2}L^{2}\cos^{2}(\theta)}{M + m}$$
(3.17)

$$\Delta_2 = M + m - \frac{m^2 L^2 \cos^2(\theta)}{I + mL^2}$$
(3.18)

Then the state space model for the system can be written as

$$\begin{bmatrix} \dot{x}_{1} \\ \dot{x}_{2} \\ \dot{x}_{3} \\ \dot{x}_{4} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & A_{42} & A_{43} & A_{44} \end{bmatrix} \cdot \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{2} \\ 0 \\ B_{4} \end{bmatrix} F$$
(3.19)

where

$$A_{22} = -\frac{b}{\Delta_2} \tag{3.20}$$

$$A_{23} = \frac{m^2 L^2 g \cos(\theta)}{\Delta_2 \left(I + mL^2\right)} \cdot \frac{\sin(\theta)}{\theta}$$
(3.21)

$$A_{24} = \frac{mL\sin(\theta)}{\Delta_2}\dot{\theta}$$
(3.22)

$$A_{42} = \frac{mLb\cos(\theta)}{(M+m)\Delta_1}$$
(3.23)

$$A_{43} = -\frac{mgL}{\Delta_1} \cdot \frac{\sin(\theta)}{\theta}$$
(3.24)

$$A_{44} = -\frac{m^2 L^2 \cos(\theta) \sin(\theta) \dot{\theta}}{(M+m)\Delta_1}$$
(3.25)

$$B_2 = \frac{1}{\Delta_2} \tag{3.26}$$

$$B_4 = -\frac{mL\cos(\theta)}{(M+m)\Delta_1}$$
(3.27)

It should be noted that this state space formulation is not a process of linearization, but rather a process of state-dependent parameterization. To avoid the

division by zero, the $\frac{\sin(x_3)}{x_3}$ term is substituted when $x_3 = 0$ by the limit

$$\lim_{x_3 \to 0} \frac{\sin(x_3)}{x_3} = 1$$
(3.28)

Assume the following system parameters are given

$$M = 0.5kg, m = 0.5kg, b = 0.1N \cdot \frac{\sec}{m}, L = 0.3m, I = 0.06kg \cdot m^2$$

The following design parameters are chosen:

Mixed NLQR- H_{∞} Design (Predominant NLQR)

 $C = [0.01 \quad 0.01 \quad 0.01 \quad 0.01], D = [0.01], Q = I_4, R = 1, \alpha = 1, \beta = 0, \gamma = -1$

Mixed NLQR- $H_{\scriptscriptstyle \infty}$ Design (Predominant $H_{\scriptscriptstyle \infty}$)

$$C = [1 \ 1 \ 1 \ 1], D = [1], Q = 0.01 \times I_4, R = 0.01, \alpha = 1, \beta = 0, \gamma = -10$$

NLQR-Very Strict Passivity

 $C = [1 \ 1 \ 1 \ 1], D = [1], Q = I_4, R = 1, \alpha = 0.01, \beta = 1, \gamma = 0.01$

Assume the initial condition $x_1 = 1, x_2 = 0, x_3 = \pi / 4, x_4 = 0$, all of the above mixed criteria control performance results are shown in Fig.3.2-3.6, in comparison with the traditional Linear Quadratic Regulator (LQR) technique based on linearization. From

these figures, we find that the novel State Dependent LMI control has better performance compared with the traditional LQR technique based on linearization. Especially, Fig.3.2 and Fig.3.3 show that the traditional LQR technique loses control of position and velocity of the cart respectively. The predominant H_{∞} control has the fastest response time. From Fig.3.6, we can verify that the predominant NLQR control is much more input energy efficient.



Fig.3.2. Position trajectory of the inverted pendulum



Fig.3.3. Velocity trajectory of the inverted pendulum



Fig.3.4. Angle "theta" trajectory of the inverted pendulum



Fig.3.5. Angular velocity trajectory of the inverted pendulum



Fig.3.6. Control input

3.2 The State Dependent LMI Control of Discrete Time Nonlinear Systems with General Performance Criteria

In this section, we discuss the nonlinear state feedback control problem for discrete time nonlinear control systems using the State Dependent Linear Matrix Inequalities approach. We characterize the solution of the nonlinear discrete time control system with State Dependent LMI, which is essentially equivalent to the classical Hamilton Jacobi Inequalities. General performance criteria are used to design the controller in order to guarantee quadratic optimality with inherent stability property in combination with H_{∞} or a passivity type of disturbance attenuation [83].

3.2.1 System Model and Performance Index

Consider the input-affine discrete-time nonlinear system represented by the following differential equation

$$x_{k+1} = A(x_k)x_k + B(x_k) \cdot u_k + F(x_k) \cdot w_k = A_k \cdot x_k + B_k \cdot u_k + F_k \cdot w_k$$
(3.29)

where

 $x_k \in \Re^n$ state vector $u_k \in \Re^m$ applied input $w_k \in \Re^q$: L_2 type of disturbance A_k, B_k, F_k known coefficient matrices of appropriate dimensions, which can
be functions of x_k

Notice that the simplified notation for time varying matrices A_k, B_k , etc. is used to denote the state dependent matrices. The performance output $z_k \in \Re^p$ is

$$z_k = C(x_k) \cdot x_k + D(x_k) \cdot w_k = C_k \cdot x_k + D_k \cdot w_k$$
(3.30)

where C_k, D_k are, in general, state dependent coefficient matrices of appropriate dimensions.

It is assumed that the state feedback is available. If the state variables are not available from the measurement, nonlinear estimators can be set up to obtain the state estimates. The nonlinear state feedback control input is given by

$$u_k = K(x_k) \cdot x_k = K_k \cdot x_k \tag{3.31}$$

The optimal control problem we consider is to determine an admissible control u_k to satisfy the performance objective (2.60). The general performance criteria is given as

$$V_{k+1} - V_k + x_k^T Q_k x_k + u_k^T R_k u_k + \alpha \cdot z_k^T z_k - \beta \cdot z_k^T w_k + \gamma \cdot w_k^T w_k \le 0$$
(2.60)

with $Q_k > 0, R_k > 0$ functions of x_k .

3.2.2 Discrete Time State Dependent LMI Control

Theorem 3.2: Given the system model (3.29), performance output (3.30), control equation (3.31) and performance index (2.60), if there exist matrices $M_k = P_k^{-1} > 0$ and Y_k for all $k \ge 0$, such that the following State Dependent LMIs hold:

$$\begin{bmatrix} M_{k} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 \\ * & * & M_{k} & 0 & 0 & 0 \\ * & * & * & I_{n} & 0 & 0 \\ * & * & * & * & I_{m} & 0 \\ * & * & * & * & * & I_{p} \end{bmatrix} \ge 0$$
(3.32)

where

 $\Xi_{12} = -\alpha M_k C_k^T D_k + 0.5 \cdot \beta M_k C_k^T$

$$\Xi_{13} = M_{k}A_{k} + Y_{k}^{T}B_{k}^{T}$$

$$\Xi_{14} = M_{k}Q_{k}^{T/2}$$

$$\Xi_{15} = Y_{k}^{T}R_{k}^{T/2}$$

$$\Xi_{16} = \alpha^{1/2}M_{k}C_{k}^{T}$$

$$\Xi_{22} = -\gamma I - \alpha D_{k}^{T}D_{k} + 0.5 \cdot \beta \left(D_{k} + D_{k}^{T}\right)$$

$$\Xi_{23} = F_{k}^{T}$$
(3.33)

and

$$M_{k+1} \ge M_k \tag{3.34}$$

then the general performance critieria inequality (2.60) is satisfied. The nonlinear feedback gain of the controller is given by

$$K_k = Y_k \cdot P_k \tag{3.35}$$

Proof

By applying system model (3.29), performance output (3.30) and state feedback input (3.31), the performance criteria (2.60) becomes

$$(A_{k} \cdot x_{k} + B_{k} \cdot u_{k} + F_{k} \cdot w_{k})^{T} P_{k+1} (A_{k} \cdot x_{k} + B_{k} \cdot u_{k} + F_{k} \cdot w_{k}) - x_{k}^{T} P_{k} x_{k} + x_{k}^{T} Q_{k} x_{k} + u_{k}^{T} R_{k} u_{k} + \alpha \cdot (C_{k} \cdot x_{k} + D_{k} \cdot w_{k})^{T} (C_{k} \cdot x_{k} + D_{k} \cdot w_{k}) - \beta \cdot (C_{k} \cdot x_{k} + D_{k} \cdot w_{k})^{T} w_{k} + \gamma \cdot w_{k}^{T} w_{k} \leq 0$$

$$(3.36)$$

Equivalently,

$$\begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \Psi \begin{bmatrix} x_k & w_k \end{bmatrix}^T = \begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \le 0$$
(3.37)

where

$$\Psi_{11} = (A_k + B_k K_k)^T P_{k+1} (A_k + B_k K_k) - P_k + Q_k + K_k^T R_k K_k + \alpha C_k^T C_k$$

$$\Psi_{12} = (A_k + B_k K_k)^T P_{k+1} F_k + \alpha C_k^T D_k - 0.5 \cdot \beta C_k^T$$

$$\Psi_{22} = F_k^T P_{k+1} F_k + \alpha D_k^T D_k + \gamma I - 0.5 \cdot \beta (D_k + D_k^T)$$
(3.38)

Therefore, (3.38) is equivalent to matrix $\Psi \leq 0$, which is equivalent to the inequality

$$\begin{bmatrix} P_{k} - Q_{k} - K_{k}^{T} R_{k} K_{k} - \alpha C_{k}^{T} C_{k} & -\alpha C_{k}^{T} D_{k} + 0.5 \cdot \beta C_{k}^{T} \\ -\alpha D_{k}^{T} C_{k} + 0.5 \cdot \beta C_{k} & -\gamma I + 0.5 \cdot \beta \left(D_{k} + D_{k}^{T} \right) - \alpha D_{k}^{T} D_{k} \end{bmatrix}$$

$$-\begin{bmatrix} \left(A_{k} + B_{k} K_{k} \right)^{T} \\ F_{k}^{T} \end{bmatrix} P_{k+1} \begin{bmatrix} \left(A_{k} + B_{k} K_{k} \right) & F_{k} \end{bmatrix} \ge 0$$

$$(3.39)$$

By adding and subtracting the same term in (3.39), the following inequality results

$$\begin{bmatrix} P_{k} - Q_{k} - K_{k}^{T} R_{k} K_{k} - \alpha C_{k}^{T} C_{k} & -\alpha C_{k}^{T} D_{k} + 0.5 \cdot \beta C_{k}^{T} \\ -\alpha D_{k}^{T} C_{k} + 0.5 \cdot \beta C_{k} & -\gamma I + 0.5 \cdot \beta \left(D_{k} + D_{k}^{T} \right) - \alpha D_{k}^{T} D_{k} \end{bmatrix}$$
$$-\begin{bmatrix} \left(A_{k} + B_{k} K_{k}\right)^{T} \\ F_{k}^{T} \end{bmatrix} \left(P_{k+1} - P_{k}\right) \begin{bmatrix} \left(A_{k} + B_{k} K_{k}\right) & F_{k} \end{bmatrix} - \begin{bmatrix} \left(A_{k} + B_{k} K_{k}\right)^{T} \\ F_{k}^{T} \end{bmatrix} P_{k} \begin{bmatrix} \left(A_{k} + B_{k} K_{k}\right) & F_{k} \end{bmatrix} \ge 0$$

$$(3.40)$$

Therefore, subject to $P_{k+1} \le P_k$, (3.40) can be rewritten as

$$\begin{bmatrix} P_{k} - Q_{k} - K_{k}^{T} R_{k} K_{k} - \alpha C_{k}^{T} C_{k} & -\alpha C_{k}^{T} D_{k} + 0.5 \cdot \beta C_{k}^{T} \\ -\alpha D_{k}^{T} C_{k} + 0.5 \cdot \beta C_{k} & -\gamma I + 0.5 \cdot \beta \left(D_{k} + D_{k}^{T} \right) - \alpha D_{k}^{T} D_{k} \end{bmatrix}$$

$$-\begin{bmatrix} \left(A_{k} + B_{k} K_{k} \right)^{T} \\ F_{k}^{T} \end{bmatrix} P_{k} \begin{bmatrix} \left(A_{k} + B_{k} K_{k} \right) & F_{k} \end{bmatrix} \ge 0$$

$$(3.41)$$

By applying Schur complement result, we obtain

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ * & \Gamma_{22} & \Gamma_{23} \\ * & * & \Gamma_{33} \end{bmatrix} \ge 0$$
(3.42)

where

$$\Gamma_{11} = P_k - Q_k - K_k^T R_k K_k - \alpha C_k^T C_k$$

$$\Gamma_{12} = -\alpha C_k^T D_k + 0.5 \cdot \beta C_k^T$$

$$\Gamma_{13} = \left(A_k + B_k K_k\right)^T P_k$$

$$\Gamma_{22} = -\gamma I + 0.5 \cdot \beta \left(D_k + D_k^T\right) - \alpha D_k^T D_k$$

$$\Gamma_{23} = F_k^T P_k$$

$$\Gamma_{33} = P_k$$
(3.43)

By pre-multiplying and post-multiplying the matrix with block diagonal matrix $diag\{M_k, I, M_k\}$, where $M_k = P_k^{-1}$, the following inequality follows

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ * & \Theta_{22} & \Theta_{23} \\ * & * & \Theta_{33} \end{bmatrix} \ge 0$$
(3.44)

where

$$\Theta_{11} = M_k - M_k \left(Q_k + K_k^T R_k K_k + \alpha C_k^T C_k \right) M_k$$

$$\Theta_{12} = -\alpha M_k C_k^T D_k + 0.5 \cdot \beta M_k C_k^T$$

$$\Theta_{13} = M_k \left(A_k + B_k K_k \right)^T$$

$$\Theta_{22} = -\gamma I + 0.5 \cdot \beta \left(D_k + D_k^T \right) - \alpha D_k^T D_k$$

$$\Theta_{23} = F_k^T$$

$$\Theta_{33} = M_k$$

(3.45)

Equivalently,

$$\begin{bmatrix} M_{k} & \Theta_{12} & \Theta_{13} \\ * & \Theta_{22} & \Theta_{23} \\ * & * & M_{k} \end{bmatrix} - \begin{bmatrix} M_{k}Q_{k}^{T/2} & M_{k}K_{k}^{T}R_{k}^{T/2} & \alpha^{1/2}M_{k}C_{k}^{T} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot I_{n+m+p}^{-1} \cdot \begin{bmatrix} Q_{k}^{1/2}M_{k} & 0 & 0 \\ R_{k}^{1/2}K_{k}M_{k} & 0 & 0 \\ \alpha^{1/2}C_{k}M_{k} & 0 & 0 \end{bmatrix} \ge 0$$

$$(3.46)$$

Finally, by applying Schur complement again, the following LMI result is obtained

$$\begin{bmatrix} M_{k} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 \\ * & * & M_{k} & 0 & 0 & 0 \\ * & * & * & I & 0 & 0 \\ * & * & * & * & I & 0 \\ * & * & * & * & * & I & 0 \\ * & * & * & * & * & * & I \end{bmatrix} \ge 0$$

$$(3.47)$$

where

$$\begin{split} \Xi_{12} &= -\alpha M_k C_k^T D_k + 0.5 \cdot \beta M_k C_k^T \\ \Xi_{13} &= M_k A_k + Y_k^T B_k^T \\ \Xi_{14} &= M_k Q_k^{T/2} \end{split}$$

$$\Xi_{15} = Y_k^T R_k^{T/2}
\Xi_{16} = \alpha^{1/2} M_k C_k^T
\Xi_{22} = -\gamma I - \alpha D_k^T D_k + 0.5 \cdot \beta \left(D_k + D_k^T \right)
\Xi_{23} = F_k^T$$
(3.48)

and

$$M_{k+1} \ge M_k \tag{3.49}$$

Hence, the performance criteria (2.60) is guaranteed to be satisfied, if the inequalities (3.47) and (3.49) hold.

Remark: For the chosen performance criterion among those in Tab.2.2, the LMI (3.47) and (3.48) need to be solved at each time step and the state feedback control gain (3.35) needs to be applied to control system (3.29) to achieve the desired performance.

3.2.3 Application to the Inverted Pendulum on a Cart

The inverted pendulum on a cart system will again be used for testing the new control algorithms. A dynamical model of the inverted pendulum problem has been derived previously in (3.16). For discrete time state space model, we denote the following state variables:

$$x_{1,k} = x(kT), x_{2,k} = \dot{x}(kT), x_{3,k} = \theta(kT), x_{4,k} = \dot{\theta}(kT)$$

By applying the Euler discretization method with sampling period T, and using the notation

$$\Omega_{1,k} = I + mL^2 - \frac{m^2 L^2 \cos^2(x_{3,k})}{M + m}$$

$$\Omega_{2,k} = M + m - \frac{m^2 L^2 \cos^2(x_{3,k})}{I + mL^2}$$
(3.50)

the discrete-time system equation can be written as

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \\ x_{4,k+1} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & T \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix} u_k$$
(3.51)

where

 u_k is the k^{th} sampling instant value of the input force F and

$$\begin{aligned} a_{22} &= 1 - T \frac{b}{\Omega_{2,k}} \\ a_{23} &= T \frac{m^2 L^2 g \cos(x_{3,k})}{\Omega_{2,k} (I + mL^2)} \frac{\sin(x_{3,k})}{x_{3,k}} \\ a_{24} &= T \frac{mL \sin(x_{3,k})}{\Omega_{2,k}} x_{4,k} \\ a_{42} &= T \frac{mLb \cos(x_{3,k})}{(M + m)\Omega_{1,k}} \\ a_{43} &= -T \frac{mgL}{\Omega_{1,k}} \frac{\sin(x_{3,k})}{x_{3,k}} \\ a_{44} &= 1 - T \frac{m^2 L^2 \cos(x_{3,k}) \sin(x_{3,k}) x_{4,k}}{(M + m)\Omega_{1,k}} \\ b_2 &= \frac{T}{\Omega_{2,k}} \\ b_4 &= -T \frac{mL \cos(x_{3,k})}{(M + m)\Omega_{1,k}} \end{aligned}$$
(3.52)

It should be noted here again that this state space formulation does not involve a process of linearization, but a process of state-dependent parameterization. To avoid the division by zero, the term $\frac{\sin(x_{3,k})}{x_{3,k}}$ is substituted when $x_{3,k} = 0$ by the limit $\sin(x_{3,k})$

$$\lim_{x_{3,k}\to 0} \frac{\sin(x_{3,k})}{x_{3,k}} = 1, \qquad (3.53)$$

as was done before. The following system parameters are assumed

$$M = 0.5kg, m = 0.5kg, b = 0.1N \cdot \frac{\sec}{m}, L = 0.3m, I = 0.06kg \cdot m^2$$

The following design parameters are chosen to satisfy different mixed criteria: Mixed NLQR- H_{∞} Design (Predominant NLQR)

$$C = [0.01 \quad 0.01 \quad 0.01], D = [0.01], Q = I_4, R = 1, \alpha = 1, \beta = 0, \gamma = -5$$

Mixed NLQR- H_{∞} Design (Predominant H_{∞})

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = 0.01 \times I_4, R = 0.01, \alpha = 1, \beta = 0, \gamma = -5$$

NLQR-Very Strict Passivity

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = I_4, R = 1, \alpha = 0.01, \beta = 1, \gamma = 0.01$$

The following initial conditions are assumed

$$x_{1,0} = 1, x_{2,0} = 0, x_{3,0} = \pi / 4, x_{4,0} = 0$$

All of the above mixed criteria control performance results are shown in the Fig.3.7-3.11, in comparison with the traditional Linear Quadratic Regulator (LQR) technique based on linearization. From these figures, we find that the novel State Dependent LMI control has better performance compared with the traditional LQR technique based on linearization. Especially, Figs. 3.7-3.9 show that the traditional LQR technique loses control of state variables. It should also be noted that predominant NLQR and predominant H_{∞} control techniques lead to faster response times than the NLQR-passivity technique. Fig.3.11 shows that the highest magnitude of control is needed by the predominant H_{∞} control and the lowest control magnitude is needed by the linearization based traditional LQR technique.



Fig.3.7. Position trajectory of the inverted pendulum



Fig.3.8. Velocity trajectory of the inverted pendulum



Fig.3.9. Angle "theta" trajectory of the inverted pendulum



Fig.3.10. Angular velocity trajectory of the inverted pendulum



Fig.3.11. Control input

3.3 Nonlinear Control of Stochastic Systems with Actuator Failures

Designing a robust controller for nonlinear discrete time systems with actuator failures is of special importance in applications such as flight controls, unmanned vehicle controls, nuclear reactors, robotics, etc. Actuator failures may cause severe system performance deterioration or even instability, which should be avoided in these critical situations.

Adaptive control techniques are most commonly used to address the issues of system with actuator failures in existing theory and applications by assuming the system model is Linear Time Invariant (LTI), aimed at compensating such uncertainties with adaptive tuning of controller parameters based on system response errors. Recently, there have been results in control of systems with actuator or component failures in different applications including flight control [33, 43, 44, 69, 70], spacecraft control [71], robotics [11], etc.

In order to address this important issue, a novel robust control technique is proposed for discrete time nonlinear systems with actuator failures, which is based on state dependent linear matrix inequality approach. Previous studies in state dependent linear matrix inequality control can be found in [78, 83]. Without the linearization of the nonlinear system, state feedback control technique is developed for nonlinear systems with actuator failures characterized by inputs randomly failing to send an actuation signal. This controller is optimally robust for actuator failures in achieving general performance criteria to secure quadratic optimality with the inherent asymptotic stability property together with quadratic dissipative type of disturbance reduction. By solving a state dependent linear matrix inequality, a sufficient condition for the control solution can be found which satisfies the general performance criteria. The results of this paper unify existing results on nonlinear quadratic regulator, H_{∞} and passivity control to provide a more flexible and less conservative robust control design for systems with random actuator failures [87].

3.3.1 System Model and Performance Index

Consider the following discrete time nonlinear system dynamics

$$x_{k+1} = A(x_k)x_k + B_k(x_k)\Gamma_k u_k + F_k(x_k)w_k$$

= $A_k x_k + B_k \Gamma_k u_k + F_k w_k$ (3.54)

where

 $x_k \in \Re^n$

state vector

 $u_k \in \Re^m$ applied input $w_k \in \Re^q$ l_2 type of system noise A_k, B_k, F_k known coefficient matrices of appropriate dimensions,
which can be functions of x_k .

Denote
$$\Gamma_k = diag\left(\left[\gamma_k^1, ..., \gamma_k^m\right]\right)$$
 (3.55)

$$w_k = \left[w_k^1, \dots, w_k^q \right]^T \tag{3.56}$$

The noise process $\{w_k\}$ is white, zero mean, uncorrelated with the initial state value x_0 , and has covariance W_k :

$$w_k \sim (0, W_k), E\left[w_k w_j^T\right] = W_k \delta_{k-j}, E\left[w_k x_0^T\right] = 0$$
 (3.57)

The scalar binary Bernoulli distributed random variables γ_k^i , which are used to represent the failure of the *i*th actuator, have mean π_i and variance $\pi_i(1-\pi_i)$ whose possible outcomes {1,0} ("1" corresponding to the healthy actuator and "0" corresponding to the failed one) are defined probabilistically as $P(\gamma_k^i = 1) = \pi_i$ and $P(\gamma_k^i = 0) = 1 - \pi_i$. This formulation involves only hard actuator failures, i.e. either the actuator works or it fails, and there is no other alternative considered in this work.

The performance output $z_k \in \Re^p$ is chosen as

$$z_{k} = C(x_{k})x_{k} + D(x_{k})u_{k} + G(x_{k})w_{k}$$

= $C_{k}x_{k} + D_{k}u_{k} + G_{k}w_{k}$ (3.58)

where C_k , D_k , G_k are, in general, state dependent matrices of appropriate dimensions. The nonlinear state feedback control input is given by $u_k = K(x_k)x_k = K_k x_k$. (3.59) Consider the quadratic energy function $V_k = x_k^T P_k x_k$ (3.60)

For the following performance criteria inequality

$$E\{V_{k+1}|x_{k}, x_{k-1}, ..., x_{0}\} - V_{k} + \alpha z_{k}^{T} z_{k} - \beta z_{k}^{T} w_{k} + \gamma w_{k}^{T} w_{k} < 0$$
(3.61)

with $Q_k > 0, R_k > 0$ being functions of x_k .

By properly specifying the value of the weighing matrices C_k , D_k , G_k and α , β , γ , mixed performance criteria can be used in nonlinear control design, which yields a mixed H_{∞} or passivity performance index. The possible performance criteria which can be used in this framework with different design parameters α , β , γ are given in Table.1. Note also that for $\alpha = 0$, $\beta = 0$, $\gamma = 0$, it follows from (3.61) that $\sum_{k=0}^{\infty} E(||x_k||^2) < \infty$ and therefore, the controlled system is mean square exponentially stable for all of the criteria given in Tab.3.1.

α	β	γ	Performance criteria
1	0	<0	H_{∞} Design
0	1	0	Passivity Design
0	1	>0	Input Strict Passivity Design
>0	1	0	Output Strict Passivity Design
>0	1	>0	Very Strict Passivity

Tab.3.1. Various performance criteria in a general framework

3.3.2 State Dependent LMI Control of Nonlinear Stochastic Systems with Actuator Failures

The following theorem provides us a novel approach for controlling the nonlinear stochastic system with actuator failures.

Theorem 3.3: Given the system model (3.54), performance output (3.58), control input (3.59) and performance index (3.61), if there exist matrices $M_k = P_k^{-1} > 0$ and Y_k for all $k \ge 0$, such that the following State Dependent LMI holds:

$$\begin{bmatrix} M_{k} & \Phi_{12} & \Phi_{13} & \Phi_{14} & Y_{k}^{T} \\ \Phi_{21} & \Phi_{22} & F_{k}^{T} & 0 & 0 \\ \Phi_{31} & F_{k} & M_{k} & 0 & 0 \\ \Phi_{41} & 0 & 0 & I & 0 \\ Y_{k} & 0 & 0 & 0 & \Phi_{L} \end{bmatrix} > 0$$
(3.62)

where

$$\Phi_{12} = -\alpha \left[C_k M_k + D_k Y_k \right]^T G_k - \frac{\beta}{2} \left[C_k M_k + D_k Y_k \right]^T$$

$$\begin{split} \Phi_{13} &= \left[A_{k} M_{k} + B_{k} \overline{\Gamma}_{k} Y_{k} \right]^{T} \\ \Phi_{14} &= \alpha^{1/2} \left[C_{k} M_{k} + D_{k} Y_{k} \right]^{T} \\ \Phi_{21} &= -\alpha G_{k}^{T} \left[C_{k} M_{k} + D_{k} Y_{k} \right] - \frac{\beta}{2} \left[C_{k} M_{k} + D_{k} Y_{k} \right] \\ \Phi_{22} &= -\alpha G_{k}^{T} G_{k} - \frac{\beta}{2} \left[G_{k}^{T} + G_{k} \right] - \gamma I \\ \Phi_{31} &= \left[A_{k} M_{k} + B_{k} \overline{\Gamma}_{k} Y_{k} \right] \\ \Phi_{41} &= \alpha^{1/2} \left[C_{k} M_{k} + D_{k} Y_{k} \right] \\ \Phi_{L} &= \lambda_{\min} \left(M_{k} \right) \left[\overline{\Gamma}_{k}^{T} B_{k}^{T} B_{k} \overline{\Gamma}_{k} \right]^{-1} = \lambda_{\min} \left(M_{k} \right) Z_{k} = \lambda_{\min} \left(M_{k} \right) \left[\Upsilon \otimes \left(B_{k}^{T} B_{k} \right) \right]^{-1} \\ Z_{k} &= \left\{ E \left[\widetilde{\Gamma}_{k}^{T} B_{k}^{T} B_{k} \overline{\Gamma}_{k} \right] \right\}^{-1} = \left[\Upsilon \otimes \left(B_{k}^{T} B_{k} \right) \right]^{-1} \\ \Upsilon &= diag \left\{ \left[\pi_{1} \left(1 - \pi_{1} \right), \dots, \pi_{m} \left(1 - \pi_{m} \right) \right] \right\} = \left[\begin{array}{c} \pi_{1} \left(1 - \pi_{1} \right) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \pi_{m} \left(1 - \pi_{m} \right) \end{array} \right] \end{split}$$
(3.63)
$$M_{k+1} \ge M_k \tag{3.64}$$

then the general performance inequality (3.61) is satisfied. The nonlinear feedback gain of the controller is given by $K_k = Y_k \cdot P_k$ (3.65)

Proof

Applying system model (3.54), performance output (3.58) and control input (3.59) to inequality (3.61), we have

$$E_{\Gamma} \left\{ E_{x} \left[\begin{bmatrix} A_{k}x_{k} + \\ B_{k}\Gamma_{k}K_{k}x_{k} \\ +F_{k}w_{k} \end{bmatrix}^{T} P_{k+1} \begin{bmatrix} A_{k}x_{k} + \\ B_{k}\Gamma_{k}K_{k}x_{k} \\ +F_{k}w_{k} \end{bmatrix} | x_{k}, x_{k-1}, \dots, x_{0} \right] \right\}$$

$$-x_{k}^{T}P_{k}x_{k} + \alpha \left[C_{k}x_{k} + D_{k}K_{k}x_{k} + G_{k}w_{k} \right]^{T} \left[C_{k}x_{k} + D_{k}K_{k}x_{k} + G_{k}w_{k} \right]$$

$$+\beta \left[C_{k}x_{k} + D_{k}K_{k}x_{k} + G_{k}w_{k} \right]^{T} w_{k} + \gamma w_{k}^{T}w_{k} < 0 \qquad (3.66)$$

By adding and subtracting the same term, we have the following equivalent inequality

$$E_{\Gamma} \left\{ E_{x} \left[\begin{bmatrix} A_{k}x_{k} + \\ B_{k}\Gamma_{k}K_{k}x_{k} \\ +F_{k}w_{k} \end{bmatrix}^{T} \begin{pmatrix} P_{k} + \\ (P_{k+1} - P_{k}) \end{pmatrix} \begin{bmatrix} A_{k}x_{k} + \\ B_{k}\Gamma_{k}K_{k}x_{k} \\ +F_{k}w_{k} \end{bmatrix} x_{k}, x_{k-1}, ..., x_{0} \right] \right\}$$

$$-x_{k}^{T}P_{k}x_{k} + \alpha \left[C_{k}x_{k} + D_{k}K_{k}x_{k} + G_{k}w_{k} \right]^{T} \left[C_{k}x_{k} + D_{k}K_{k}x_{k} + G_{k}w_{k} \right]$$

$$+\beta \left[C_{k}x_{k} + D_{k}K_{k}x_{k} + G_{k}w_{k} \right]^{T} w_{k} + \gamma w_{k}^{T}w_{k} < 0 \qquad (3.67)$$

To satisfy strict dissipation of energy, we must impose $P_{k+1} < P_k$. The sufficient condition to (3.67) is

$$E_{\Gamma} \left\{ E_{x} \left[\begin{bmatrix} A_{k} x_{k} + \\ B_{k} \Gamma_{k} K_{k} x_{k} \\ + F_{k} w_{k} \end{bmatrix}^{T} P_{k} \begin{bmatrix} A_{k} x_{k} + \\ B_{k} \Gamma_{k} K_{k} x_{k} \\ + F_{k} w_{k} \end{bmatrix} | x_{k}, x_{k-1}, \dots, x_{0} \right] \right\}$$
$$-x_{k}^{T} P_{k} x_{k} + \alpha \left[C_{k} x_{k} + D_{k} K_{k} x_{k} + G_{k} w_{k} \right]^{T} \left[C_{k} x_{k} + D_{k} K_{k} x_{k} + G_{k} w_{k} \right]$$
$$+\beta \left[C_{k} x_{k} + D_{k} K_{k} x_{k} + G_{k} w_{k} \right]^{T} w_{k} + \gamma w_{k}^{T} w_{k} < 0$$

(3.68)

and

By expanding the terms, we have

$$E_{\Gamma} \left\{ E_{x} \left[\left[A_{k} x_{k} + B_{k} \Gamma_{k} K_{k} x_{k} \right]^{T} P_{k} \left[A_{k} x_{k} + B_{k} \Gamma_{k} K_{k} x_{k} \right] | x_{k}, x_{k-1}, ..., x_{0} \right] \right\} \\ + E_{\Gamma} \left\{ E_{x} \left[\left[A_{k} x_{k} + B_{k} \Gamma_{k} K_{k} x_{k} \right]^{T} P_{k} \left[F_{k} w_{k} \right] | x_{k}, x_{k-1}, ..., x_{0} \right] \right\} \\ + E_{\Gamma} \left\{ E_{x} \left[\left[F_{k} w_{k} \right]^{T} P_{k} \left[A_{k} x_{k} + B_{k} \Gamma_{k} K_{k} x_{k} \right] | x_{k}, x_{k-1}, ..., x_{0} \right] \right\} \\ + w_{k}^{T} F_{k}^{T} P_{k} F_{k} w_{k} - x_{k}^{T} P_{k} x_{k} + \alpha \left[C_{k} x_{k} + D_{k} K_{k} x_{k} \right]^{T} \left[C_{k} x_{k} + D_{k} K_{k} x_{k} \right] \\ + \alpha \left[C_{k} x_{k} + D_{k} K_{k} x_{k} \right]^{T} \left[G_{k} w_{k} \right] + \alpha \left[G_{k} w_{k} \right]^{T} \left[C_{k} x_{k} + D_{k} K_{k} x_{k} \right] \\ + \alpha \left[G_{k} w_{k} \right]^{T} \left[G_{k} w_{k} \right] + \beta \left[C_{k} x_{k} + D_{k} K_{k} x_{k} \right]^{T} w_{k} \\ + \gamma w_{k}^{T} w_{k} < 0 \end{cases}$$

$$(3.69)$$

By applying the smoothing property of expectations

$$E\{E\{x|y\}\} = E\{x\},$$
 (3.70)

Inequality (3.69) yields the following:

$$\begin{aligned} x_{k}^{T}E_{\Gamma}\left[\left[A_{k}+B_{k}\Gamma_{k}K_{k}\right]^{T}P_{k}\left[A_{k}+B_{k}\Gamma_{k}K_{k}\right]\right]x_{k} \\ +x_{k}^{T}\left[A_{k}+B_{k}\overline{\Gamma}_{k}K_{k}\right]^{T}P_{k}F_{k}w_{k}+w_{k}^{T}F_{k}^{T}P_{k}\left[A_{k}+B_{k}\overline{\Gamma}_{k}K_{k}\right]x_{k} \\ +w_{k}^{T}F_{k}^{T}P_{k}F_{k}w_{k}-x_{k}^{T}P_{k}x_{k} \\ +\alpha x_{k}^{T}\left[C_{k}+D_{k}K_{k}\right]^{T}\left[C_{k}+D_{k}K_{k}\right]x_{k} \\ +\alpha x_{k}^{T}\left[C_{k}+D_{k}K_{k}\right]^{T}G_{k}w_{k}+\alpha w_{k}^{T}G_{k}^{T}\left[C_{k}+D_{k}K_{k}\right]x_{k} \\ +\alpha w_{k}^{T}G_{k}^{T}G_{k}w_{k}+\beta x_{k}^{T}\left[C_{k}+D_{k}K_{k}\right]^{T}w_{k}+\beta w_{k}^{T}G_{k}^{T}w_{k}+\gamma w_{k}^{T}w_{k}<0\end{aligned}$$

$$(3.71)$$

Notice that the first term in the inequality (3.71) can be written as

$$E_{\Gamma}\left[\left[A_{k}+B_{k}\Gamma_{k}K_{k}\right]^{T}P_{k}\left[A_{k}+B_{k}\Gamma_{k}K_{k}\right]\right]$$

$$=E_{\Gamma}\left[\left[A_{k}+B_{k}\left(\overline{\Gamma}_{k}+\widetilde{\Gamma}_{k}\right)K_{k}\right]^{T}P_{k}\left[A_{k}+B_{k}\left(\overline{\Gamma}_{k}+\widetilde{\Gamma}_{k}\right)K_{k}\right]\right]$$

$$=\left[A_{k}+B_{k}\overline{\Gamma}_{k}K_{k}\right]^{T}P_{k}\left[A_{k}+B_{k}\overline{\Gamma}_{k}K_{k}\right]+K_{k}^{T}E_{\Gamma}\left[\widetilde{\Gamma}_{k}^{T}B_{k}^{T}P_{k}B_{k}\widetilde{\Gamma}_{k}\right]K_{k}$$

$$(3.72)$$

Hence, we have

$$\begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} < 0$$
(3.73)

$$\Lambda_{11} = \begin{bmatrix} A_k + B_k \overline{\Gamma}_k K_k \end{bmatrix}^T P_k \begin{bmatrix} A_k + B_k \overline{\Gamma}_k K_k \end{bmatrix} + K_k^T E_{\Gamma} \begin{bmatrix} \widetilde{\Gamma}_k^T B_k^T P_k B_k \widetilde{\Gamma}_k \end{bmatrix} K_k$$

$$-P_k + \alpha \begin{bmatrix} C_k + D_k K_k \end{bmatrix}^T \begin{bmatrix} C_k + D_k K_k \end{bmatrix}$$

$$\Lambda_{12} = \begin{bmatrix} A_k + B_k \overline{\Gamma}_k K_k \end{bmatrix}^T P_k F_k + \alpha \begin{bmatrix} C_k + D_k K_k \end{bmatrix}^T G_k + \frac{\beta}{2} \begin{bmatrix} C_k + D_k K_k \end{bmatrix}^T$$

$$\Lambda_{21} = F_k^T P_k \begin{bmatrix} A_k + B_k \overline{\Gamma}_k K_k \end{bmatrix} + \alpha G_k^T \begin{bmatrix} C_k + D_k K_k \end{bmatrix} + \frac{\beta}{2} \begin{bmatrix} C_k + D_k K_k \end{bmatrix}$$

$$\Lambda_{22} = F_k^T P_k F_k + \alpha G_k^T G_k + \beta G_k^T + \gamma I$$
(3.74)

Equivalently, we have

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} A_k + B_k \overline{\Gamma}_k K_k \end{bmatrix}^T \\ F_k^T \end{bmatrix} P_k \begin{bmatrix} \begin{bmatrix} A_k + B_k \overline{\Gamma}_k K_k \end{bmatrix} F_k \end{bmatrix} > 0$$
(3.75)

where

$$\Xi_{11} = P_k - K_k^T E_{\Gamma} \Big[\tilde{\Gamma}_k^T B_k^T P_k B_k \tilde{\Gamma}_k \Big] K_k - \alpha \Big[C_k + D_k K_k \Big]^T \Big[C_k + D_k K_k \Big]$$

$$\Xi_{12} = -\alpha \Big[C_k + D_k K_k \Big]^T G_k - \frac{\beta}{2} \Big[C_k + D_k K_k \Big]^T$$

$$\Xi_{21} = -\alpha G_k^T \Big[C_k + D_k K_k \Big] - \frac{\beta}{2} \Big[C_k + D_k K_k \Big]$$

$$\Xi_{22} = -\alpha G_k^T G_k - \frac{\beta}{2} \Big[G_k^T + G_k \Big] - \gamma I$$
(3.76)

By applying Schur complement, we have

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \left[A_k + B_k \overline{\Gamma}_k K_k\right]^T \\ \Omega_{21} & \Omega_{22} & F_k^T \\ \left[A_k + B_k \overline{\Gamma}_k K_k\right] & F_k & P_k^{-1} \end{bmatrix} > 0$$
(3.77)

$$\Omega_{11} = P_k - K_k^T E_{\Gamma} \Big[\tilde{\Gamma}_k^T B_k^T P_k B_k \tilde{\Gamma}_k \Big] K_k - \alpha \Big[C_k + D_k K_k \Big]^T \Big[C_k + D_k K_k \Big]$$

$$\Omega_{12} = -\alpha \Big[C_k + D_k K_k \Big]^T G_k - \frac{\beta}{2} \Big[C_k + D_k K_k \Big]^T$$

$$\Omega_{21} = -\alpha G_k^T \Big[C_k + D_k K_k \Big] - \frac{\beta}{2} \Big[C_k + D_k K_k \Big]$$

$$\Omega_{22} = -\alpha G_k^T G_k - \frac{\beta}{2} \Big[G_k^T + G_k \Big] - \gamma I$$
(3.78)

Denote $M_k = P_k^{-1}, K_k = Y_k P_k$. (3.79)

By pre- and post-multiplying the matrix inequality with the block diagonal matrix $diag\{M_k, I, I\}$, we have

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \begin{bmatrix} A_k M_k + B_k \overline{\Gamma}_k Y_k \end{bmatrix}^T \\ \Lambda_{21} & \Lambda_{22} & F_k^T \\ \begin{bmatrix} A_k M_k + B_k \overline{\Gamma}_k Y_k \end{bmatrix} & F_k & M_k \end{bmatrix} > 0$$
(3.80)

where

$$\Lambda_{11} = M_k - Y_k^T E_{\Gamma} \Big[\tilde{\Gamma}_k^T B_k^T P_k B_k \tilde{\Gamma}_k \Big] Y_k - \alpha \Big[C_k M_k + D_k Y_k \Big]^T \Big[C_k M_k + D_k Y_k \Big]$$

$$\Lambda_{12} = -\alpha \Big[C_k M_k + D_k Y_k \Big]^T G_k - \frac{\beta}{2} \Big[C_k M_k + D_k Y_k \Big]^T$$

$$\Lambda_{21} = -\alpha G_k^T \Big[C_k M_k + D_k Y_k \Big] - \frac{\beta}{2} \Big[C_k M_k + D_k Y_k \Big]$$

$$\Lambda_{22} = -\alpha G_k^T G_k - \frac{\beta}{2} \Big[G_k^T + G_k \Big] - \gamma I$$
(3.81)

By applying Schur complement twice, we have

$$\begin{bmatrix} M_{k} & \Phi_{12} & \Phi_{13} & \Phi_{14} & Y_{k}^{T} \\ \Phi_{21} & \Phi_{22} & F_{k}^{T} & 0 & 0 \\ \Phi_{31} & F_{k} & M_{k} & 0 & 0 \\ \Phi_{41} & 0 & 0 & I & 0 \\ Y_{k} & 0 & 0 & 0 & \Phi_{55} \end{bmatrix} > 0$$

$$(3.82)$$

where

$$\Phi_{12} = -\alpha \left[C_k M_k + D_k Y_k \right]^T G_k - \frac{\beta}{2} \left[C_k M_k + D_k Y_k \right]^T$$

$$\Phi_{13} = \left[A_k M_k + B_k \overline{\Gamma}_k Y_k \right]^T$$

$$\Phi_{14} = \alpha^{1/2} \left[C_k M_k + D_k Y_k \right]^T$$

$$\Phi_{21} = -\alpha G_k^T \left[C_k M_k + D_k Y_k \right] - \frac{\beta}{2} \left[C_k M_k + D_k Y_k \right]$$

$$\Phi_{22} = -\alpha G_k^T G_k - \frac{\beta}{2} \left[G_k^T + G_k \right] - \gamma I$$

$$\Phi_{31} = \left[A_k M_k + B_k \overline{\Gamma}_k Y_k \right]$$

$$\Phi_{41} = \alpha^{1/2} \left[C_k M_k + D_k Y_k \right]$$

$$\Phi_{55} = \left[E_{\Gamma} \left[\widetilde{\Gamma}_k^T B_k^T P_k B_k \widetilde{\Gamma}_k \right] \right]^{-1}$$
(3.83)

According to Rayleigh's Inequality, the following inequality holds

$$\Phi_{55} = \left[E_{\Gamma} \left[\tilde{\Gamma}_{k}^{T} B_{k}^{T} P_{k} B_{k} \tilde{\Gamma}_{k} \right] \right]^{-1} \ge \left[\lambda_{\max} \left(P_{k} \right) E_{\Gamma} \left[\tilde{\Gamma}_{k}^{T} B_{k}^{T} B_{k} \tilde{\Gamma}_{k} \right] \right]^{-1} = \lambda_{\min} \left(M_{k} \right) \left[\overline{\tilde{\Gamma}_{k}^{T} B_{k}^{T} B_{k} \tilde{\Gamma}_{k}} \right]^{-1}$$

$$(3.84)$$

Denote
$$\Phi_L = \lambda_{\min} (M_k) Z_k = \lambda_{\min} (M_k) \left[\overline{\Gamma_k^T B_k^T B_k \widetilde{\Gamma}_k} \right]^{-1}$$
 (3.85)

Replacing Φ_{55} with Φ_L , we have the sufficient condition as follows:

$$\begin{bmatrix} M_{k} & \Phi_{12} & \Phi_{13} & \Phi_{14} & Y_{k}^{T} \\ \Phi_{21} & \Phi_{22} & F_{k}^{T} & 0 & 0 \\ \Phi_{31} & F_{k} & M_{k} & 0 & 0 \\ \Phi_{41} & 0 & 0 & I & 0 \\ Y_{k} & 0 & 0 & 0 & \Phi_{L} \end{bmatrix} > 0$$

$$(3.86)$$

Denote $Z_k = \left\{ E \left[\tilde{\Gamma}_k^T B_k^T B_k \tilde{\Gamma}_k \right] \right\}^{-1} = \left[\Upsilon \otimes \left(B_k^T B_k \right) \right]^{-1}$ (3.87) where

$$\Upsilon = diag\left\{ \left[\pi_1 (1 - \pi_1), \dots, \pi_m (1 - \pi_m) \right] \right\} = \begin{bmatrix} \pi_1 (1 - \pi_1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \pi_m (1 - \pi_m) \end{bmatrix}$$
(3.88)

which concludes the proof.

Remark 1: If Z_k in (3.86) fails to be invertible, then one can replace it with $Z_k + \varepsilon I$ where $0 < \varepsilon <<1$ to make it invertible. With this new term, (3.86) is still a sufficient condition for this control solution.

3.3.3 Application to the Inverted Pendulum on a Cart

The inverted pendulum on a cart problem is used for testing the effectiveness and robustness of the novel state dependent linear matrix inequality control for nonlinear system with actuator failures. A dynamical model of the inverted pendulum on a cart without actuator failure has been derived previously in (3.16). Considering the effect of independent actuator failure, for discrete time state space model, we denote the following state variables:

$$x_{1,k} = x(kT), x_{2,k} = \dot{x}(kT), x_{3,k} = \theta(kT), x_{4,k} = \theta(kT)$$

By applying Euler discretization method with sampling period T, and using the notation

$$\Omega_{1} = I + mL^{2} - \frac{m^{2}L^{2}\cos^{2}(x_{3,k})}{M + m}$$

$$\Omega_{2} = M + m - \frac{m^{2}L^{2}\cos^{2}(x_{3,k})}{I + mL^{2}}$$
(3.89)

the discrete-time system equation can be written as (3.90), with the Bernoulli distributed actuator failure rate $(1-\gamma_1)$.

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \\ x_{4,k+1} \end{bmatrix} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 1 & T \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix} \gamma_1 u_k + \begin{bmatrix} w_k^1 \\ w_k^2 \\ w_k^3 \\ w_k^4 \end{bmatrix}$$
(3.90)

where

 u_k is the k^{th} sampling instant value of the input force F and

$$a_{22} = 1 + T \frac{-b}{\Omega_2}$$

$$a_{23} = T \frac{m^2 L^2 g \cos(x_{3,k})}{\Omega_2 (I + mL^2)} \frac{\sin(x_{3,k})}{x_{3,k}}$$

$$a_{24} = T \frac{mL \sin(x_{3,k})}{\Omega_2} x_{4,k}$$

$$a_{42} = T \frac{mLb \cos(x_{3,k})}{(M + m)\Omega_1}$$

$$a_{43} = -T \frac{mgL}{\Omega_1} \frac{\sin(x_{3,k})}{x_{3,k}}$$

$$a_{44} = 1 - T \frac{m^2 L^2 \cos(x_{3,k}) \sin(x_{3,k}) x_{4,k}}{(M + m)\Omega_1}$$

$$b_2 = \frac{T}{\Omega_2}$$

$$b_4 = -T \frac{mL \cos(x_{3,k})}{(M + m)\Omega_1}$$
(3.91)

The following system parameters are assumed

$$M = 0.5kg, m = 0.5kg, b = 0.1N \cdot \frac{\sec}{m}, L = 0.3m, I = 0.06kg \cdot m^{2}$$

sampling time T = 0.01

The mean values of γ_1 are $\pi_1 = 0.9$.

Measurement noise covariance matrix
$$W_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times (0.9^k).$$

The following design parameters are chosen to satisfy very strict passivity performance:

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, \alpha = 0.01, \beta = 1, \gamma = 0.01$$

The following initial conditions are assumed

$$x_1 = 1, x_2 = 0, x_3 = \pi / 4, x_4 = 0$$

To demonstrate effectiveness and robustness of the proposed method under different actuator failure rates, the conditions when mean values of γ_1 : $\pi_1 = 0.9$ and $\pi_2 = 0.6$ are simulated respectively. Simulation results are shown in the Fig. 3.16-3.17. It can be seen that the new method effectively stabilizes the inverted pendulum on a cart in the case of actuators with different failure rates.



Fig.3.12. Position trajectory of the inverted pendulum



Fig.3.13. Velocity trajectory of the inverted pendulum



Fig.3.14. Angle "theta" trajectory of the inverted pendulum



Fig.3.15. Angular velocity trajectory of the inverted pendulum



Fig.3.16. Control input when $\pi_1 = 0.6$



Fig.3.17. Control input when $\pi_1 = 0.9$

3.4 Summary

Chapter 3 presents novel nonlinear state feedback control approaches based on State Dependent Linear Matrix Inequality. General performance criteria are used to design the controller and relative weighting of these criteria can be achieved by choosing different coefficient matrices. The benchmark inverted pendulum on a cart problem is used as an example to demonstrate the effectiveness of the control technique. Moreover, the novel state dependent control approach for nonlinear systems with random actuator failures is proposed in Section 3.3. The proposed methods provide us powerful alternatives to the existing nonlinear control approaches.

CHAPTER 4 ROBUST AND RESILIENT STATE DEPENDENT LINEAR MATRIX INEQUALITY CONTROL OF NONLINEAR SYSTEMS WITH GENERAL PERFORMANCE CRITERIA

Chapter 2 has briefly discussed the dissipative system theory, which is an important concept in systems, circuits and controls. The theory of dissipative systems generalizes the basic tools including the passivity theorem, bounded real lemma, Kalman-Yakubovich lemma and circle criterion. Dissipativity performance includes H_{∞} performance, passivity, positive realness, and sector bounded constraint as special case. In Chapter 3, State Dependent Linear Matrix Inequality control approaches have been introduced for nonlinear systems, which characterize the characteristics of traditional Hamilton Jacobi Equations.

Based on the theoretical analysis in the previous two chapters, we will further investigate the robust and resilient State Dependent Linear Matrix Inequality control techniques in Chapter 4, thereby extending the results in Chapter 3. The control objective is to design a controller which is optimally robust for model uncertainties and resilient against control feedback gain perturbations in achieving general performance criteria to secure quadratic optimality with inherent asymptotic stability property together with quadratic dissipative type of disturbance reduction. For the uncertain nonlinear systems model, we consider a general form of L_2 -bounded uncertainty description, without any standard structure, incorporating commonly used types of uncertainty, such as normbounded and positive real uncertainties as special cases. By solving State Dependent Linear Matrix Inequality, the sufficient condition for control solution can be found satisfying the general performance criteria [85, 86].

4.1 Robust and Resilient State Dependent LMI Control of Continuous Time Nonlinear Systems

In this section, we discuss a novel state feedback control problem for continuous time nonlinear control systems using the State Dependent Linear Matrix Inequalities approach, which is robust for model uncertainties and resilient for gain perturbations. We characterize the solution of the continuous time nonlinear system with State Dependent LMIs, which are essentially equivalent to the classical Hamilton Jacobi Inequalities. We further propose to employ general performance criteria to design the controller guaranteeing the quadratic sub-optimality with inherent stability property in combination with dissipativity type of disturbance attenuation. The results of this section unify existing results on nonlinear quadratic regulator, H_{∞} and positive real control to provide a more flexible and less conservative robust control design. The effectiveness of the proposed technique is demonstrated by simulations of the nonlinear control of inverted pendulum on a cart system [85].

4.1.1 System Model and General Performance Criteria

Consider the following nonlinear dynamical system model and performance output equation

$$\dot{x} = f(x(t), u(t), w(t)) = (A(x,t) + \Delta_A(x,t))x(t) + (B(x,t) + \Delta_B(x,t)) \cdot u(t) + (E(x,t) + \Delta_E(x,t))w(t)$$
(4.1)

$$z(t) = g(x(t), w(t)) = C \cdot x(t) + D \cdot w(t)$$

$$(4.2)$$

where

 $x(t) \in \Re^n$ state variable of dynamical system $u(t) \in \Re^m$ applied input $w(t) \in \Re^p$ L_2 type of disturbance $z(t) \in \Re^r$ performance output function smooth real vector function f,g $A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$, $E \in \mathfrak{R}^{n \times p}$, $C \in \mathfrak{R}^{r \times n}$, $D \in \mathfrak{R}^{r \times p}$ state-dependent coefficient matrices $\Delta_A \in \mathfrak{R}^{n \times n}, \Delta_B \in \mathfrak{R}^{n \times m}, \Delta_E \in \mathfrak{R}^{n \times p}$ time-varying state dependent uncertainty matrices, which may be due to modeling errors

It is assumed that the state feedback is available and the state feedback control input is given by $u(t) = \left(K(x,t) + \Delta_{K}(x,t)\right)x(t)$ (4.3)

where Δ_{K} represents the perturbation on the given matrix which may be due to computational errors or drifts in coefficient values.

The optimal control problem we consider is to determine an admissible control u to satisfy the performance objective (2.46). The general performance criteria is given as

$$\dot{V} + x^T Q x + u^T R u - (z^T M z + 2z^T S w + w^T N w) \le 0, \forall T \ge 0$$
 (2.46)

Before introducing the main result of the paper, the following model of uncertainties is introduced.

Assumption 4.1: The following general form of L_2 -bounded unstructured uncertainties is considered:

$$\Delta_A \Delta_A^T \le \gamma_A I; \Delta_B \Delta_B^T \le \gamma_B I; \Delta_E \Delta_E^T \le \gamma_E I; \Delta_K \Delta_K^T \le \gamma_K I$$
(4.4)

for $\forall x \in \Re^n$ and $t \ge 0$.

4.1.2 Robust and Resilient State Dependent LMI Control for Continuous Time Nonlinear System

The following theorem summarizes the main result of this section:

Theorem 4.1: Given the system model (4.1), performance output (4.2) and control equation (4.3), if there exist matrices $X = P^{-1} > 0$ and Y for all t > 0, such that the following state dependent Linear Matrix Inequality holds:

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & X & Y^T \\ * & \Upsilon_{22} & 0 & 0 \\ * & * & \Upsilon_{33} & 0 \\ * & * & * & \Upsilon_{44} \end{bmatrix} < 0$$
(4.5)

where

$$\begin{split} \Upsilon_{11} &= W + \left[\gamma_{A} + 2\gamma_{B} + \gamma_{E} \right] I + BB^{T} = XA^{T} + AX + Y^{T}B^{T} + BY + BB^{T} + \left[\gamma_{A} + 2\gamma_{B} + \gamma_{E} \right] I \\ \Upsilon_{12} &= E - XC^{T}MD - XC^{T}S \\ \Upsilon_{22} &= -D^{T}MD - 2D^{T}S - N + I \\ \Upsilon_{33} &= -\left\{ I + \left[3 + \lambda_{\max} \left(R \right) \right] \gamma_{K}I + Q + C^{T}MC \right\}^{-1} \\ \Upsilon_{44} &= -\left\{ I + R^{2} + R \right\}^{-1} \end{split}$$
(4.6)

Then the performance index inequality (2.46) is satisfied. The nonlinear feedback control gain is given by $K = Y \cdot P$ (4.7)

Proof

In the proof below, the time and state argument will be dropped for notational simplicity. By applying system and performance output equations (4.1), (4.2), and state feedback input equation (4.3), the performance index can be formed as follows:

$$x^{T} \left\{ A + \Delta_{A} + \left(B + \Delta_{B} \right) \left(K + \Delta_{K} \right) \right\}^{T} Px + w^{T} \left[E + \Delta_{E} \right]^{T} Px + x^{T} P \left\{ A + \Delta_{A} + \left(B + \Delta_{B} \right) \left(K + \Delta_{K} \right) \right\} x + x^{T} P \left[E + \Delta_{E} \right] w + x^{T} \dot{P} x + x^{T} Qx + x^{T} \left[K + \Delta_{K} \right]^{T} R \left[K + \Delta_{K} \right] x - \left[Cx + Dw \right]^{T} M \left[Cx + Dw \right] - 2 \left[Cx + Dw \right]^{T} Sw - w^{T} Nw < 0$$

$$(4.8)$$

Equivalently,

$$\begin{bmatrix} x^T & w^T \end{bmatrix} \Psi \begin{bmatrix} x & w \end{bmatrix}^T = \begin{bmatrix} x^T & w^T \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$
(4.9)

where

$$\Psi_{11} = \dot{P} + \left[A + \Delta_A + \left(B + \Delta_B\right)\left(K + \Delta_K\right)\right]^T P + P\left[A + \Delta_A + \left(B + \Delta_B\right)\left(K + \Delta_K\right)\right] + Q + \left[K + \Delta_K\right]^T R\left[K + \Delta_K\right] + C^T M C$$

$$\Psi_{12} = P\left[E + \Delta_E\right] - C^T M D - C^T S$$

$$\Psi_{22} = -D^T M D - 2D^T S - N$$
(4.10)

Pre-multiplying and post-multiplying the matrix Ψ with the block $diag\{X,I\}$, where $X = P^{-1}$, $Y = K \cdot P^{-1} = KX$. Then the following matrix inequality holds:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ * & \Lambda_{22} \end{bmatrix} < 0 \tag{4.11}$$

where

$$\Lambda_{11} = X \Big[A + \Delta_A + (B + \Delta_B) (K + \Delta_K) \Big]^T + \Big[A + \Delta_A + (B + \Delta_B) (K + \Delta_K) \Big] X$$

+ $X\dot{P}X + XQX + X \Big[K + \Delta_K \Big]^T R \Big[K + \Delta_K \Big] X + XC^T MCX$
$$\Lambda_{12} = \Big[E + \Delta_E \Big] - XC^T MD - XC^T S$$

$$\Lambda_{22} = -D^T MD - 2D^T S - N$$
(4.12)

Applying Lemma 1.2, we have $X\dot{P}X < 0$.

Denote
$$W = XA^T + AX + Y^TB^T + BY$$
 (4.13)

The sufficient condition for matrix inequality (4.11) to be held is to change term $\Lambda_{_{11}}$ as

follows:

$$\Lambda_{11} = XA^{T} + AX + Y^{T}B^{T} + BY + X \{\Delta_{A} + \Delta_{B}K + B\Delta_{K} + \Delta_{B}\Delta_{K}\}^{T} + \{\Delta_{A} + \Delta_{B}K + B\Delta_{K} + \Delta_{B}\Delta_{K}\}X + XQX + Y^{T}RY + X\Delta_{K}^{T}RY + Y^{T}R\Delta_{K}X + X\Delta_{K}^{T}R\Delta_{K}X + XC^{T}MCX$$

$$= W + \{\Delta_{A} + \Delta_{B}K\}X + X \{\Delta_{A} + \Delta_{B}K\}^{T} + X \{B\Delta_{K} + \Delta_{B}\Delta_{K}\}^{T} + \{B\Delta_{K} + \Delta_{B}\Delta_{K}\}X + \{X\Delta_{K}^{T}RY + Y^{T}R\Delta_{K}X\} + \{XQX + Y^{T}RY + X\Delta_{K}^{T}R\Delta_{K}X + XC^{T}MCX\}$$

$$(4.14)$$

By applying Lemma 1.1 to (4.14) and using Assumption 4.1, we obtain

$$\{\Delta_{A} + \Delta_{B}K\} X + X \{\Delta_{A} + \Delta_{B}K\}^{T} = X \begin{bmatrix} I & K^{T} \end{bmatrix} \begin{bmatrix} \Delta_{A}^{T} \\ \Delta_{B}^{T} \end{bmatrix} + \begin{bmatrix} \Delta_{A} & \Delta_{B} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} X$$

$$\leq \alpha_{1} \begin{bmatrix} \Delta_{A} & \Delta_{B} \end{bmatrix} \begin{bmatrix} \Delta_{A}^{T} \\ \Delta_{B}^{T} \end{bmatrix} + \alpha_{1}^{-1}X \begin{bmatrix} I & K^{T} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} X \leq \alpha_{1} (\gamma_{A} + \gamma_{B})I + \alpha_{1}^{-1}X \begin{bmatrix} I & K^{T} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} X$$

$$X \Delta_{K}^{T}B^{T} + B\Delta_{K}X \leq \alpha_{2}X \Delta_{K}^{T}\Delta_{K}X + \alpha_{2}^{-1}BB^{T} \leq \alpha_{2}\gamma_{K}X^{2} + \alpha_{2}^{-1}BB^{T}$$

$$X \Delta_{K}^{T}\Delta_{B}^{T} + \Delta_{B}\Delta_{K}X \leq \alpha_{3}X \Delta_{K}\Delta_{K}^{T}X + \alpha_{3}^{-1}\Delta_{B}\Delta_{B}^{T} \leq \alpha_{3}\gamma_{K}X^{2} + \alpha_{3}^{-1}\gamma_{B}I$$

$$X \Delta_{K}^{T}RY + Y^{T}R\Delta_{K}X \leq \alpha_{4}X \Delta_{K}^{T}\Delta_{K}X + \alpha_{4}^{-1}Y^{T}R^{2}Y \leq \alpha_{4}\gamma_{K}X^{2} + \alpha_{4}^{-1}Y^{T}R^{2}Y$$

$$X \Delta_{K}^{T}R\Delta_{K}X \leq X \Delta_{K}^{T}\Delta_{K}X \cdot \lambda_{max}(R) \leq \lambda_{max}(R)\gamma_{K}X^{2}$$

$$(4.15)$$

Therefore, we have

$$\Lambda_{11} \leq W + \alpha_1 \left(\gamma_A + \gamma_B \right) I + \alpha_1^{-1} X \begin{bmatrix} I & K^T \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} X + \alpha_2 \gamma_K X^2 + \alpha_2^{-1} B B^T + \alpha_3 \gamma_K X^2 + \alpha_3^{-1} \gamma_B I$$

+ $\alpha_4 \gamma_K X^2 + \alpha_4^{-1} Y^T R^2 Y + \left\{ X Q X + Y^T R Y + \lambda_{\max} \left(R \right) \gamma_K X^2 + X C^T M C X \right\}$
$$\Lambda_{12} = \begin{bmatrix} E + \Delta_E \end{bmatrix} - X C^T M D - X C^T S$$

$$\Lambda_{22} = -D^T M D - 2D^T S - N$$
(4.16)

Using Lemma 1.1 and Assumption 4.1, we have

$$\begin{bmatrix} 0 & \Delta_E \\ \Delta_E^T & 0 \end{bmatrix} \leq \begin{bmatrix} \alpha_5 \Delta_E \Delta_E^T & 0 \\ 0 & \alpha_5^{-1}I \end{bmatrix} \leq \begin{bmatrix} \alpha_5 \gamma_E I & 0 \\ 0 & \alpha_5^{-1}I \end{bmatrix}$$
(4.17)

Therefore, applying (4.17), (4.11) is implied by:

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix} < 0 \tag{4.18}$$

where

$$\begin{split} \Theta_{11} &= W + \left[\alpha_{1} \gamma_{A} + \alpha_{1} \gamma_{B} + \alpha_{5} \gamma_{E} + \alpha_{3}^{-1} \gamma_{B} \right] I + \alpha_{1}^{-1} X \left[I \quad K^{T} \right] \begin{bmatrix} I \\ K \end{bmatrix} X + \alpha_{4}^{-1} Y^{T} R^{2} Y \\ &+ \left[\alpha_{2} + \alpha_{3} + \alpha_{4} + \lambda_{\max} \left(R \right) \right] \gamma_{K} X^{2} + \alpha_{2}^{-1} B B^{T} + \left\{ X Q X + Y^{T} R Y + X C^{T} M C X \right\} \\ &= W + \left[\alpha_{1} \gamma_{A} + \alpha_{1} \gamma_{B} + \alpha_{5} \gamma_{E} + \alpha_{3}^{-1} \gamma_{B} \right] I \\ &+ X \left\{ \alpha_{1}^{-1} I + \left[\alpha_{2} + \alpha_{3} + \alpha_{4} + \lambda_{\max} \left(R \right) \right] \gamma_{K} I + Q + C^{T} M C \right\} X + Y^{T} \left\{ \alpha_{1}^{-1} + \alpha_{4}^{-1} R^{2} + R \right\} Y + \alpha_{2}^{-1} B B^{T} \\ \Theta_{12} &= E - X C^{T} M D - X C^{T} S \\ \Theta_{22} &= -D^{T} M D - 2D^{T} S - N + \alpha_{5}^{-1} I \end{split}$$

$$(4.19)$$

By applying the Schur complement, we have the final linear matrix inequality solution

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & X & Y^{T} \\ * & \Upsilon_{22} & 0 & 0 \\ * & * & \Upsilon_{33} & 0 \\ * & * & * & \Upsilon_{44} \end{bmatrix} < 0$$
(4.20)

where

$$\begin{split} \Upsilon_{11} &= W + \left[\alpha_{1} \gamma_{A} + \alpha_{1} \gamma_{B} + \alpha_{5} \gamma_{E} + \alpha_{3}^{-1} \gamma_{B} \right] I + \alpha_{2}^{-1} B B^{T} \\ &= X A^{T} + A X + Y^{T} B^{T} + B Y + \alpha_{2}^{-1} B B^{T} + \left[\alpha_{1} \gamma_{A} + \alpha_{1} \gamma_{B} + \alpha_{5} \gamma_{E} + \alpha_{3}^{-1} \gamma_{B} \right] I \\ \Upsilon_{12} &= E - X C^{T} M D - X C^{T} S \\ \Upsilon_{22} &= -D^{T} M D - 2 D^{T} S - N + \alpha_{5}^{-1} I \\ \Upsilon_{33} &= -\left\{ \alpha_{1}^{-1} I + \left[\alpha_{2} + \alpha_{3} + \alpha_{4} + \lambda_{\max} \left(R \right) \right] \gamma_{K} I + Q + C^{T} M C \right\}^{-1} \\ \Upsilon_{44} &= -\left\{ \alpha_{1}^{-1} + \alpha_{4}^{-1} R^{2} + R \right\}^{-1} \end{split}$$
(4.21)

Since positive constants $\alpha_1, ..., \alpha_5$ are arbitrary, choosing all of them as 1, we obtain (4.5). Therefore, if LMI (4.5) holds, the inequality (2.46) is satisfied. This concludes the proof of the theorem.

Remark 4.1: At this point, it is to be noted that other choices of constants $\alpha_1, ..., \alpha_5$ are

possible and can be tried if the value 1 for all these constants does not work.

4.1.3 Application to the Inverted Pendulum on a Cart

The inverted pendulum on a cart problem is used for testing the novel robust and resilient State Dependent LMI approach with this system to compare the performance.

The following system parameters are assumed:

$$M = 0.5kg, m = 0.5kg, b = 0.1N \cdot \frac{\sec}{m}, L = 0.3m, I = 0.06kg \cdot m^2$$

Sampling Time: $T = 0.01 \sec$

Denote the following state variables:

$$x_1 = x(t), x_2 = \dot{x}(t), x_3 = \theta(t), x_4 = \theta(t)$$

The following initial conditions are assumed:

$$x_1 = 1, x_2 = 0, x_3 = \pi / 4, x_4 = 0$$

The following design parameters are chosen to satisfy different mixed criteria:

Mixed NLQR- H_{∞} Design (Predominant NLQR)

$$C = [0.01 \quad 0.01 \quad 0.01], D = [0.01], Q = I_4, R = 1, M = 1, S = 0, N = -1$$

Mixed NLQR- H_{∞} Design (Predominant H_{∞})

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = 0.01 \times I_4, R = 0.01, M = 1, S = 0, N = -10$$

Mixed NLQR- H_{∞} -Positive Real Design (NLQR-Passivity)

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = I_4, R = 1, M = 0.01, S = 1, N = 0.01$$

All of the above mixed criteria control performance results are shown in the Figs.4.1-4.5, in comparison with the traditional Linear Quadratic Regulator (LQR) technique based on linearization. From these figures, we find that the novel State Dependent LMI control has

better performance compared with the traditional LQR technique based on linearization. Especially, Fig.4.3 and Fig.4.4 show that the traditional LQR technique loses control of the anlge and angular velocity of the pendulum, respectively. Fig.4.5 shows that the highest magnitude of control is needed by the predominant H_{∞} control and the lowest control magnitude is needed by the linearization based LQR technique.



Fig.4.1. Position trajectory of the inverted pendulum





Fig.4.3. Angle "theta" trajectory of the inverted pendulum



Fig.4.4. Angular velocity trajectory of the inverted pendulum



Fig.4.5. Control input

4.2 Robust and Resilient State Dependent LMI Control of Discrete Time Nonlinear Systems

In this section, we discuss a novel state feedback control problem for discrete time nonlinear control systems using the State Dependent Linear Matrix Inequalities approach, which is robust for model uncertainties and resilient for gain perturbations. We characterize the solution of the nonlinear discrete time control system with State Dependent LMI, which is essentially equivalent to the classical Hamilton Jacobi Inequalities. We further propose to employ general performance criteria to design the controller guaranteeing the quadratic sub-optimality with inherent stability property in combination with dissipativity type of disturbance attenuation. The results of this section unify existing results on nonlinear quadratic regulator, H_{∞} and positive real control to provide a more flexible robust control design. The effectiveness of the proposed technique is demonstrated by simulations of the nonlinear control of the inverted pendulum on a cart system [86].

4.2.1 System Model and General Performance Criteria

Consider the following nonlinear dynamical system equation and performance output equation

$$\begin{aligned} x_{k+1} &= f\left(x_{k}, u_{k}, w_{k}\right) \\ &= \left(A(x_{k}) + \Delta_{A}(x_{k})\right)x_{k} + \left(B(x_{k}) + \Delta_{B}(x_{k})\right) \cdot u_{k} + \left(E\left(x_{k}\right) + \Delta_{E}\left(x_{k}\right)\right)w_{k} \end{aligned}$$

$$= \left(A_{k} + \Delta_{A}\right)x_{k} + \left(B_{k} + \Delta_{B}\right)u_{k} + \left(E_{k} + \Delta_{E}\right)w_{k}$$

$$(4.22)$$

$$z_k = g(x_k, u_k) = C_k \cdot x_k + D_k \cdot w_k \tag{4.23}$$

where

$x_k \in \mathfrak{R}^n$	state variable of dynamical system
$u_k \in \Re^m$	applied input
$W_k \in \mathfrak{R}^p$	l_2 type of disturbance
$\boldsymbol{Z}_k \in \boldsymbol{\mathfrak{R}}^r$	performance output function
f,g	smooth real vector function

 $A_k \in \Re^{n \times n} \,, \; B_k \in \Re^{n \times m} \,, \; E_k \in \Re^{n \times p} \,, \; C_k \in \Re^{r \times n} \,, \; D_k \in \Re^{r \times p}$

state-dependent coefficient matrices

$$\Delta_A \in \mathfrak{R}^{n \times n}, \ \Delta_B \in \mathfrak{R}^{n \times m}, \ \Delta_E \in \mathfrak{R}^{n \times p}$$

time-varying uncertainty matrices timevarying state dependent uncertainty matrices, which may be due to modeling errors

It is assumed that the state feedback is available and the state feedback control input is

given by
$$u_k = \left(K(x_k) + \Delta_K(x_k)\right) x_k = \left(K_k + \Delta_K\right) x_k$$
(4.24)

The optimal control problem we consider is to determine an admissible control u to satisfy the performance objective (2.56). The general performance criteria is given as

$$V_{k+1} - V_k + x_k^T Q x_k + u_k^T R u_k - \left(z_k^T M z_k + 2 z_k^T S w_k + w_k^T N w_k \right) \le 0, \forall k \ge 0$$
(2.56)

Before introducing the main result of the paper, the following model of uncertainties is introduced.

Assumption 4.2: The following general form of l_2 -bounded unstructured uncertainties are

considered:
$$\Delta_A \Delta_A^T \le \gamma_A I; \Delta_B \Delta_B^T \le \gamma_B I; \Delta_E \Delta_E^T \le \gamma_E I; \Delta_K \Delta_K^T \le \gamma_K I$$
 (4.25)

for $\forall x_k \in \Re^n$ and $k \ge 0$.

4.2.2 Robust and Resilient State Dependent LMI Control for Discrete Time Nonlinear System

The following theorem summarizes the main results of this section:

Theorem 4.2: Given the system model (4.22), performance output (4.23) and control equation (4.24), if there exist matrices $X_k = P_k^{-1} > 0$ and Y_k for all k > 0, such that the following state dependent linear matrix inequality holds:

If
$$M < 0$$
, then
$$\begin{bmatrix}
X_{k} & \Upsilon_{12} & \Upsilon_{13} & Y_{k}^{T} & \Upsilon_{15} & X_{k} \\
* & \Upsilon_{22} & E^{T} & 0 & 0 & 0 \\
* & * & \Upsilon_{33} & 0 & 0 & 0 \\
* & * & * & \Upsilon_{44} & 0 & 0 \\
* & * & * & * & \Upsilon_{55} & 0 \\
* & * & * & * & * & \Upsilon_{66}
\end{bmatrix} > 0$$
(4.26)
If $M = 0$, then
$$\begin{bmatrix}
X_{k} & \Upsilon_{12} & \Upsilon_{13} & Y_{k}^{T} & X_{k} \\
* & \Upsilon_{22} & E^{T} & 0 & 0 \\
* & * & \Upsilon_{33} & 0 & 0 \\
* & * & * & \Upsilon_{44} & 0 \\
* & * & * & & \Upsilon_{66}
\end{bmatrix} > 0$$
(4.27)

where

$$\begin{split} \Upsilon_{12} &= X_{k}C_{k}^{T}MD_{k} + X_{k}C_{k}^{T}S \\ \Upsilon_{13} &= X_{k}A_{k}^{T} + Y_{k}^{T}B_{k}^{T} \\ \Upsilon_{15} &= X_{k}C_{k}^{T} \\ \Upsilon_{22} &= D_{k}^{T}S + S^{T}D_{k} + D_{k}^{T}MD_{k} + N + I \\ \Upsilon_{33} &= X_{k} + (2\gamma_{B} + \gamma_{E} + 1)I + B_{k}B_{k}^{T} \\ \Upsilon_{44} &= R^{-1} \\ \Upsilon_{55} &= -M^{-1} \\ \Upsilon_{66} &= Q^{-1} - (\gamma_{A} + 2\gamma_{K})^{-1}I \end{split}$$
(4.28)

Then the performance index inequality (2.56) is satisfied. The nonlinear feedback control

gain is given by

$$K_k = Y_k \cdot P_k \tag{4.29}$$

Proof

In the proof below, the time and state argument will be dropped for notational simplicity. By applying system and performance output equations (4.22), (4.23), and state feedback input equation (4.24), the performance index can be formed as follows:

$$\left\{ x_k^T \left[A_k + \Delta_A + \left(B_k + \Delta_B \right) \left(K_k + \Delta_K \right) \right]^T + w_k^T \left[E_k + \Delta_E \right]^T \right\} \cdot P_{k+1} \cdot \left\{ \left[A_k + \Delta_A + \left(B_k + \Delta_B \right) \left(K_k + \Delta_K \right) \right] x_k + \left[E_k + \Delta_E \right] w_k \right\} + \left[-x_k^T P_k x_k + x_k^T Q x_k + x_k^T \left[K_k + \Delta_K \right]^T R \left[K_k + \Delta_K \right] x_k - \left[C_k x_k + D w_k \right]^T M \left[C_k x_k + D w_k \right] - 2 \left[C_k x_k + D_k w_k \right]^T S w_k - w_k^T N w_k \le 0$$

$$(4.30)$$

which can be expanded as

$$x_{k}^{T} \{ (A_{k} + \Delta_{A}) + (B_{k} + \Delta_{B})(K_{k} + \Delta_{K}) \}^{T} \cdot P_{k+1} \cdot \{ (A_{k} + \Delta_{A}) + (B_{k} + \Delta_{B})(K_{k} + \Delta_{K}) \} x_{k} + w_{k}^{T} [E_{k} + \Delta_{E}]^{T} P_{k+1} \{ (A_{k} + \Delta_{A}) + (B_{k} + \Delta_{B})(K_{k} + \Delta_{K}) \} x_{k} + x_{k}^{T} \{ (A_{k} + \Delta_{A}) + (B_{k} + \Delta_{B})(K_{k} + \Delta_{K}) \}^{T} P_{k+1} [E_{k} + \Delta_{E}] w_{k} + w_{k}^{T} [E_{k} + \Delta_{E}]^{T} P_{k+1} [E_{k} + \Delta_{E}] w_{k} + w_{k}^{T} [E_{k} + \Delta_{E}]^{T} P_{k+1} [E_{k} + \Delta_{E}] w_{k} + x_{k}^{T} P_{k} x_{k} + x_{k}^{T} Q x_{k} + x_{k}^{T} [K_{k} + \Delta_{K}]^{T} R [K_{k} + \Delta_{K}] x_{k} - [C_{k} x_{k} + D_{k} w_{k}]^{T} M [C_{k} x_{k} + D_{k} w_{k}] - 2 [C_{k} x_{k} + D_{k} w_{k}]^{T} S w_{k} - w_{k}^{T} N w_{k} \leq 0$$

$$(4.31)$$

Equivalently,

$$\begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \Psi \begin{bmatrix} x_k & w_k \end{bmatrix}^T = \begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \le 0$$
(4.32)

where

$$\Psi_{11} = \left\{ \left(A_{k} + \Delta_{A} \right) + \left(B_{k} + \Delta_{B} \right) \left(K_{k} + \Delta_{K} \right) \right\}^{T} \cdot P_{k+1} \cdot \left\{ \left(A_{k} + \Delta_{A} \right) + \left(B_{k} + \Delta_{B} \right) \left(K_{k} + \Delta_{K} \right) \right\} + Q - P_{k} + \left[K_{k} + \Delta_{K} \right]^{T} R \left[K_{k} + \Delta_{K} \right] - C_{k}^{T} M C_{k}$$

$$\Psi_{12} = \left\{ \left(A_{k} + \Delta_{A} \right) + \left(B_{k} + \Delta_{B} \right) \left(K_{k} + \Delta_{K} \right) \right\}^{T} P_{k+1} \left[E_{k} + \Delta_{E} \right] - C_{k}^{T} M D_{k} - C_{k}^{T} S$$

$$\Psi_{22} = \left[E_{k} + \Delta_{E} \right]^{T} P_{k+1} \left[E_{k} + \Delta_{E} \right] - D_{k}^{T} M D_{k} - \left(D_{k}^{T} S + S^{T} D_{k} \right) - N$$
(4.33)

Denote the following terms:

$$A = (A_k + \Delta_A) + (B_k + \Delta_B)(K_k + \Delta_K)$$

$$K = K_k + \Delta_K$$

$$E = E_k + \Delta_E$$

(4.34)

Then (4.32) is equivalent to

$$\begin{bmatrix} A^{T}P_{k+1}A - P_{k} & A^{T}P_{k+1}E \\ * & E^{T}P_{k+1}E \end{bmatrix} + \begin{bmatrix} Q + K^{T}RK - C_{k}^{T}MC_{k} & -C_{k}^{T}MD_{k} - C_{k}^{T}S \\ * & -D_{k}^{T}S - S^{T}D_{k} - D_{k}^{T}MD_{k} - N \end{bmatrix} \leq 0$$

$$(4.35)$$

By adding and subtracting P_k term, we have

$$\begin{bmatrix} \mathbf{A}^{T} \\ \mathbf{E}^{T} \end{bmatrix} (P_{k+1} - P_{k} + P_{k}) \begin{bmatrix} \mathbf{A} & \mathbf{E} \end{bmatrix} - \begin{bmatrix} I \\ \mathbf{0} \end{bmatrix} P_{k} \begin{bmatrix} I & \mathbf{0} \end{bmatrix} + \begin{bmatrix} Q + \mathbf{K}^{T} R \mathbf{K} - C_{k}^{T} M C_{k} & -C_{k}^{T} M D_{k} - C_{k}^{T} S \\ * & -D_{k}^{T} S - S^{T} D_{k} - D_{k}^{T} M D_{k} - N \end{bmatrix} \leq 0$$

$$(4.36)$$

Imposing the property $P_{k+1} \le P_k$, the sufficient condition for (4.36) is given as follows:

$$\begin{bmatrix} \mathbf{A}^{T} \\ \mathbf{E}^{T} \end{bmatrix} P_{k} \begin{bmatrix} \mathbf{A} & \mathbf{E} \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix} P_{k} \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} Q + \mathbf{K}^{T} R \mathbf{K} - C_{k}^{T} M C_{k} & -C_{k}^{T} M D_{k} - C_{k}^{T} S \\ * & -D_{k}^{T} S - S^{T} D_{k} - D_{k}^{T} M D_{k} - N \end{bmatrix} \leq 0$$

$$(4.37)$$

Hence, we have

$$\begin{bmatrix} \left(\mathbf{A}^{T} P_{k} \mathbf{A} - P_{k} + Q + \mathbf{K}^{T} R \mathbf{K} - C_{k}^{T} M C_{k}\right) & \left(\mathbf{A}^{T} P_{k} \mathbf{E} - C_{k}^{T} M D_{k} - C_{k}^{T} S\right) \\ * & \left(\mathbf{E}^{T} P_{k} \mathbf{E} - D_{k}^{T} S - S^{T} D_{k} - D_{k}^{T} M D_{k} - N\right) \end{bmatrix} < 0$$
(4.38)

Equivalently, we have

$$\begin{bmatrix} P_{k} - Q - \mathbf{K}^{T}R\mathbf{K} + C_{k}^{T}MC_{k} & C_{k}^{T}MD_{k} + C_{k}^{T}S \\ * & D_{k}^{T}S + S^{T}D_{k} + D_{k}^{T}MD_{k} + N \end{bmatrix} - \begin{bmatrix} \mathbf{A}^{T} \\ \mathbf{E}^{T} \end{bmatrix} P_{k} \begin{bmatrix} \mathbf{A} & \mathbf{E} \end{bmatrix} > 0$$

$$(4.39)$$

Applying Schur complement, we have

$$\begin{bmatrix} \left(P_{k} - Q - \mathbf{K}^{T}R\mathbf{K} + C_{k}^{T}MC_{k}\right) & C_{k}^{T}MD_{k} + C_{k}^{T}S & \mathbf{A}^{T} \\ * & \left(D_{k}^{T}S + S^{T}D_{k} + D_{k}^{T}MD_{k} + N\right) & \mathbf{E}^{T} \\ * & * & P_{k}^{-1} \end{bmatrix} > 0 \quad (4.40)$$

Taking M < 0 (the case where M = 0 will be considered later), we apply Schur complement twice to (4.40), then

$$\begin{bmatrix} P_{k} - Q & C_{k}^{T} M D_{k} + C_{k}^{T} S & A^{T} & K^{T} & C_{k}^{T} \\ * & D_{k}^{T} S + S^{T} D_{k} + D_{k}^{T} M D_{k} + N & E^{T} & 0 & 0 \\ * & * & P_{k}^{-1} & 0 & 0 \\ * & * & * & R^{-1} & 0 \\ * & * & * & * & -M^{-1} \end{bmatrix} > 0$$
(4.41)

Let $X_k = P_k^{-1}$, by pre- and post-multiplying the above matrix inequality by $diag \{X_k \ I \ I \ I \ I\}$, we have

$$\begin{bmatrix} X_{k} - X_{k}QX_{k} & X_{k}C_{k}^{T}MD_{k} + X_{k}C_{k}^{T}S & X_{k}A^{T} & X_{k}K^{T} & X_{k}C_{k}^{T} \\ * & D_{k}^{T}S + S^{T}D_{k} + D_{k}^{T}MD_{k} + N & E^{T} & 0 & 0 \\ * & * & X_{k} & 0 & 0 \\ * & * & * & R^{-1} & 0 \\ * & * & * & * & -M^{-1} \end{bmatrix} > 0 \quad (4.42)$$

By applying Schur complement again, we have

$$\begin{bmatrix} X_{k} & X_{k}C_{k}^{T}MD_{k} + X_{k}C_{k}^{T}S & X_{k}A^{T} & X_{k}K^{T} & X_{k}C^{T} & X_{k} \\ * & D_{k}^{T}S + S^{T}D_{k} + D_{k}^{T}MD_{k} + N & E^{T} & 0 & 0 & 0 \\ * & * & X_{k} & 0 & 0 & 0 \\ * & * & * & R^{-1} & 0 & 0 \\ * & * & * & * & R^{-1} & 0 \\ * & * & * & * & R^{-1} & 0 \\ * & * & * & * & * & Q^{-1} \end{bmatrix} > 0$$
(4.43)

Denote $Y_k = K_k X_k$

(4.44)

By replacing the variables using (4.34), we have

$$\begin{bmatrix} X_{k} \begin{pmatrix} X_{k}C_{k}^{T}MD_{k} + \\ X_{k}C_{k}^{T}S \end{pmatrix} & X_{k} \begin{bmatrix} (A_{k} + \Delta_{A}) + \\ (B_{k} + \Delta_{B})(K_{k} + \Delta_{K}) \end{bmatrix}^{T} & X_{k} \begin{bmatrix} X_{k} - X_{k} \\ X_{k}C_{k}^{T} & X_{k} \\ * \begin{pmatrix} D_{k}^{T}S + S^{T}D_{k} + \\ D_{k}^{T}MD_{k} + N \end{pmatrix} & \begin{bmatrix} E_{k} + \Delta_{E} \end{bmatrix}^{T} & 0 & 0 & 0 \\ * & * & X_{k} & 0 & 0 & 0 \\ * & * & & R^{-1} & 0 & 0 \\ * & * & & * & -M^{-1} & 0 \\ * & * & & * & & * & Q^{-1} \end{bmatrix} > 0$$

$$(4.45)$$

Equivalently, we have

By applying Lemma 1.1 and Assumption 4.2, we have

$$\begin{bmatrix} 0 & 0 & X_{k} \Delta_{K}^{T} \Delta_{B}^{T} & X_{k} \Delta_{K}^{T} & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} = \begin{bmatrix} X_{k} \Delta_{K}^{T} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ \Delta_{B} \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} X_{k} \Delta_{K}^{T} \\ 0 \\ \Delta_{B} \\ I \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} X_{k} \Delta_{K}^{T} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{1} \gamma_{K} X_{k} X_{k} & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & \alpha_{1}^{-1} \gamma_{B} I & 0 & 0 & 0 \\ * & * & * & * & \alpha_{1}^{-1} I & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$$

$$(4.47)$$

Similarly, by applying Lemma 1.1 and Assumption 4.2, we have

$$\begin{bmatrix} 0 & 0 & X_{k}K^{T}\Delta_{B}^{T} & 0 & 0 & 0 \\ * & 0 & \Delta_{E}^{T} & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ \end{bmatrix} = \begin{bmatrix} 0 & 0 & Y_{k}^{T}\Delta_{B}^{T} & 0 & 0 & 0 \\ * & 0 & \Delta_{E}^{T} & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$$

$$\begin{bmatrix} Y_{k}^{T} & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \Delta_{B}^{T} & 0 & 0 & 0 \\ 0 & 0 & \Delta_{E}^{T} & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta_{B} & \Delta_{E} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_{k} & 0 & 0 & 0 & 0 & 0 \\ Y_{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_{2}Y_{k}^{T}Y_{k} & 0 & 0 & 0 & 0 & 0 \\ * & \alpha_{2}I & 0 & 0 & 0 & 0 \\ * & * & \alpha_{2}^{-1}(\gamma_{B} + \gamma_{E})I & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$$

$$(4.48)$$

$$\begin{bmatrix} 0 & 0 & X_{k} \Delta_{A}^{T} & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \leq \begin{bmatrix} \alpha_{3} \gamma_{A} X_{k} X_{k} & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & * & \alpha_{3}^{-1} I & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & X_{k} \Delta_{K}^{T} B_{k}^{T} & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix} \leq \begin{bmatrix} \alpha_{4} \gamma_{K} X_{k} X_{k} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$$

$$(4.49)$$

By applying (4.47)-(4.50), the sufficient condition for inequality (4.46) is given below

$$\begin{bmatrix} X_{k} & \left(X_{k}C_{k}^{T}MD_{k}+X_{k}C_{k}^{T}S\right) & \left(X_{k}A_{k}^{T}+Y_{k}^{T}B_{k}^{T}\right) & X_{k}K_{k}^{T} & X_{k}C_{k}^{T} & X_{k}\\ * & \left(D_{k}^{T}S+S^{T}D_{k}+D_{k}^{T}MD_{k}+N\right) & E_{k}^{T} & 0 & 0 & 0\\ * & * & X_{k} & 0 & 0 & 0\\ * & * & * & R^{-1} & 0 & 0\\ * & * & * & * & -M^{-1} & 0\\ * & & * & * & * & Q^{-1} \end{bmatrix}$$
(4.51)
$$+\begin{bmatrix} \Omega_{11} & 0 & 0 & 0 & 0 & 0\\ * & \Omega_{22} & 0 & 0 & 0 & 0\\ * & \Omega_{33} & 0 & 0 & 0\\ * & * & & A_{1}^{-1}I & 0 & 0\\ * & * & * & * & 0 & 0\\ * & * & * & * & 0 & 0\\ * & * & * & * & 0 & 0 \end{bmatrix} > 0$$

where

$$\Omega_{11} = (\alpha_1 \gamma_K + \alpha_1 \gamma_A + \alpha_4 \gamma_K) X_k X_k + \alpha_2 Y_k^T Y_k$$

$$\Omega_{22} = \alpha_2 I$$

$$\Omega_{33} = \alpha_1^{-1} \gamma_B I + \alpha_2^{-1} (\gamma_B + \gamma_E) I + \alpha_3^{-1} I + \alpha_4^{-1} B_k B_k^T$$
(4.52)

Finally, by applying Schur complement twice, we have

$$\begin{bmatrix} X_{k} & \Upsilon_{12} & \Upsilon_{13} & Y_{k}^{T} & \Upsilon_{15} & X_{k} \\ * & \Upsilon_{22} & E^{T} & 0 & 0 & 0 \\ * & * & \Upsilon_{33} & 0 & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 & 0 \\ * & * & * & * & \Upsilon_{55} & 0 \\ * & * & * & * & * & \Upsilon_{56} \end{bmatrix} > 0$$

$$(4.53)$$

where

$$\begin{split} \Upsilon_{12} &= X_{k}C_{k}^{T}MD_{k} + X_{k}C_{k}^{T}S \\ \Upsilon_{13} &= X_{k}A_{k}^{T} + Y_{k}^{T}B_{k}^{T} \\ \Upsilon_{15} &= X_{k}C_{k}^{T} \\ \Upsilon_{22} &= D_{k}^{T}S + S^{T}D_{k} + D_{k}^{T}MD_{k} + N + \alpha_{2}I \\ \Upsilon_{33} &= X_{k} + \alpha_{1}^{-1}\gamma_{B}I + \alpha_{2}^{-1}(\gamma_{B} + \gamma_{E})I + \alpha_{3}^{-1}I + \alpha_{4}^{-1}B_{k}B_{k}^{T} \\ \Upsilon_{44} &= R^{-1} + (\alpha_{1}^{-1} - \alpha_{2}^{-1})I \\ \Upsilon_{55} &= -M^{-1} \\ \Upsilon_{66} &= Q^{-1} - (\alpha_{1}\gamma_{A} + \alpha_{2}\gamma_{K} + \alpha_{4}\gamma_{K})^{-1}I \end{split}$$
(4.54)

Notice that (4.53) is derived under the condition that M < 0. However, when strict positive realness criteria are chosen for control design, then condition M = 0 must be satisfied. In this case, LMI condition (4.53) should be replaced by

$$\begin{bmatrix} X_{k} & \Upsilon_{12} & \Upsilon_{13} & Y_{k}^{T} & X_{k} \\ * & \Upsilon_{22} & E^{T} & 0 & 0 \\ * & * & \Upsilon_{33} & 0 & 0 \\ * & * & * & \Upsilon_{44} & 0 \\ * & * & * & * & \Upsilon_{66} \end{bmatrix} > 0$$
(4.55)

Since positive constants $\alpha_1, ..., \alpha_5$ are arbitrary, choosing all of them as 1, we obtain (4.26) and (4.27). Therefore, if LMI (4.26) or (4.27) holds under different conditions on Q, the inequality (2.56) is satisfied.

Remark 4.5: At this point, it is to be noted that other choices of constants $\alpha_1, ..., \alpha_4$ are possible and can be tried if the value 1 for all these constants does not work.

4.2.3 Application to the Inverted Pendulum on a Cart

We test the novel robust and resilient State Dependent LMI approach with the inverted pendulum on a cart to compare the performance of different controllers.

The following system parameters are assumed

$$M = 0.5kg, m = 0.5kg, b = 0.1N \cdot \frac{\sec}{m}, L = 0.3m, I = 0.06kg \cdot m^2$$

Sampling Time: $T = 0.01 \sec$

Denote the following state variables:

$$x_{1,k} = x(kT), x_{2,k} = \dot{x}(kT), x_{3,k} = \theta(kT), x_{4,k} = \theta(kT)$$

The following initial conditions are assumed:

$$x_1 = 1, x_2 = 0, x_3 = \pi / 4, x_4 = 0$$

The following design parameters are chosen to satisfy different mixed criteria:

Mixed NLQR- H_{∞} Design (Predominant NLQR)

$$C = \begin{bmatrix} 0.01 & 0.01 & 0.01 \end{bmatrix}, D = \begin{bmatrix} 0.01 \end{bmatrix}, Q = I_4, R = 1, M = -1, S = 0, N = 5$$

Mixed NLQR- H_{∞} Design (Predominant H_{∞})

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = 0.01 \times I_4, R = 0.01, M = -1, S = 0, N = 5$$

Mixed NLQR- H_{∞} -Positive Real Design (NLQR-Passivity)

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = I_4, R = 1, M = -0.01, S = 0.5, N = 0.01$$

All of the above mixed criteria control performance results are shown in the Figs.4.6-4.10, in comparison with the traditional Linear Quadratic Regulator (LQR)

technique based on linearization. From these figures, we find that the novel state dependent LMI control has better performance compared with the traditional LQR technique based on linearization. Especially, Fig.4.6 and Fig.4.7 show that the traditional LQR technique loses control of the position and velocity of the cart, respectively. It should also be noted that predominant NLQR and predominant H_{∞} control techniques lead to faster response times than the NLQR-passivity technique. Fig.4.10 shows that the highest magnitude of control is needed by the predominant H_{∞} control and the lowest control magnitude is needed by the linearization based LQR technique.



Fig.4.6. Position trajectory of the inverted pendulum



Fig.4.7. Velocity trajectory of the inverted pendulum



Fig.4.8. Angle "theta" trajectory of the inverted pendulum



Fig.4.9. Angular velocity trajectory of the inverted pendulum



Fig.4.10. Control input
4.3 Summary

This chapter addresses nonlinear system control design with general nonlinear quadratic regulator and quadratic dissipative criteria to achieve asymptotic stability, quadratic optimality and strict quadratic dissipativeness. For systems with unstructured but bounded uncertainty, the Linear Matrix Inequality based sufficient conditions are derived for the solution of general performance criteria control. The relative weighting matrices of these criteria can be achieved by choosing different coefficient matrices. The optimal control can be obtained by solving LMI. The inverted pendulum on a cart is used as examples to demonstrate the effectiveness and robustness of the proposed methods. The simulation studies show that the proposed methods provide satisfactory alternatives to the existing control techniques.

CHAPTER 5 $H_2 - H_{\infty}$ STATE DEPENDENT RICCATI EQUATION CONTROL APPROACH

The Hamilton Jacobi Equation (HJE) and Hamilton Jacobi Inequality (HJI) are first order partial differential equations and inequalities, which characterize the solutions of optimal control of nonlinear systems. However, it is well-known of the difficulty to solve HJE and HJI in closed forms for more than a few state variables. As powerful alternatives to HJE / HJI techniques, the State Dependent Linear Matrix Inequality (SDLMI) and the State Dependent Riccati Equation (SDRE) techniques provide very effective algorithms for synthesizing the nonlinear feedback controls. As discussed in Chapters 3 and 4, the futher enhancements developed for the SDLMI provide an effective method to synthesize nonlinear feedback control in achieving nonlinear quadratic regulator (NLQR), H_{∞} and positive realness performance criteria.

The SDRE control has emerged as general design method since the mid-1990s, to provide a systematic and effective control design framework for nonlinear systems. Motivated by linear quadratic regulator control by Algebraic Riccati Equation (ARE), *Cloutier et al.* extends the result to nonlinear quadratic regulator problem by using state dependent coefficient matrices [7, 8]. A discrete SDRE method is developed in [14]. Due to the computational advantage, stability and effectiveness in control, the SDRE method is of meaningful and practical importance and has a wide range of applications, including robotics, missiles, aircraft, satellite / spacecraft, unmanned aerial vehicles (UAVs), ship systems, autonomous underwater vehicles, automotives, process control, chaotic systems, biomedical systems, guidance and navigation, etc. A recent survey of the development of SDRE method can be found in [6].

Traditionally, the SDRE method has been used for solving the nonlinear quadratic regulation problem. In this chapter, novel $H_2 - H_{\infty}$ State Dependent Riccati Equation control approaches are presented with the purpose of providing a more general control framework for a nonlinear system. By solving the generalized SDRE, the optimal control solution is found to satisfy mixed performance criteria guaranteeing quadratic optimality with inherent stability property in combination with H_{∞} type of disturbance reduction. The effectiveness of the proposed technique is demonstrated by simulations involving the control of the inverted pendulum on a cart system.

5.1 $H_2 - H_{\infty}$ Control of Continuous Time Nonlinear Systems Using SDRE Approach

A novel $H_2 - H_{\infty}$ State Dependent Riccati Equation control approach is presented in this section with the purpose of providing a generalized control design framework for continuous time nonlinear systems. By solving the generalized Riccati Equations, the optimal control solution is found to satisfy mixed performance criteria guaranteeing quadratic optimality with inherent stability property in combination with H_{∞} type of disturbance reduction. The effectiveness of the proposed technique is demonstrated by simulations involving the control of inverted pendulum on a cart [88].

5.1.1 System Model and Performance Index

Consider the input affine continuous time nonlinear system given by the following differential equation:

$$\dot{x}(t) = A(x) \cdot x + B(x) \cdot u + F(x) \cdot w$$
(5.1)

where

$x \in \Re^n$	state of the dynamical system
$u \in \mathfrak{R}^m$	applied input
$w \in \Re^q$	L_2 type disturbance
A, B, F	known coefficient matrices of appropriate dimensions, which can be
	functions of x

The performance output $z \in \Re^p$ is generalized as follows:

$$z = C(x) \cdot x + D(x) \cdot u + G(x) \cdot w$$
(5.2)

where the coefficient matrices can also be functions of x.

It is assumed that the state feedback is available. Otherwise, estimated state variable can be obtained from nonlinear state estimator. The nonlinear state feedback control input is given by

$$u = K(x) \cdot x \tag{5.3}$$

The optimal control problem we consider is to determine an admissible control u to satisfy the performance objective (2.52). The mixed H_2 and H_{∞} performance criteria is given as

$$\dot{V} + x^T Q x + u^T R u + z^T z + \gamma \cdot w^T w \le 0$$

$$V(T) + \int_0^T \left[x^T Q x + u^T R u + z^T z \right] dt \le V(0) - \gamma \int_0^T w^T w \, dt$$
(2.52)

By properly specifying the values of the weighing matrices Q, R, C, D, G, this performance criterion can be used in nonlinear control design, which yields a mixed H_2 and H_{∞} performance.

5.1.2 $H_2 - H_{\infty}$ State Dependent Riccati Equation Control for Continuous Time

Nonlinear Systems

In the following, we will drop the argument x as we have done before to simplify the notation. The following theorem summarizes the main results of the paper:

Theorem 5.1: Given the system dynamics (5.1), performance output (5.2) and control input (5.3); the performance index (2.52) can be achieved by using the control feedback

$$K^{o} = -\left[R - D^{T}G\left(G^{T}G + \gamma I\right)^{-1}G^{T}D + D^{T}D\right]^{-1} \times \left[B^{T}P - D^{T}G\left(G^{T}G + \gamma I\right)^{-1}\left(PF + C^{T}G\right)^{T} + D^{T}C\right]$$
(5.4)

where P is obtained from the generalized SDRE equation:

$$\begin{bmatrix} PA + A^{T}P + Q + C^{T}C - (PF + C^{T}G)(G^{T}G + \gamma I)^{-1}(PF + C^{T}G)^{T} \end{bmatrix} - \begin{bmatrix} PB - (PF + C^{T}G)(G^{T}G + \gamma I)^{-1}G^{T}D + C^{T}D \end{bmatrix} \times \begin{bmatrix} R - D^{T}G(G^{T}G + \gamma I)^{-1}G^{T}D + D^{T}D \end{bmatrix}^{-1} \times \begin{bmatrix} B^{T}P - D^{T}G(G^{T}G + \gamma I)^{-1}(PF + C^{T}G)^{T} + D^{T}C \end{bmatrix} = 0$$

$$(5.5)$$

Proof

By applying system model (5.1), performance output (5.2) and control input (5.3), performance index (2.52) becomes

$$x^{T}P(Ax + BKx + Fw) + (Ax + BKx + Fw)^{T}Px + x^{T}\dot{P}x + x^{T}Qx$$

+ $x^{T}K^{T}RKx + (Cx + DKx + Gw)^{T}(Cx + DKx + Gw) + \gamma w^{T}w \le 0$
(5.6)

Equivalently, we have

$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \le 0$$
(5.7)

$$\Xi_{11} = P(A+BK) + (A+BK)^{T} P + \dot{P} + Q + K^{T}RK + (C+DK)^{T} (C+DK)$$

$$\Xi_{12} = PF + (C+DK)^{T} G$$

$$\Xi_{22} = G^{T}G + \gamma I$$
(5.8)

By applying the Schur complement result, we have the following inequality:

$$P(A+BK)+(A+BK)^{T}P+\dot{P}+Q+K^{T}RK+(C+DK)^{T}(C+DK)-\left[PF+(C+DK)^{T}G\right]^{X}\left[G^{T}G+\gamma I\right]^{-1}\left[PF+(C+DK)^{T}G\right]^{T} \leq 0$$
(5.9)

Equivalently, we have

$$P(A+BK) + (A+BK)^{T} P + Q + K^{T}RK + (C+DK)^{T} (C+DK) - [PF + (C+DK)^{T} G] \times [G^{T}G + \gamma I]^{-1} [PF + (C+DK)^{T} G]^{T} \le -\dot{P}$$
(5.10)

In order to guarantee the controller stability, the quadratic energy function $V = x^T P x$ must be decreasing. Therefore, we have $-\dot{P} > 0$. Since we are trying to minimize *P*, the minimum value is achieved when the inequality (5.10) is satisfied as the following equation:

$$P(A+BK)+(A+BK)^{T}P+Q+K^{T}RK+(C+DK)^{T}(C+DK)-$$

$$\left[PF+(C+DK)^{T}G\right]\times\left[G^{T}G+\gamma I\right]^{-1}\left[PF+(C+DK)^{T}G\right]^{T}=0$$
(5.11)

By collecting terms with the same powers of K, we have

$$\begin{bmatrix} PA + A^{T}P + Q + C^{T}C - (PF + C^{T}G)(G^{T}G + \gamma I)^{-1}(PF + C^{T}G)^{T} \end{bmatrix}$$

+
$$\begin{bmatrix} PB - (PF + C^{T}G)(G^{T}G + \gamma I)^{-1}G^{T}D + C^{T}D \end{bmatrix}K$$

+
$$K^{T}\begin{bmatrix} B^{T}P - D^{T}G(G^{T}G + \gamma I)^{-1}(PF + C^{T}G)^{T} + D^{T}C \end{bmatrix}$$

+
$$K^{T}\begin{bmatrix} R - D^{T}G(G^{T}G + \gamma I)^{-1}G^{T}D + D^{T}D \end{bmatrix}K = 0$$
(5.12)

(5.18)

Equivalently, (5.12) can be simplified as

$$0 = \Upsilon + K^T \Omega^T + \Omega K + K^T \Phi K$$
(5.13)

where

$$\Upsilon = \left[PA + A^{T}P + Q + C^{T}C - \left(PF + C^{T}G\right)\left(G^{T}G + \gamma I\right)^{-1}\left(PF + C^{T}G\right)^{T} \right]$$

$$\Omega = \left[PB - \left(PF + C^{T}G\right)\left(G^{T}G + \gamma I\right)^{-1}G^{T}D + C^{T}D \right]$$

$$\Phi = \left[R - D^{T}G\left(G^{T}G + \gamma I\right)^{-1}G^{T}D + D^{T}D \right]$$
(5.14)

By completing the squares in controller gain K, we have

$$0 = \Upsilon + \left(K - K^{o}\right)^{T} \Phi\left(K - K^{o}\right) - K^{oT} \Phi K^{o}$$
(5.15)

For (5.13) to be equal to (5.15), we must have

$$-K^{oT}\Phi K = \Omega K \tag{5.16}$$

Therefore, the optimal feedback gain is found to be

$$K^{o} = -\Phi^{-1}\Omega^{T} =$$

$$-\left[R - D^{T}G\left(G^{T}G + \gamma I\right)^{-1}G^{T}D + D^{T}D\right]^{-1}$$

$$\times \left[B^{T}P - D^{T}G\left(G^{T}G + \gamma I\right)^{-1}\left(PF + C^{T}G\right)^{T} + D^{T}C\right]$$
(5.17)

When $K = K^{\circ}$, the minimum *P* is defined by the positive definite solution of the following Generalized SDRE:

$$0 = \Upsilon - K^{o^{T}} \Phi K^{o}$$

= $\left[PA + A^{T}P + Q + C^{T}C - (PF + C^{T}G)(G^{T}G + \gamma I)^{-1}(PF + C^{T}G)^{T} \right] - \left[PB - (PF + C^{T}G)(G^{T}G + \gamma I)^{-1}G^{T}D + C^{T}D \right] \times \left[R - D^{T}G(G^{T}G + \gamma I)^{-1}G^{T}D + D^{T}D \right]^{-1} \times \left[B^{T}P - D^{T}G(G^{T}G + \gamma I)^{-1}(PF + C^{T}G)^{T} + D^{T}C \right]$

which concludes the proof.

Remark 5.1: As a special case, when no noise is present, then we do not have H_{∞} component in the performance index, i.e. nonlinear quadratic regulator control, then the following controller can be derived as a special case:

If we neglect the noise term, then system equation becomes

as

$$\dot{x} = A(x)x + B(x)u \tag{5.19}$$

For F and G identically equal to zero, the optimal feedback control gain can be derived

$$K^o = -R^{-1}B^T P \tag{5.20}$$

P is defined by the positive definite solution of the following State Dependent Riccati Equation:

$$0 = PA + A^{T}P + Q - PBR^{-1}B^{T}P$$
(5.21)

Therefore, the conventional SDRE solution proposed by *Cloutier et al.* [7, 8] is derived as a special case of our results.

Remark 5.2: Computationally, this method is fairly easy to implement, and standard SDRE solvers can be used to approach the solution of the generalized SDRE to find the optimal control gain. To facilitate the computation process, the following notation is introduced:

$$\Delta = \left(G^{T}G + \gamma I\right)^{-1}$$

$$\Gamma = \left[R - D^{T}G\left(G^{T}G + \gamma I\right)^{-1}G^{T}D + D^{T}D\right]^{-1} = \left[R - D^{T}G\Delta G^{T}D + D^{T}D\right]^{-1}$$

$$\Psi = B - F\left(G^{T}G + \gamma I\right)^{-1}G^{T}D = B - F\Delta G^{T}D$$

$$\Theta = -C^{T}G\left(G^{T}G + \gamma I\right)^{-1}G^{T}D + C^{T}D = -C^{T}G\Delta G^{T}D + C^{T}D$$

$$\alpha = F\Delta G^{T}C + \Psi\Gamma\Psi^{T}$$

$$\beta = C^{T}G\Delta G^{T}C + \Theta\Gamma\Theta^{T}$$
(5.22)

Then, the equivalent form of the generalized SDRE (5.18) is given as

$$0 = P[A - \alpha] + [A - \alpha]^{T} P + [Q - C^{T}C - \beta] - P[F\Delta F^{T} + \Psi\Gamma\Psi^{T}]P \qquad (5.23)$$

where the equivalent form of (5.17) is given as following:

$$K^{o} = -\Gamma \times \left[\Psi^{T} P + \Theta^{T} \right]$$
(5.24)

5.1.3 Application to the Inverted Pendulum on a Cart

The dynamics of the inverted pendulum problem can be found in (3.16). By choosing $x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta}$ and the state space model for the system can be written as (3.19). Assume the following system parameters are given

$$M = 0.5kg, m = 0.5kg, b = 0.1N \cdot \frac{\sec}{m}, L = 0.3m, I = 0.06kg \cdot m^2$$

The following design parameters are chosen:

Classical SDRE Design (NLQR only)

 $C = [1 \ 1 \ 1 \ 1], D = [1], Q = I_4, R = 1$

Mixed NLQR- H_{∞} Design (Predominant H_{∞})

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, G = \begin{bmatrix} 1 \end{bmatrix}, Q = 0.5 \times I_4, R = 0.5, \gamma = -0.9$$

Mixed NLQR- H_2 Design (Predominant H_2)

$$C = [0.8 \quad 0.8 \quad 0.8], D = [0.8], G = [0.1], Q = I_4, R = 1, \gamma = -0.2$$

Assume the initial condition

$$x_1 = 1, x_2 = 0, x_3 = \pi / 4, x_4 = 0$$

All of the above mixed criteria control performance results are shown in Fig.'s 5.1-5.5. It should be noted that the traditional Linear Quadratic Regulator (LQR) technique based on linearization loses control of this nonlinear systems. Simulation results show that the

classical SDRE control has the fastest response time. Predominant H_2 control shows very similar response with the classical SDRE control. Predominant H_{∞} shows better capability of disturbance rejection.



Fig.5.1. Position trajectory of the inverted pendulum



Fig.5.2. Velocity trajectory of the inverted pendulum



Fig.5.3. Angle "theta" trajectory of the inverted pendulum



Fig.5.4. Angular velocity trajectory of the inverted pendulum



Fig.5.5. Control input

5.2 $H_2 - H_{\infty}$ Control of Discrete Time Nonlinear Systems Using SDRE Approach

A novel $H_2 - H_{\infty}$ State Dependent Riccati Equation control approach is presented in this section with the purpose of providing a generalized control design framework for discrete time nonlinear systems. By solving the generalized Riccati Equations, the optimal control solution is found to satisfy mixed performance criteria guaranteeing quadratic optimality with inherent stability property in combination with H_{∞} type of disturbance reduction. In case when SDRE is not available to be solved, two powerful control alternatives: suboptimal $H_2 - H_{\infty}$ State Dependent Riccati Difference Equation control and the State Dependent Linear Matrix Inequality control are also proposed. The effectiveness of the proposed techniques is demonstrated by simulations involving the control of inverted pendulum [89].

5.2.1 System Model and Performance Index

Consider the input affine discrete time nonlinear system given by the following difference equation:

$$x_{k+1} = A(x_k)x_k + B(x_k)u_k + F(x_k)w_k$$

= $A_k x_k + B_k u_k + F_k w_k$ (5.25)

where

 $x_k \in \Re^n$: state vector

- $u_k \in \mathfrak{R}^m$: applied input
- $w_k \in \Re^q$: l_2 type of disturbance

 A_k, B_k, F_k : known coefficient matrices of appropriate dimensions, which can be functions of x_k

Note that the simplified notation for time varying matrices A_k, B_k , etc. is used as was done before to denote the state dependent matrices. The performance output function $z_k \in \Re^p$ is generalized as follows:

$$z_{k} = C(x_{k})x_{k} + D(x_{k})u_{k} + G(x_{k})w_{k} = C_{k}x_{k} + D_{k}u_{k} + G_{k}w_{k}$$
(5.26)

where C_k , D_k are state dependent coefficient matrices of appropriate dimensions in general. It is assumed that the state feedback is available. Otherwise, estimates of the state variables can be obtained from nonlinear state estimator. The nonlinear state feedback control input is given by

$$u_k = K(x_k) \cdot x_k = K_k \cdot x_k \tag{5.27}$$

The optimal control problem we consider is to determine an admissible control u to satisfy the performance objective (2.62). The mixed H_2 and H_{∞} performance criteria is given as

$$V_{k+1} - V_k + x_k^T Q_k x_k + u_k^T R_k u_k + z_k^T z_k + \gamma \cdot w_k^T w_k \le 0$$

$$V_N + \sum_{k=0}^{N-1} \left[x_k^T Q_k x_k + u_k^T R_k u_k + z_k^T z_k \right] \le V_0 - \gamma \cdot \sum_{k=0}^{N-1} w_k^T w_k$$
(2.62)

By properly specifying the value of the weighing matrices Q_k, R_k, C_k, D_k, G_k , this performance criterion can be used in nonlinear control design, which yields a mixed Nonlinear Quadratic Regulator (NLQR) in combination with H_{∞} performance index.

5.2.2 $H_2 - H_{\infty}$ State Dependent Riccati Equation Control for Discrete Time Nonlinear Systems

Theorem 5.2 $(H_2 - H_{\infty} \text{ SDRE Control})$: Given the system dynamics (5.25), performance output (5.26), control input (5.27); the performance index (2.62) can be achieved by using the control feedback

$$K_{k}^{o} = -\begin{cases} R_{k} + B_{k}^{T} P_{k} B_{k} + D_{k}^{T} D_{k} - \\ \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \end{cases}^{-1}$$

$$\times \begin{cases} B_{k}^{T} P_{k} A_{k} + D_{k}^{T} C_{k} - \\ \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right)^{T} \end{cases}^{-1}$$
(5.28)

where P_k is obtained from the generalized SDRE equation:

$$P_{k} = \left\{ A_{k}^{T} P_{k} A_{k} + C_{k}^{T} C_{k} + Q_{k} - \left[A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right] \cdot \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left[A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right]^{T} \right\} \\ - \left\{ A_{k}^{T} P_{k} B_{k} + C_{k}^{T} D_{k} - \left(A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \right\} \times \\ \left\{ R_{k} + B_{k}^{T} P_{k} B_{k} + D_{k}^{T} D_{k} - \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \right\}^{-1} \times \\ \left\{ B_{k}^{T} P_{k} A_{k} + D_{k}^{T} C_{k} - \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right)^{T} \right\}$$

$$(5.29)$$

Proof

By applying system equation (5.25), performance output equation (5.26), control input (5.27), performance index (2.62) becomes

$$\begin{bmatrix} (A_{k} + B_{k}K_{k})x_{k} + F_{k}w_{k} \end{bmatrix}^{T} P_{k+1} \begin{bmatrix} (A_{k} + B_{k}K_{k})x_{k} + F_{k}w_{k} \end{bmatrix} -x_{k}^{T}P_{k}x_{k} + x_{k}^{T}Qx_{k} + u_{k}^{T}Ru_{k} + \begin{bmatrix} (E_{k}x_{k} + D_{k}u_{k} + G_{k}w_{k})^{T} \end{bmatrix} \begin{bmatrix} (E_{k}x_{k} + D_{k}u_{k} + G_{k}w_{k})^{T} + \gamma w_{k}^{T}w_{k} \le 0 \end{bmatrix}$$

$$(5.30)$$

Equivalently, we have

$$\begin{bmatrix} x_k^T & w_k^T \end{bmatrix} \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \le 0$$
(5.31)

$$\Xi_{11} = (A_{k} + B_{k}K_{k})^{T} P_{k+1}(A_{k} + B_{k}K_{k}) - P_{k} + Q_{k} + K_{k}^{T}R_{k}K_{k} + (C_{k} + D_{k}K_{k})^{T}(C_{k} + D_{k}K_{k})$$

$$\Xi_{12} = (A_{k} + B_{k}K_{k})^{T} P_{k+1}F_{k} + (C_{k} + D_{k}K_{k})^{T} G_{k}$$

$$\Xi_{22} = F_{k}^{T}P_{k+1}F_{k} + G_{k}^{T}G_{k} + \gamma I$$
(5.32)

Therefore, we have

$$\begin{bmatrix} \Delta_{11} & \Delta_{12} \\ * & \Delta_{22} \end{bmatrix} \ge 0 \tag{5.33}$$

where

$$\Delta_{11} = P_{k} - \left[\left(A_{k} + B_{k} K_{k} \right)^{T} P_{k+1} \left(A_{k} + B_{k} K_{k} \right) + Q_{k} + K_{k}^{T} R_{k} K_{k} + \left(C_{k} + D_{k} K_{k} \right)^{T} \left(C_{k} + D_{k} K_{k} \right) \right]$$

$$\Delta_{12} = - \left[\left(A_{k} + B_{k} K_{k} \right)^{T} P_{k+1} F_{k} + \left(C_{k} + D_{k} K_{k} \right)^{T} G_{k} \right]$$

$$\Delta_{22} = - \left[F_{k}^{T} P_{k+1} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]$$
(5.34)

By applying Schur complement, the following inequality holds:

$$P_{k} - \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k+1} \left(A_{k} + B_{k}K_{k} \right) + Q_{k} + K_{k}^{T}R_{k}K_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T} \left(C_{k} + D_{k}K_{k} \right) \right] \\ + \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k+1}F_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T}G_{k} \right] \times \left[F_{k}^{T}P_{k+1}F_{k} + G_{k}^{T}G_{k} + \gamma I \right]^{-1} \times \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k+1}F_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T}G_{k} \right]^{T} \ge 0$$

Equivalently, we have

$$P_{k} \geq \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k+1} \left(A_{k} + B_{k}K_{k} \right) + Q_{k} + K_{k}^{T}R_{k}K_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T} \left(C_{k} + D_{k}K_{k} \right) \right] \\ - \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k+1}F_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T}G_{k} \right] \times \left[F_{k}^{T}P_{k+1}F_{k} + G_{k}^{T}G_{k} + \gamma I \right]^{-1} \times \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k+1}F_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T}G_{k} \right]^{T}$$

(5.36)

(5.35)

Since we are trying to minimize P_k , the minimum value is achieved when the inequality above becomes an equality. Since the iterative solution starts at P_{∞} runs backward in time, we have $P_{k+1} = P_k$ if the iteration converges. Therefore, the difference equation becomes algebraic as follows:

$$P_{k} = \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k} \left(A_{k} + B_{k}K_{k} \right) + Q_{k} + K_{k}^{T}R_{k}K_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T} \left(C_{k} + D_{k}K_{k} \right) \right] \\ - \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k}F_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T}G_{k} \right] \times \left[F_{k}^{T}P_{k}F_{k} + G_{k}^{T}G_{k} + \gamma I \right]^{-1} \times \left[\left(A_{k} + B_{k}K_{k} \right)^{T} P_{k}F_{k} + \left(C_{k} + D_{k}K_{k} \right)^{T}G_{k} \right]^{T} \right]^{T}$$

By collecting terms, we have

$$P_{k} = \left\{ A_{k}^{T} P_{k} A_{k} + C_{k}^{T} C_{k} + Q_{k} - \left[A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right] \cdot \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left[A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right]^{T} \right\} + K_{k}^{T} \left\{ B_{k}^{T} P_{k} A_{k} + D_{k}^{T} C_{k} - \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right)^{T} \right\} + \left\{ A_{k}^{T} P_{k} B_{k} + C_{k}^{T} D_{k} - \left(A_{k}^{T} P_{k} F + C_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \right\} K_{k} + K_{k}^{T} \left\{ R_{k} + B_{k}^{T} P_{k} B_{k} + D_{k}^{T} D_{k} - \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \right\} K_{k}$$

(5.38)

(5.37)

Equivalently, the equation can be simply written as

$$P_k = \Upsilon_k + K_k^T \Omega_k^T + \Omega_k K_k + K_k^T \Phi_k K_k$$
(5.39)

where

$$\Upsilon_{k} = \left\{ A_{k}^{T} P_{k} A_{k} + C_{k}^{T} C_{k} + Q_{k} - \left[A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right] \cdot \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} - \gamma^{2} I \right]^{-1} \left[A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right]^{T} \right\}$$

$$\Omega_{k} = \left\{ A_{k}^{T} P_{k} B_{k} + C_{k} D_{k} - \left(A_{k}^{T} P_{k} F + C_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} - \gamma^{2} I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \right\}$$

$$\Phi_{k} = \left\{ R_{k} + B_{k}^{T} P_{k} B_{k} + D_{k}^{T} D_{k} - \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} - \gamma^{2} I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \right\}$$
(5.40)

By completing the square in controller gain K_k , we have

$$P_{k} = \Upsilon_{k} + \left(K_{k} - K_{k}^{o}\right)^{T} \Phi_{k} \left(K_{k} - K_{k}^{o}\right) - K_{k}^{oT} \Phi_{k} K_{k}^{o}$$

$$(5.41)$$

For (5.41) to be equal to (5.39), we must have

$$-K_k^{oT} \Phi_k K_k = \Omega_k K_k \tag{5.42}$$

Therefore, the optimal feedback gain

$$K_{k}^{o} = -\Phi_{k}^{-1}\Omega_{k}^{T} = -\left\{R_{k} + B_{k}^{T}P_{k}B_{k} + D_{k}^{T}D_{k} - \left(B_{k}^{T}P_{k}F_{k} + D_{k}^{T}G_{k}\right)\left[F_{k}^{T}P_{k}F_{k} + G_{k}^{T}G_{k} + \gamma I\right]^{-1}\left(B_{k}^{T}P_{k}F_{k} + D_{k}^{T}G_{k}\right)^{T}\right\}^{-1} \times \left\{B_{k}^{T}P_{k}A_{k} + D_{k}^{T}C_{k} - \left(B_{k}^{T}P_{k}F_{k} + D_{k}^{T}G_{k}\right)\left[F_{k}^{T}P_{k}F_{k} + G_{k}^{T}G_{k} + \gamma I\right]^{-1}\left(A_{k}^{T}P_{k}F_{k} + C_{k}^{T}G_{k}\right)^{T}\right\}$$

$$(5.43)$$

when $K_k = K_k^o$, the minimum P_k is defined by the positive definite solution of the following generalized State Dependent Riccati Equation:

$$P_{k} = \Upsilon_{k} - K_{k}^{oT} \Phi_{k} K_{k}^{o} = \left\{ A_{k}^{T} P_{k} A_{k} + C_{k}^{T} C_{k} + Q_{k} - \left[A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right] \cdot \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left[A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right]^{T} \right\} - \left\{ A_{k}^{T} P_{k} B_{k} + C_{k}^{T} D_{k} - \left(A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \right\} \times \left\{ R_{k} + B_{k}^{T} P_{k} B_{k} + D_{k}^{T} D_{k} - \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right)^{T} \right\}^{-1} \times \left\{ B_{k}^{T} P_{k} A_{k} + D_{k}^{T} C_{k} - \left(B_{k}^{T} P_{k} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(A_{k}^{T} P_{k} F_{k} + C_{k}^{T} G_{k} \right)^{T} \right\}$$

$$(5.44)$$

The equation (5.44) is the generalized discrete SDRE equation. By solving P_k from (5.44), the $H_2 - H_{\infty}$ SDRE control solution can be achieved by (5.43).

Remark 5.1: In the special case, where we do not have H_{∞} component in the performance index, i.e. nonlinear quadratic regulator control, then the above derivation yields the following:

$$x_{k+1} = A_k x_k + B_k u_k \tag{5.45}$$

The optimal feedback control gain is

$$K_{k}^{o} = -\left\{R_{k} + B_{k}^{T}P_{k}B_{k}\right\}^{-1} \times B_{k}^{T}P_{k}A_{k}$$
(5.46)

where P_k is defined by the positive definite solution of the following generalized State Dependent Riccati Equation:

$$P_{k} = A_{k}^{T} P_{k} A_{k} - \left\{ A_{k}^{T} P_{k} B_{k} \right\} \times \left\{ R_{k} + B_{k}^{T} P_{k} B_{k} \right\}^{-1} \times \left\{ B_{k}^{T} P_{k} A_{k} \right\} + Q_{k}$$
(5.47)

Therefore, the conventional discrete SDRE solution [14] is derived, which is a special case of our results.

Remark 5.2: However, the generalized SDRE (5.44) can be numerically difficult to solve. To facilitate the computation process, the following two theorems provide two alternative solutions to the generalized SDRE in Theorem 5.2. Theorem 5.3 provides us a suboptimal solution by solving the difference State Dependent Riccati Equation (5.49), instead of (5.44). Theorem 5.4 provides us a State Dependent Linear Matrix Inequality approach. *Theorem 5.3* (Suboptimal $H_2 - H_{\infty}$ SDRE Control): Given the system dynamics (5.25), performance output (5.26), control input (5.27) and performance index (2.62) can be achieved by using the control feedback

$$K_{k+1} = -\begin{cases} R_{k} + B_{k}^{T} P_{k+1} B_{k} + D_{k}^{T} D_{k} - \\ \left(B_{k}^{T} P_{k+1} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k+1} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(B_{k}^{T} P_{k+1} F_{k} + D_{k}^{T} G_{k} \right)^{T} \end{cases}^{-1}$$

$$\times \begin{cases} B_{k}^{T} P_{k+1} A_{k} + D_{k}^{T} C_{k} - \\ \left(B_{k}^{T} P_{k+1} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k+1} F_{k} + G_{k}^{T} G_{k} + \gamma I \right]^{-1} \left(A_{k}^{T} P_{k+1} F_{k} + C_{k}^{T} G_{k} \right)^{T} \end{cases}^{-1}$$
(5.48)

where P_k is obtained as the steady solution to the following difference SDRE equation:

$$P_{k,i+1} = \begin{cases} A_{k}^{T} P_{k,i} A_{k} + C_{k}^{T} C_{k} + Q_{k} - \\ \left[A_{k}^{T} P_{k,i} F_{k} + C_{k}^{T} G_{k} \right] \cdot \left[F_{k}^{T} P_{k,i} F_{k} + G_{k}^{T} G_{k} - \gamma^{2} I \right]^{-1} \left[A_{k}^{T} P_{k,i} F_{k} + C_{k}^{T} G_{k} \right]^{T} \end{cases} \\ - \begin{cases} A_{k}^{T} P_{k,i} B_{k} + C_{k}^{T} D_{k} - \\ \left(A_{k}^{T} P_{k,i} F_{k} + C_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k,i} F_{k} + G_{k}^{T} G_{k} - \gamma^{2} I \right]^{-1} \left(B_{k}^{T} P_{k,i} F_{k} + D_{k}^{T} G_{k} \right)^{T} \end{cases} \times \\ \begin{cases} R_{k} + B_{k}^{T} P_{k,i} B_{k} + D_{k}^{T} D_{k} - \\ \left(B_{k}^{T} P_{k,i} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k,i} F_{k} + G_{k}^{T} G_{k} - \gamma^{2} I \right]^{-1} \left(B_{k}^{T} P_{k,i} F_{k} + D_{k}^{T} G_{k} \right)^{T} \end{cases} \\ \end{cases} \end{cases}$$

$$(5.49) \begin{cases} R_{k} + B_{k}^{T} P_{k,i} B_{k} + D_{k}^{T} G_{k} - \\ \left(B_{k}^{T} P_{k,i} F_{k} + D_{k}^{T} G_{k} \right) \left[F_{k}^{T} P_{k,i} F_{k} + G_{k}^{T} G_{k} - \gamma^{2} I \right]^{-1} \left(B_{k}^{T} P_{k,i} F_{k} + D_{k}^{T} G_{k} \right)^{T} \end{cases} \\ \end{cases} \end{cases}$$

At time step k, the difference equation (5.49) is iterated starting with an arbitrary initial condition $P_{k,0} > 0$ until $P_{k,i}$ converges to $P_{k,i+1}$, for i = 1, 2, 3, ... Hence, the solution to the generalized SDRE equation (5.44) can be found using this method. In practical applications, we can choose $P_{k,0} = I$, (5.50)

as the starting value for iterations of calculating P_k .

Theorem 5.4 (State Dependent LMI Control): Given the system dynamics (5.25), performance output (5.26), control equation (5.27) and performance index (2.62), if there exist matrices $M_k = P_k^{-1} > 0$ and Y_k for all $k \ge 0$, such that the following State Dependent LMI hold:

$$\begin{bmatrix} M_{k} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 \\ * & * & M_{k} & 0 & 0 & 0 \\ * & * & * & I_{n} & 0 & 0 \\ * & * & * & * & I_{m} & 0 \\ * & * & * & * & * & I_{p} \end{bmatrix} \ge 0$$
(5.51)

where

 $\Xi_{12} = -\alpha M_k C_k^T D_k + 0.5 \cdot \beta M_k C_k^T$

$$\begin{aligned} \Xi_{13} &= M_k A_k + Y_k^T B_k^T \\ \Xi_{14} &= M_k Q_k^{T/2} \\ \Xi_{15} &= Y_k^T R_k^{T/2} \\ \Xi_{16} &= \alpha^{1/2} M_k C_k^T \\ \Xi_{22} &= -\gamma I - \alpha D_k^T D_k + 0.5 \cdot \beta \left(D_k + D_k^T \right) \\ \Xi_{23} &= F_k^T \end{aligned}$$
(5.52)

and
$$M_{k+1} \ge M_k$$
, where max π_k s.t. $M_k \ge \pi_k I$ (5.53)

then the performance criteria inequality (2.62) is satisfied. The nonlinear feedback gain of the controller is given by $K_k = Y_k \cdot P_k$

(5.54)

Proof

Inequality (5.33) is equivalent to the following inequality:

$$\begin{bmatrix}
\begin{pmatrix}
P_{k} - Q_{k} - K_{k}^{T} R_{k} K_{k} - \\
(C_{k} + D_{k} K_{k})^{T} (C_{k} + D_{k} K_{k})
\end{pmatrix} - (C_{k} + D_{k} K_{k})^{T} G_{k} \\
* -\gamma I - G_{k}^{T} G_{k}
\end{bmatrix}$$

$$-\begin{bmatrix}
\begin{pmatrix}
A_{k} + B_{k} K_{k} \\
F_{k}^{T}
\end{bmatrix} P_{k+1} \begin{bmatrix}
(A_{k} + B_{k} K_{k}) & F_{k}\end{bmatrix} \ge 0$$
(5.55)

By adding and subtracting the same term in (5.55), the following inequality results

$$\begin{bmatrix} \begin{pmatrix} P_{k} - Q_{k} - K_{k}^{T} R_{k} K_{k} - \\ (C_{k} + D_{k} K_{k})^{T} (C_{k} + D_{k} K_{k}) \end{pmatrix} & -(C_{k} + D_{k} K_{k})^{T} G_{k} \\ * & -\gamma I - G_{k}^{T} G_{k} \end{bmatrix}$$
$$-\begin{bmatrix} \begin{pmatrix} A_{k} + B_{k} K_{k} \end{pmatrix}^{T} \\ F_{k}^{T} \end{bmatrix} (P_{k+1} - P_{k}) \begin{bmatrix} (A_{k} + B_{k} K_{k}) & F_{k} \end{bmatrix} -$$
$$\begin{bmatrix} \begin{pmatrix} A_{k} + B_{k} K_{k} \end{pmatrix}^{T} \\ F_{k}^{T} \end{bmatrix} P_{k} \begin{bmatrix} (A_{k} + B_{k} K_{k}) & F_{k} \end{bmatrix} \ge 0$$
(5.56)

Therefore, subject to $P_{k+1} \leq P_k$, (5.61) can be rewritten as

$$\begin{bmatrix} \begin{pmatrix} P_{k} - Q_{k} - K_{k}^{T} R_{k} K_{k} - \\ (C_{k} + D_{k} K_{k})^{T} (C_{k} + D_{k} K_{k}) \end{pmatrix} & -(C_{k} + D_{k} K_{k})^{T} G_{k} \\ * & -\gamma I - G_{k}^{T} G_{k} \end{bmatrix}$$

$$-\begin{bmatrix} \begin{pmatrix} A_{k} + B_{k} K_{k} \end{pmatrix}^{T} \\ F_{k}^{T} \end{bmatrix} P_{k} \begin{bmatrix} (A_{k} + B_{k} K_{k}) & F_{k} \end{bmatrix} \ge 0$$
(5.57)

By applying Schur complement result, we obtain

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ * & \Gamma_{22} & \Gamma_{23} \\ * & * & \Gamma_{33} \end{bmatrix} \ge 0$$
(5.58)

where

$$\Gamma_{11} = P_{k} - Q_{k} - K_{k}^{T} R_{k} K_{k} - (C_{k} + D_{k} K_{k})^{T} (C_{k} + D_{k} K_{k})$$

$$\Gamma_{12} = -(C_{k} + D_{k} K_{k})^{T} G_{k}$$

$$\Gamma_{13} = (A_{k} + B_{k} K_{k})^{T}$$

$$\Gamma_{22} = -\gamma I - G_{k}^{T} G_{k}$$

$$\Gamma_{23} = F_{k}^{T}$$

$$\Gamma_{33} = P_{k}^{-1}$$
(5.59)

By pre-multiplying and post-multiplying the matrix with block diagonal matrix $diag\{M_k, I, I\}$, where $M_k = P_k^{-1}$, the following inequality follows

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} \\ * & \Theta_{22} & \Theta_{23} \\ * & * & \Theta_{33} \end{bmatrix} \ge 0$$
(5.60)

where

$$\begin{split} \Theta_{11} &= M_{k} - M_{k} \left(Q_{k} + K_{k}^{T} R_{k} K_{k} - \left(C_{k} + D_{k} K_{k} \right)^{T} \left(C_{k} + D_{k} K_{k} \right) \right) M_{k} \\ \Theta_{12} &= -M_{k} \left(C_{k} + D_{k} K_{k} \right)^{T} G_{k} = -M_{k} C_{k}^{T} G_{k} - Y_{k}^{T} D_{k}^{T} G_{k} \\ \Theta_{13} &= M_{k} \left(A_{k} + B_{k} K_{k} \right)^{T} = M_{k} A_{k}^{T} + Y_{k}^{T} B_{k}^{T} \\ \Theta_{22} &= -\gamma I - G_{k}^{T} G_{k} \\ \Theta_{23} &= F_{k}^{T} \end{split}$$

$$\Theta_{33} = M_k \tag{5.61}$$

Finally, by applying Schur complement, the following LMI result is obtained

$$\begin{bmatrix} M_{k} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 \\ * & * & M_{k} & 0 & 0 & 0 \\ * & * & * & I_{n} & 0 & 0 \\ * & * & * & * & I_{m} & 0 \\ * & * & * & * & * & I_{p} \end{bmatrix} \ge 0$$
(5.62)

where

$$\begin{aligned} \Xi_{12} &= -M_k C_k^T G_k - Y_k^T D_k^T G_k \\ \Xi_{13} &= M_k A_k^T + Y_k^T B_k^T \\ \Xi_{14} &= M_k Q_k^{T/2} \\ \Xi_{15} &= Y_k^T R_k^{T/2} \\ \Xi_{16} &= M_k \left(C_k + D_k K_k \right)^T = M_k C_k^T + Y_k^T D_k^T \\ \Xi_{22} &= -\gamma I - G_k^T G_k \\ \Xi_{23} &= F_k^T \end{aligned}$$
(5.63)

Hence, if the LMI (5.62) holds, inequality (2.62) is satisfied. Therefore, if the generalized SDRE (5.44) cannot be solved, the State Dependent LMI provides an alternative solution to the generalized SDRE.

Remark 5.3: Maximizing π_k in (5.53) minimizes a bound on P_k and therefore forces the solution to be close to the one given in the SDRE (5.44).

5.2.3 Application to the Inverted Pendulum on a Cart

The dynamics of the inverted pendulum problem can be found in (3.16). By choosing $x_{1,k} = x(kT), x_{2,k} = \dot{x}(kT), x_{3,k} = \theta(kT), x_{4,k} = \dot{\theta}(kT)$ and the state space model for the system can be written as (3.51). The following system parameters are assumed

$$M = 0.5kg, m = 0.5kg, b = 0.1N \cdot \frac{\sec}{m}, L = 0.3m, I = 0.06kg \cdot m^2$$

The following design parameters are chosen to satisfy different mixed criteria: Classical SDRE Design (NLQR only)

 $C = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = I_4, R = 1$

Suboptimal $H_2 - H_{\infty}$ SDRE Design (Difference SDRE)

$$C = \begin{bmatrix} 0.01 & 0.01 & 0.01 \end{bmatrix}, D = \begin{bmatrix} 0.1 \end{bmatrix}, G = \begin{bmatrix} 0.01 \end{bmatrix}, Q = I_4, R = 0.5, \gamma = -0.01, P_0 = I_4$$

State Dependent $H_2 - H_{\infty}$ LMI Design (Predominant H_2)

$$C = [0.01 \quad 0.01 \quad 0.01 \quad 0.01], D = [0.01], G = [0.01], Q = I_4, R = 1, \gamma = -5$$

State Dependent $H_2 - H_{\infty}$ LMI Design (Predominant H_{∞})

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, G = \begin{bmatrix} 1 \end{bmatrix}, Q = 0.01 \times I_4, R = 0.01, \gamma = -5$$

Linear Quadratic Regulator Based on Linearization (LQR)

$$Q = diag([10, 10, 50, 2]), R = 1$$

The following initial conditions are assumed:

 $x_1 = 1, x_2 = 0, x_3 = \pi / 4, x_4 = 0$

All of the above mixed criteria control performance results are shown in the Figs.5.6–5.10, in comparison with the traditional Linear Quadratic Regulator (LQR) technique based on linearization. From these figures, we find that the $H_2 - H_{\infty}$ SDRE has better performance compared with the traditional LQR technique based on linearization. Especially, Figs.5.6, 5.7 and 5.8 show that the traditional LQR technique loses control of the state variables. Fig.5.10 shows that the lowest control magnitude is needed by the linearization based LQR technique.



Fig.5.6. Position trajectory of the inverted pendulum



Fig.5.7. Velocity trajectory of the inverted pendulum



Fig.5.8. Angle "theta" trajectory of the inverted pendulum



Fig.5.9. Angular velocity trajectory of the inverted pendulum



Fig.5.10. Control input

5.3 Summary

This chapter presents novel $H_2 - H_{\infty}$ control of nonlinear systems with SDRE approach. The optimal control solution can be obtained by solving the generalized State Dependent Riccati Equation. It is shown that the conventional SDRE is the special case of the new generalized SDRE, when the performance index is limited to nonlinear quadratic regulator. In case when SDRE is no available to be solved, two powerful control alternatives: suboptimal $H_2 - H_{\infty}$ State Dependent Riccati Difference Equation control and the State Dependent Linear Matrix Inequality control are also proposed. The Inverted pendulum on a cart, a benchmark under-actuated system is used as an illustrative example to demonstrate the effectiveness and robustness of the proposed methods.

CHAPTER 6 ROBUST MULTI-CRITERIA OPTIMAL FUZZY CONTROL OF NONLINEAR SYSTEMS

Fuzzy control systems have recently shown growing popularity in nonlinear system control applications. A fuzzy control system is essentially an effective way to decompose the task of nonlinear system control into a group of local linear controls based on a set of design-specific model rules. Fuzzy control also provides a mechanism to blend these local linear control problems all together to achieve overall control of the original nonlinear system. In this regard, fuzzy control technique has its unique advantage over other kinds of nonlinear control techniques. Latest research on fuzzy control system design is aimed to improve the optimality and robustness of the controller performance by combining the advantage of modern control theory with the Takagi-Sugeno fuzzy model.

In Chapter 6, we address the nonlinear state feedback control design of nonlinear fuzzy control systems using the Linear Matrix Inequality (LMI) approach. We characterize the solution of the nonlinear control system with the LMI, which provides a sufficient condition for satisfying various performance criteria. A preliminary investigation into the LMI approach to nonlinear fuzzy control systems can be found in works of [66]-[68], [75]. The purpose behind this novel approach is to convert a nonlinear system control problem into a convex optimization problem which is solved by a Linear Matrix Inequality at each time. The recent development in numerical techniques for convex optimization provides efficient algorithms for solving LMIs. If a solution can be expressed in an LMI form, then there exist optimization algorithms providing efficient

global numerical solutions. Therefore if the LMI is feasible, then the LMI control technique provides globally stable solutions satisfying the corresponding mixed performance criteria at each time. We further propose to employ mixed performance criteria to design the controller guaranteeing the quadratic sub-optimality with inherent stability property in combination with dissipativity type of disturbance attenuation.

6.1 Robust Multi-Criteria Optimal Fuzzy Control of Continuous Time Nonlinear Systems This section presents novel fuzzy control designs of continuous time nonlinear systems with multiple performance criteria. The purpose behind this work is to improve the traditional fuzzy controller performance to satisfy several performance criteria simultaneously to secure quadratic optimality with inherent stability property together with dissipativity type of disturbance reduction. The Takagi-Sugeno type fuzzy model is used in our control system design. By solving the Linear Matrix Inequality, the control solution can be found to satisfy mixed performance criteria. The effectiveness of the proposed technique is demonstrated by simulation of the control of inverted pendulum system [82].

6.1.1 Continuous Time Takagi-Sugeno System Model

The importance of the Takagi-Sugeno fuzzy system model is that it provides an effective way to decompose a complicated nonlinear system into local dynamical relations and express those local dynamics of each fuzzy implication rule by a linear system model. The overall fuzzy nonlinear system model is achieved by fuzzy "blending" of the linear system models, so that the overall nonlinear control performance is achieved. The i^{th} rule of the Takagi-Sugeno fuzzy model can be expressed by the following forms:

MODEL RULE *i* :

IF $\varphi_1(t)$ is M_{i1} , $\varphi_2(t)$ is M_{i2} , ..., and $\varphi_p(t)$ is M_{ip} ,

THEN, the input-affine continuous-time fuzzy system equation is:

$$\begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) + F_i w(t) \\ y(t) = C_i x(t) + D_i u(t) + Z_i w(t) \\ i = 1, 2, 3, ..., r \end{cases}$$
(6.1)

where

r

- $x(t) \in \Re^n$ state vector
- $u(t) \in \Re^m$ control input vector
- $y(t) \in \Re^q$ performance output vector
- $w(t) \in \Re^s$ L_2 type of disturbance

total number of the model rules

 M_{ii} fuzzy set

 $A_i \in \Re^{n \times n}, B_i \in \Re^{n \times m}, F_i \in \Re^{n \times s}, C_i \in \Re^{q \times n}, D_i \in \Re^{q \times m}, Z_i \in \Re^{q \times s}$

coefficient matrices

 $\varphi_1, \dots, \varphi_p$ known premise variables which can be functions of state variables,

external disturbance and time

It is assumed that the premises are not the function of the input vector u(t), which is needed to avoid the defuzzification process of fuzzy controller. If we use $\varphi(t)$ to denote the vector containing all the individual elements $\varphi_1(t),...,\varphi_p(t)$, then the overall fuzzy system is

$$\dot{x}(t) = \frac{\sum_{i=1}^{r} g_i(\varphi(t)) \left\{ A_i x(t) + B_i u(t) + F_i w(t) \right\}}{\sum_{i=1}^{r} g_i(\varphi(t))} = \sum_{i=1}^{r} h_i(\varphi(t)) \left\{ A_i x(t) + B_i u(t) + F_i w(t) \right\}$$
(6.2)

$$y(t) = \frac{\sum_{i=1}^{r} g_i(\varphi(t)) \{C_i x(t) + D_i u(t) + Z_i w(t)\}}{\sum_{i=1}^{r} g_i(\varphi(t))} = \sum_{i=1}^{r} h_i(\varphi(t)) \{C_i x(t) + D_i u(t) + Z_i w(t)\}$$
(6.3)

where

$$\varphi(t) = \left[\varphi_1(t), \varphi_2(t), \dots, \varphi_p(t)\right]$$
(6.4)

$$g_{i}(\varphi(t)) = \prod_{j=1}^{p} M_{ij}(\varphi_{j}(t))$$
(6.5)

$$h_{i}(\varphi(t)) = \frac{g_{i}(\varphi(t))}{\sum_{i=1}^{r} g_{i}(\varphi(t))}$$
(6.6)

for all time t. The term $M_{ij}(\varphi_j(t))$ is the grade membership of $\varphi_j(t)$ in M_{ij} .

Since

$$\begin{cases} \sum_{i=1}^{r} g_i(\varphi(t)) > 0\\ g_i(\varphi(t)) \ge 0, i = 1, 2, 3, ..., r \end{cases}$$
(6.7)

we have

$$\begin{cases} \sum_{i=1}^{r} h_i(\varphi(t)) = 1\\ h_i(\varphi(t)) \ge 0, i = 1, 2, 3, ..., r \end{cases}$$
(6.8)

for all time t.

It is assumed that the state is available for feedback and the nonlinear state feedback control input is given by

$$u(t) = -\sum_{i=1}^{r} h_i(\varphi(t)) K_i x(t)$$
(6.9)

Substituting this into the system and performance output equations, we have

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\varphi(t)) h_j(\varphi(t)) \left\{ A_i - B_i K_j \right\} x(t) + \sum_{i=1}^{r} h_i(\varphi(t)) F_i w(t)$$
(6.10)

$$y(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\varphi(t)) h_j(\varphi(t)) \left\{ C_i - D_i K_j \right\} x(t) + \sum_{i=1}^{r} h_i(\varphi(t)) Z_i w(t)$$
(6.11)

Using the notation

$$G_{ij} = A_i - B_i K_j \tag{6.12}$$

$$H_{ij} = C_i - D_i K_j \tag{6.13}$$

then the system equation becomes

$$\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\varphi(t)) h_j(\varphi(t)) \cdot G_{ij} \cdot x(t) + \sum_{i=1}^{r} h_i(\varphi(t)) F_i w(t)$$
(6.14)

$$y(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\varphi(t)) h_j(\varphi(t)) \cdot H_{ij} \cdot x(t) + \sum_{i=1}^{r} h_i(\varphi(t)) Z_i w(t)$$
(6.15)

The optimal control problem we consider is to determine an admissible control u to satisfy the performance objective (2.50). The general performance criteria is given as

$$\dot{V}(t) + x^{T}(t)Qx(t) + u^{T}(t)Ru(t) + \alpha \cdot y^{T}(t)y(t) - \beta \cdot y^{T}(t)w(t) + \gamma \cdot w^{T}(t)w(t) \le 0$$
(2.50)

with Q > 0, R > 0 functions of x.

6.1.2 Fuzzy LMI Control of Continuous Time Nonlinear System with General

Performance Criteria

The following theorem summarizes the main results of this section:

Theorem 6.1: Given the system model (6.10), performance output (6.11) and control input (6.9), if there exist matrices $S = P^{-1} > 0$ for all $t \ge 0$, such that the following LMI holds:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} \\ * & \Lambda_{22} & \Lambda_{23} & 0 & 0 \\ * & * & I & 0 & 0 \\ * & * & * & R^{-1} & 0 \\ * & * & * & * & I \end{bmatrix} \ge 0$$
(6.16)

where

$$\Lambda_{11} = -\frac{1}{2} \Big[SA_i^T - M_j B_i^T + SA_j^T - M_i^T B_j^T + A_i S - B_i M_j + A_j S - B_j M_i \Big]
\Lambda_{12} = -\frac{1}{2} \Big(F_i + F_j \Big) + \frac{\beta}{4} \Big[SC_i^T - M_j^T D_i^T + SC_j^T - M_i^T D_j^T \Big]
\Lambda_{13} = \frac{1}{2} \alpha^{1/2} \Big[SC_i^T - M_j^T D_i^T + SC_j^T - M_i^T D_j^T \Big]
\Lambda_{14} = \frac{1}{2} \Big(M_i^T + M_j^T \Big)
\Lambda_{15} = SQ^{T/2}
\Lambda_{22} = -\gamma I + \frac{1}{2} \beta \cdot \Big(Z_i + Z_j \Big)^T
\Lambda_{23} = \frac{1}{2} \alpha^{1/2} \Big[Z_i + Z_j \Big]^T$$
(6.17)

using the notation

$$M_i = K_i P^{-1} = K_i S ag{6.18}$$

then inequality (6.17) is satisfied.

Proof

By applying system model (6.10) and (6.14), performance output (6.11) and (6.15), and state feedback input (6.9), the performance index inequality (2.50) becomes

$$\begin{bmatrix} \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(t))h_{j}(\phi(t)) \cdot G_{ij} \cdot x(t) \\ + \sum_{i=1}^{r} h_{i}(\phi(t))F_{i}w(t) \end{bmatrix}^{T} \cdot P \cdot x(t) + x^{T}(t) \cdot P \cdot \begin{bmatrix} \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(t))h_{j}(\phi(t)) \cdot G_{ij} \cdot x(t) \\ + \sum_{i=1}^{r} h_{i}(\phi(t))F_{i}w(t) \end{bmatrix}^{T} x^{T}(t)Qx(t) + \begin{bmatrix} -\sum_{i=1}^{r} h_{i}(\phi(t))K_{i}x(t) \end{bmatrix}^{T} R \begin{bmatrix} -\sum_{i=1}^{r} h_{i}(\phi(t))K_{i}x(t) \end{bmatrix}^{T} R \begin{bmatrix} \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(t))h_{j}(\phi(t)) \cdot H_{ij} \cdot x(t) \\ + \sum_{i=1}^{r} h_{i}(\phi(t))Z_{i}w(t) \end{bmatrix}^{T} \cdot \begin{bmatrix} \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(t))h_{j}(\phi(t)) \cdot H_{ij} \cdot x(t) \\ + \sum_{i=1}^{r} h_{i}(\phi(t))Z_{i}w(t) \end{bmatrix}^{T} \cdot W(t) + \gamma \cdot w^{T}(t)w(t) \leq 0$$

$$(6.19)$$

Inequality (6.19) is equivalent to

$$\begin{bmatrix} x^{T}(t) & w^{T}(t) \end{bmatrix} \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ * & \Delta_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \le 0$$
(6.20)

where

$$\Delta_{11} = \left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right)^{T}P + P\left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right) + Q + \left[\sum_{i}h_{i}K_{i}\right]^{T}R\left[\sum_{i}h_{i}K_{i}\right] + \alpha \cdot \left[\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right]^{T}\left[\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right] \\\Delta_{12} = P\left(\sum_{i}h_{i}F_{i}\right) + \alpha \cdot \left[\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right]^{T}\left[\sum_{i}h_{i}Z_{i}\right] - \frac{\beta}{2} \cdot \left[\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right]^{T} \\\Delta_{22} = \gamma I + \alpha \cdot \left[\sum_{i}h_{i}Z_{i}\right]^{T}\left[\sum_{i}h_{i}Z_{i}\right] - \beta \cdot \left[\sum_{i}h_{i}Z_{i}\right]^{T}$$
(6.21)

Inequality (6.20) can be rewritten as

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} - \alpha \cdot \begin{bmatrix} \sum_{i} \sum_{j} h_{i} h_{j} H_{ij} \end{bmatrix}^{T} \\ \begin{bmatrix} \sum_{i} \sum_{j} h_{i} h_{j} H_{ij} \end{bmatrix} \begin{bmatrix} \sum_{i} \sum_{j} h_{i} h_{j} H_{ij} \end{bmatrix} \begin{bmatrix} \sum_{i} h_{i} Z_{i} \end{bmatrix} \ge 0 \quad (6.22)$$

where

$$\Psi_{11} = -\left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right)^{T}P - P\left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right) - Q - \left[\sum_{i}h_{i}K_{i}\right]^{T}R\left[\sum_{i}h_{i}K_{i}\right]$$

$$\Psi_{12} = -P\left(\sum_{i}h_{i}F_{i}\right) + \frac{\beta}{2} \cdot \left[\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right]^{T}$$

$$\Psi_{22} = -\gamma I + \beta \cdot \left[\sum_{i}h_{i}Z_{i}\right]^{T}$$
(6.23)

By applying Schur complement to inequality (6.23), we have

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \alpha^{1/2} \left[\sum_{i} \sum_{j} h_{i} h_{j} H_{ij} \right]^{T} \\ * & \Psi_{22} & \alpha^{1/2} \left[\sum_{i} h_{i} Z_{i} \right]^{T} \\ * & * & I \end{bmatrix} \ge 0$$
(6.24)

Similarly, inequality (6.24) can also be written as

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \alpha^{1/2} \left[\sum_{i} \sum_{j} h_{i} h_{j} H_{ij} \right]^{T} \\ * & \Phi_{22} & \alpha^{1/2} \left[\sum_{i} h_{i} Z_{i} \right]^{T} \\ * & * & I \end{bmatrix} - \begin{bmatrix} \left[\sum_{i} h_{i} K_{i} \right]^{T} \\ 0 \\ 0 \end{bmatrix} R \begin{bmatrix} \left[\sum_{i} h_{i} K_{i} \right]^{T} & 0 & 0 \end{bmatrix} \ge 0$$

where

$$\Phi_{11} = -\left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right)^{T}P - P\left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right) - Q$$

$$\Phi_{12} = -P\left(\sum_{i}h_{i}F_{i}\right) + \frac{\beta}{2} \cdot \left[\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right]^{T}$$

$$\Phi_{22} = -\gamma I + \beta \cdot \left[\sum_{i}h_{i}Z_{i}\right]^{T}$$
(6.26)

By applying Schur complement again to (6.25), we have

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & \alpha^{1/2} \left(\sum_{i} \sum_{j} h_{i} h_{j} H_{ij} \right)^{T} & \left[\sum_{i} h_{i} K_{i} \right]^{T} \\ * & \Phi_{22} & \alpha^{1/2} \left[\sum_{i} h_{i} Z_{i} \right]^{T} & 0 \\ * & * & I & 0 \\ * & * & R^{-1} \end{bmatrix} \ge 0$$
(6.27)

Equivalently, we have

$$\sum_{i} \sum_{j} h_{i} h_{j} \cdot \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} \\ * & \Xi_{22} & \Xi_{23} & 0 \\ * & * & I & 0 \\ * & * & * & R^{-1} \end{bmatrix} \ge 0$$
(6.28)

where

$$\Xi_{11} = -\frac{1}{2} \Big[\Big(A_i - B_i K_j \Big) + \Big(A_j - B_j K_i \Big) \Big]^T \cdot P - \frac{1}{2} P \cdot \Big[\Big(A_i - B_i K_j \Big) + \Big(A_j - B_j K_i \Big) \Big] - Q$$

$$\Xi_{12} = -\frac{1}{2} P \Big(F_i + F_j \Big) + \frac{\beta}{4} \Big[\Big(C_i - D_i K_j \Big) + \Big(C_j - D_j K_i \Big) \Big]^T$$

$$\Xi_{13} = \frac{1}{2} \alpha^{1/2} \Big[\Big(C_i - D_i K_j \Big) + \Big(C_j - D_j K_i \Big) \Big]^T$$

$$\Xi_{14} = \frac{1}{2} \Big(K_i + K_j \Big)^T$$

$$\Xi_{22} = -\gamma I + \frac{1}{2} \beta \cdot \Big(Z_i + Z_j \Big)^T$$

$$\Xi_{23} = \frac{1}{2} \alpha^{1/2} \Big[Z_i + Z_j \Big]^T$$
(6.29)

Therefore, we have the following LMI

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} \\ * & \Xi_{22} & \Xi_{23} & 0 \\ * & * & I & 0 \\ * & * & * & R^{-1} \end{bmatrix} \ge 0$$
(6.30)

By multiplying both sides of the LMI above by the block diagonal matrix $diag\{S, I, I, I\}$, where $S = P^{-1}$, and using the notation
$$M_i = K_i P^{-1} = K_i S ag{6.31}$$

we obtain

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & \Theta_{23} & 0 \\ * & * & I & 0 \\ * & * & * & R^{-1} \end{bmatrix} \ge 0$$
(6.32)

where

$$\Theta_{11} = -\frac{1}{2} \Big[SA_i^T - M_j B_i^T + SA_j^T - M_i^T B_j^T + A_i S - B_i M_j + A_j S - B_j M_i \Big] - SQS$$

$$\Theta_{12} = -\frac{1}{2} \Big(F_i + F_j \Big) + \frac{\beta}{4} \Big[SC_i^T - M_j^T D_i^T + SC_j^T - M_i^T D_j^T \Big]$$

$$\Theta_{13} = \frac{1}{2} \alpha^{1/2} \Big[SC_i^T - M_j^T D_i^T + SC_j^T - M_i^T D_j^T \Big]$$

$$\Theta_{14} = \frac{1}{2} \Big(M_i^T + M_j^T \Big)$$

$$\Theta_{22} = -\gamma I + \frac{1}{2} \beta \cdot \Big(Z_i + Z_j \Big)^T$$

$$\Theta_{23} = \frac{1}{2} \alpha^{1/2} \Big[Z_i + Z_j \Big]^T$$
(6.33)

By applying Schur complement again, the final LMI is derived

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} \\ * & \Lambda_{22} & \Lambda_{23} & 0 & 0 \\ * & * & I & 0 & 0 \\ * & * & * & R^{-1} & 0 \\ * & * & * & * & I \end{bmatrix} \ge 0$$
(6.34)

where

$$\begin{split} \Lambda_{11} &= -\frac{1}{2} \Big[SA_{i}^{T} - M_{j}B_{i}^{T} + SA_{j}^{T} - M_{i}^{T}B_{j}^{T} + A_{i}S - B_{i}M_{j} + A_{j}S - B_{j}M_{i} \Big] \\ \Lambda_{12} &= -\frac{1}{2} \Big(F_{i} + F_{j} \Big) + \frac{\beta}{4} \Big[SC_{i}^{T} - M_{j}^{T}D_{i}^{T} + SC_{j}^{T} - M_{i}^{T}D_{j}^{T} \Big] \\ \Lambda_{13} &= \frac{1}{2} \alpha^{1/2} \Big[SC_{i}^{T} - M_{j}^{T}D_{i}^{T} + SC_{j}^{T} - M_{i}^{T}D_{j}^{T} \Big] \\ \Lambda_{14} &= \frac{1}{2} \Big(M_{i}^{T} + M_{j}^{T} \Big) \\ \Lambda_{15} &= SQ^{T/2} \end{split}$$

$$\Lambda_{22} = -\gamma I + \frac{1}{2} \beta \cdot \left(Z_i + Z_j \right)^T$$

$$\Lambda_{23} = \frac{1}{2} \alpha^{1/2} \left[Z_i + Z_j \right]^T$$
(6.35)

Hence, if the LMI (6.35) holds, inequality (2.50) is satisfied. This concludes the proof of the theorem.

Remark 6.1: For the chosen performance criterion, the LMI (6.16) need to be solved at each time to find matrices S, M, by using relation (6.18), we can find the feedback control gain, therefore, the feedback control can be found to satisfy the chosen criterion.

6.1.3 Application to the Inverted Pendulum on a Cart

The inverted pendulum on a cart system is used for testing the effectiveness of the proposed algorithm. The system model of the inverted pendulum on a cart problem has been derived in [4], which is a simplified version of the previous state space model (3.16):

$$\begin{cases} \dot{x}_{1} = x_{2} + \varepsilon_{1} \cdot w \\ \dot{x}_{2} = \frac{g \sin(x_{1}) - amLx_{2}^{2} \cdot \sin(2x_{1})/2 - a\cos(x_{1})u}{4L/3 - amL\cos^{2}(x_{1})} + \varepsilon_{2} \cdot w \end{cases}$$
(6.36)

where

- x_2 angular velocity of the pendulum
- g gravity constant
- *m* mass of the pendulum
- *M* mass of the cart
- L length to the pendulum center of mass, length of the pendulum equals 2L

u	external	force,	control	input to	the system
---	----------	--------	---------	----------	------------

w L_2 type of disturbance

a constant
$$a = 1/(m+M)$$

 $\varepsilon_1, \varepsilon_2$ weighting coefficients of the disturbance

Due to the system nonlinearity, we approximate the system using the following two-rule fuzzy model:

RULE 1:

IF $|x_1|$ is close to zero,

THEN $\dot{x}(t) = A_1 x(t) + B_1 u(t) + F_1 w(t)$.

RULE 2:

IF $|x_1|$ is close to $\pi/2$,

THEN $\dot{x}(t) = A_2 x(t) + B_2 u(t) + F_2 w(t)$.

where

$$A_{1} = \begin{bmatrix} 0 & 1 \\ \frac{g}{4L/3 - amL} & 0 \end{bmatrix}, B_{1} = \begin{bmatrix} 0 \\ -\frac{a}{4L/3 - amL} \end{bmatrix}, F_{1} = \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \end{bmatrix}$$
$$A_{2} = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(4L/3 - amL\delta^{2})} & 0 \end{bmatrix}, B_{2} = \begin{bmatrix} 0 \\ -\frac{a\delta}{4L/3 - amL\delta^{2}} \end{bmatrix}, F_{2} = \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \end{bmatrix} \text{ with } \delta = \cos(80^{\circ})$$

The following values are used in our simulation:

$$m = 2kg, M = 8kg, L = 0.5m, g = 9.8m / s^2, \varepsilon_1 = 1, \varepsilon_2 = 0$$

sampling time T = 0.001, $x_1(0) = \pi / 6$, $x_2(0) = -\pi / 6$ as the initial conditions. The membership functions of Rule 1 and Rule 2 are shown in Fig.6.1.



Fig.6.1. Membership functions of Rule 1 and Rule 2

The following design parameters are chosen to satisfy:

Mixed NLQR- H_{∞} criteria:

 $C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = diag\{\begin{bmatrix} 100 & 1 \end{bmatrix}\}, R = 1, \alpha = 1, \beta = 0, \gamma = -5$

Mixed NLQR-passivity criteria:

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = diag\{\begin{bmatrix} 100 & 1 \end{bmatrix}\}, R = 1, \alpha = 1, \beta = 5, \gamma = 0$$

The mixed criteria control performance results are shown in the Figs.6.2-6.4. From these figures, we find that the novel fuzzy LMI control has satisfactory performance. The new technique controls the inverted pendulum very well under the effect of finite energy disturbance. It should also be noted that the LMI fuzzy control with mixed performance criteria satisfies global asymptotic stability.



Fig.6.2. Angle trajectory of the inverted pendulum



Fig.6.3. Angular velocity trajectory of the inverted pendulum



Fig.6.4. Control input applied to the inverted pendulum

6.2 Robust Multi-Criteria Optimal Fuzzy Control of Discrete Time Nonlinear Systems This section presents novel fuzzy control designs of discrete time nonlinear systems with multiple performance criteria. The purpose behind this work is to improve the traditional fuzzy controller performance to satisfy several performance criteria simultaneously to secure quadratic optimality with inherent stability property together with dissipativity type of disturbance reduction. The Takagi-Sugeno type fuzzy model is used in our control system design. By solving the Linear Matrix Inequality, the control solution can be found to satisfy mixed performance criteria. The effectiveness of the proposed technique is demonstrated by simulation of the control of inverted pendulum on a cart system [84]. 6.2.1 Discrete Time Takagi-Sugeno System Model

At time step k, the i^{th} rule of the Takagi-Sugeno fuzzy model can be expressed by the following forms:

MODEL RULE *i* :

IF $\varphi_1(k)$ is M_{i1} , $\varphi_2(k)$ is M_{i2} , ..., and $\varphi_p(k)$ is M_{ip} ,

THEN, the input-affine discrete-time fuzzy system equation is:

$$\begin{cases} x(k+1) = A_i x(k) + B_i u(k) + F_i w(k) \\ y(k) = C_i x(k) + D_i u(k) + Z_i w(k) \end{cases}$$

 $i = 1, 2, 3, ..., r$
(6.37)

where

$x(k) \in \mathfrak{R}^n$	state vector	
$u(k) \in \Re^m$	control input vector	
$y(k) \in \Re^q$	performance output vector	
$w(k) \in \Re^s$	l_2 type of disturbance	
r	total number of the model rules	
M_{ij}	fuzzy set	
$A_i \in \mathfrak{R}^{n \times n}, B_i \in \mathfrak{R}^{n \times m}, F_i \in \mathfrak{R}^{n \times s}, C_i \in \mathfrak{R}^{q \times n}, D_i \in \mathfrak{R}^{q \times m}, Z_i \in \mathfrak{R}^{q \times s}$		

coefficient matrices

 $\varphi_1,...,\varphi_p$ known premise variables which can be functions of state variables, external disturbance and time

It is assumed that the premises are not the function of the input vector u(k), which is needed to avoid the defuzzification process of fuzzy controller. If we use $\varphi(k)$ to denote the vector containing all the individual elements $\varphi_1(k), ..., \varphi_p(k)$, then the overall fuzzy system is

$$x(k+1) = \frac{\sum_{i=1}^{r} g_i(\varphi(k)) \{A_i x(k) + B_i u(k) + F_i w(k)\}}{\sum_{i=1}^{r} g_i(\varphi(k))} = \sum_{i=1}^{r} h_i(\varphi(k)) \{A_i x(k) + B_i u(k) + F_i w(k)\}$$

$$y(k) = \frac{\sum_{i=1}^{r} g_i(\varphi(k)) \{C_i x(k) + D_i u(k) + Z_i w(k)\}}{\sum_{i=1}^{r} g_i(\varphi(k))} = \sum_{i=1}^{r} h_i(\varphi(k)) \{C_i x(k) + D_i u(k) + Z_i w(k)\}$$

(6.39)

where

$$\varphi(k) = \left[\varphi_1(k), \varphi_2(k), ..., \varphi_p(k)\right]$$
(6.40)

$$g_i(\varphi(k)) = \prod_{j=1}^p M_{ij}(\varphi_j(k))$$
(6.41)

$$h_i(\varphi(k)) = \frac{g_i(\varphi(k))}{\sum_{i=1}^r g_i(\varphi(k))}$$
(6.42)

for all k. The term $M_{ij}(\varphi_j(k))$ is the grade membership of $\varphi_j(k)$ in M_{ij} .

Since

$$\begin{cases} \sum_{i=1}^{r} g_i(\varphi(k)) > 0\\ g_i(\varphi(k)) \ge 0, i = 1, 2, 3, ..., r \end{cases}$$
(6.43)

we have

$$\begin{cases} \sum_{i=1}^{r} h_i(\varphi(k)) = 1\\ h_i(\varphi(k)) \ge 0, i = 1, 2, 3, ..., r \end{cases}$$
(6.44)

for all k.

It is assumed that the state feedback is available and the nonlinear state feedback control input is given by

$$u(k) = -\sum_{i=1}^{r} h_i(\varphi(k)) K_i x(k)$$
(6.45)

Substituting this into the system and performance output equation, we have

$$x(k+1) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\varphi(k)) h_j(\varphi(k)) \left\{ A_i - B_i K_j \right\} x(k) + \sum_{i=1}^{r} h_i(\varphi(k)) F_i w(k)$$
(6.46)

$$y(k) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\varphi(k)) h_j(\varphi(k)) \left\{ C_i - D_i K_j \right\} x(k) + \sum_{i=1}^{r} h_i(\varphi(k)) Z_i w(k)$$
(6.47)

Using the notation

$$G_{ij} = A_i - B_i K_j \tag{6.48}$$

$$H_{ij} = C_i - D_i K_j \tag{6.49}$$

then the system equation becomes

$$x(k+1) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\varphi(k)) h_j(\varphi(k)) \cdot G_{ij} \cdot x(k) + \sum_{i=1}^{r} h_i(\varphi(k)) F_i w(k)$$
(6.50)

$$y(k) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\varphi(k)) h_j(\varphi(k)) \cdot H_{ij} \cdot x(k) + \sum_{i=1}^{r} h_i(\varphi(k)) Z_i w(k)$$
(6.51)

The optimal control problem we consider is to determine an admissible control u to satisfy the performance objective (2.56). The general performance criteria is given as

$$V(k+1) - V(k) + x^{T}(k)Qx(k) + u^{T}(k)Ru(k) + \alpha \cdot y^{T}(k)y(k) - \beta \cdot y^{T}(k)w(k) + \gamma \cdot w^{T}(k)w(k) \le 0$$
(2.60)

with Q > 0, R > 0 functions of x(k).

6.2.2 Fuzzy LMI Control of Discrete Time Nonlinear System with General Performance Criteria

The following theorem summarizes the main results of the paper:

Theorem 6.2: Given the closed loop system (6.50), performance output (6.51), if there exist matrices $S = P^{-1} > 0$ for all $k \ge 0$, such that the following LMI holds:

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & 0 & 0 \\ * & * & S & 0 & 0 & 0 \\ * & * & * & I & 0 & 0 \\ * & * & * & * & R^{-1} & 0 \\ * & * & * & * & * & I \end{bmatrix} \ge 0$$
(6.52)

where

$$\begin{aligned} \Xi_{11} &= S \\ \Xi_{12} &= \frac{\beta}{4} \Big[C_i S - D_i Y_j + C_j S - D_j Y_i \Big]^T \\ \Xi_{13} &= \frac{1}{2} \Big[A_i S - B_i Y_j + A_j S - B_j Y_i \Big]^T \\ \Xi_{14} &= \frac{1}{2} \alpha^{1/2} \cdot \Big[C_i S - D_i Y_j + C_j S - D_j Y_i \Big]^T \\ \Xi_{15} &= \frac{1}{2} \Big[Y_i + Y_j \Big]^T \\ \Xi_{16} &= SQ^{T/2} \\ \Xi_{22} &= -\gamma I + \frac{\beta}{2} \Big(Z_i + Z_j \Big)^T \\ \Xi_{23} &= \frac{1}{2} \alpha^{1/2} \cdot \Big(F_i + F_j \Big)^T \\ \Xi_{24} &= \frac{1}{2} \alpha^{1/2} \cdot \Big[Z_i + Z_j \Big]^T \end{aligned}$$
(6.53)

and

$$S(k+1) > S(k) \tag{6.54}$$

where $S(k) = P^{-1}(k)$, then (2.60) is satisfied with the feedback control gain being found

by
$$K(k) = Y(k) \cdot P(k)$$
(6.55)

Proof

The performance index inequality (2.60) can be explicitly written as

$$\begin{split} & \left[\sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(k))h_{j}(\phi(k)) \cdot G_{ij} \cdot x(k) \right]^{T} \cdot P \cdot \left[\sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(k))h_{j}(\phi(k)) \cdot G_{ij} \cdot x(k) \right] \\ & + \sum_{i=1}^{r} h_{i}(\phi(k))F_{i}w(k) \\ & - x^{T}(k)Px(k) + x^{T}(k)Qx(k) \\ & + \left[-\sum_{i=1}^{r} h_{i}(\phi(k))K_{i}x(k) \right]^{T} R \left[-\sum_{i=1}^{r} h_{i}(\phi(k))K_{i}x(k) \right] \\ & + \alpha \left[\sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(k))h_{j}(\phi(k)) \cdot H_{ij} \cdot x(k) \\ & + \sum_{i=1}^{r} h_{i}(\phi(k))Z_{i}w(k) \right]^{T} \cdot \left[\sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(k))h_{j}(\phi(k)) \cdot H_{ij} \cdot x(k) \\ & + \sum_{i=1}^{r} h_{i}(\phi(k))Z_{i}w(k) \right]^{T} \cdot \left[\sum_{i=1}^{r} \sum_{j=1}^{r} h_{i}(\phi(k))h_{j}(\phi(k)) \cdot H_{ij} \cdot x(k) \\ & + \sum_{i=1}^{r} h_{i}(\phi(k))Z_{i}w(k) \right]^{T} w(k) + \gamma w^{T}(k)w(k) \leq 0 \end{split}$$

Equivalently,

$$\begin{bmatrix} x^{T}(k) & w^{T}(k) \end{bmatrix} \begin{bmatrix} -P + Q & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} x^{T}(k) & w^{T}(k) \end{bmatrix} \begin{bmatrix} \left[\sum_{i} \sum_{j} h_{i}h_{j}G_{ij} \right] & \left[\sum_{i} h_{i}F_{i} \right] \right]^{T} \cdot P \cdot \begin{bmatrix} \left[\sum_{i} \sum_{j} h_{i}h_{j}G_{ij} \right] & \left[\sum_{i} h_{i}F_{i} \right] \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + x^{T}(k) \begin{bmatrix} \sum_{i} h_{i}K_{i} \end{bmatrix}^{T} R \begin{bmatrix} \sum_{i} h_{i}K_{i} \end{bmatrix} x(k) + \alpha \cdot \begin{bmatrix} x^{T}(k) & w^{T}(k) \end{bmatrix} \begin{bmatrix} \sum_{i} \sum_{j} h_{i}h_{j}H_{ij} & \sum_{i} h_{i}Z_{i} \end{bmatrix}^{T} \cdot \begin{bmatrix} \sum_{i} \sum_{j} h_{i}h_{j}H_{ij} & \sum_{i} h_{i}Z_{i} \end{bmatrix}^{T} \cdot \begin{bmatrix} \sum_{i} \sum_{j} h_{i}h_{j}H_{ij} & \sum_{i} h_{i}Z_{i} \end{bmatrix}^{T} w(k) = \rho \cdot \begin{bmatrix} x^{T}(k) & w^{T}(k) \end{bmatrix} \begin{bmatrix} \sum_{i} \sum_{j} h_{i}h_{j}H_{ij} & \sum_{i} h_{i}Z_{i} \end{bmatrix}^{T} w(k) \le 0$$

$$(6.57)$$

which can be written, after collecting terms, as

$$\begin{bmatrix} x^{T}(k) & w^{T}(k) \end{bmatrix} \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ * & \Upsilon_{22} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} - \begin{bmatrix} x^{T}(k) & w^{T}(k) \end{bmatrix} \begin{bmatrix} \left[\sum_{i} \sum_{j} h_{i}h_{j}G_{ij} \right] & \left[\sum_{i} h_{i}F_{i} \right] \right]^{T} \cdot P \cdot \begin{bmatrix} \left[\sum_{i} \sum_{j} h_{i}h_{j}G_{ij} \right] & \left[\sum_{i} h_{i}F_{i} \right] \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} - \alpha \cdot \begin{bmatrix} x^{T}(k) & w^{T}(k) \end{bmatrix} \begin{bmatrix} \sum_{i} \sum_{j} h_{i}h_{j}H_{ij} & \sum_{i} h_{i}Z_{i} \end{bmatrix}^{T} \cdot \begin{bmatrix} \sum_{i} \sum_{j} h_{i}h_{j}H_{ij} & \sum_{i} h_{i}Z_{i} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \ge 0$$
(6.58)

where

$$\Upsilon_{11} = P - Q - \left[\sum_{i} h_{i} K_{i}\right]^{T} R \left[\sum_{i} h_{i} K_{i}\right]$$
$$\Upsilon_{12} = \frac{\beta}{2} \left[\sum_{i} \sum_{j} h_{i} h_{j} H_{ij}\right]^{T}$$
$$\Upsilon_{22} = -\gamma I + \beta \left[\sum_{i} h_{i} Z_{i}\right]^{T}$$
(6.59)

Equivalently, we have

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ * & \Upsilon_{22} \end{bmatrix} - \left[\left(\sum_{i} \sum_{j} h_{i} h_{j} G_{ij} \right) \left(\sum_{i} h_{i} F_{i} \right) \right]^{T} \cdot P \cdot \left[\left(\sum_{i} \sum_{j} h_{i} h_{j} G_{ij} \right) \left(\sum_{i} h_{i} F_{i} \right) \right]$$

$$-\alpha \cdot \left[\sum_{i} \sum_{j} h_{i} h_{j} H_{ij} - \sum_{i} h_{i} Z_{i} \right]^{T} \cdot \left[\sum_{i} \sum_{j} h_{i} h_{j} H_{ij} - \sum_{i} h_{i} Z_{i} \right]^{T} \cdot \left[\sum_{i} \sum_{j} h_{i} h_{j} H_{ij} - \sum_{i} h_{i} Z_{i} \right] \ge 0$$

$$(6.60)$$

By applying Schur complement, we obtain

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right)^{T} \\ * & \Upsilon_{22} & \left(\sum_{i}h_{i}F_{i}\right)^{T} \\ * & * & P^{-1} \end{bmatrix}^{T} \\ -\alpha \cdot \left[\sum_{i}\sum_{j}h_{i}h_{j}H_{ij} & \sum_{i}h_{i}Z_{i}\right]^{T} \cdot \left[\sum_{i}\sum_{j}h_{i}h_{j}H_{ij} & \sum_{i}h_{i}Z_{i}\right]^{2} \ge 0$$

$$(6.61)$$

By applying Schur complement again, we obtain

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right)^{T} & \alpha^{1/2}\left(\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right)^{T} \\ * & \Upsilon_{22} & \left(\sum_{i}h_{i}F_{i}\right)^{T} & \alpha^{1/2}\left(\sum_{i}h_{i}Z_{i}\right)^{T} \\ * & * & P^{-1} & 0 \\ * & * & * & I \end{bmatrix} \ge 0$$
(6.62)

Equivalently, the following inequality holds

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right)^{T} & \alpha^{1/2}\left(\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right)^{T} \\ * & \Psi_{22} & \left(\sum_{i}h_{i}F_{i}\right)^{T} & \alpha^{1/2}\left(\sum_{i}h_{i}Z_{i}\right)^{T} \\ * & * & P^{-1} & 0 \\ * & * & * & I \end{bmatrix}$$
(6.63)
$$\begin{bmatrix} \left(\sum_{i}h_{i}K_{i}\right)^{T} \\ 0 \\ 0 \\ 0 \end{bmatrix} R \begin{bmatrix} \left(\sum_{i}h_{i}K_{i}\right) & 0 & 0 & 0 \end{bmatrix} \ge 0$$

where

$$\Psi_{11} = P - Q$$

$$\Psi_{12} = \frac{\beta}{2} \left[\sum_{i} \sum_{j} h_{i} h_{j} H_{ij} \right]^{T}$$

$$\Psi_{22} = -\gamma I + \beta \left[\sum_{i} h_{i} Z_{i} \right]^{T}$$
(6.64)

By applying Schur complement one more time, we have

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \left(\sum_{i}\sum_{j}h_{i}h_{j}G_{ij}\right)^{T} & \alpha^{1/2}\left(\sum_{i}\sum_{j}h_{i}h_{j}H_{ij}\right)^{T} & \left(\sum_{i}h_{i}K_{i}\right)^{T} \\ * & \Psi_{22} & \left(\sum_{i}h_{i}F_{i}\right)^{T} & \alpha^{1/2}\left(\sum_{i}h_{i}Z_{i}\right)^{T} & 0 \\ * & * & P^{-1} & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & R^{-1} \end{bmatrix} \ge 0 \quad (6.65)$$

By factoring out the $\sum_{i} \sum_{j} h_i(\varphi_k) h_j(\varphi_k)$ term, we have

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 \\ * & * & P^{-1} & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & R^{-1} \end{bmatrix} \ge 0$$
(6.66)

$$\Omega_{11} = P - Q$$

$$\Omega_{12} = \frac{\beta}{4} \Big[H_{ij} + H_{ji} \Big]^{T}$$

$$\Omega_{13} = \frac{1}{2} \Big[G_{ij} + G_{ji} \Big]^{T}$$

$$\Omega_{14} = \frac{1}{2} \alpha^{1/2} \cdot \Big[H_{ij} + H_{ji} \Big]^{T}$$

$$\Omega_{15} = \frac{1}{2} \Big[K_{i} + K_{j} \Big]^{T}$$

$$\Omega_{22} = -\gamma I + \frac{\beta}{2} \Big(Z_{i} + Z_{j} \Big)^{T}$$

$$\Omega_{23} = \frac{1}{2} \Big(F_{i} + F_{j} \Big)^{T}$$

$$\Omega_{24} = \frac{1}{2} \alpha^{1/2} \cdot \Big[Z_{i} + Z_{j} \Big]^{T}$$
(6.67)

By pre-multiplying and post-multiplying the matrix with the block diagonal matrix *block diag* {*S*,*I*,*I*,*I*,*I*}, where $S = P^{-1}$, and applying Schur complement again, the following LMI result is obtained

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16} \\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & 0 & 0 \\ * & * & S & 0 & 0 & 0 \\ * & * & * & I & 0 & 0 \\ * & * & * & * & R^{-1} & 0 \\ * & * & * & * & * & I \end{bmatrix} \ge 0$$
(6.68)

where

$$\begin{split} \Xi_{11} &= S \\ \Xi_{12} &= \frac{\beta}{4} \Big[C_i S - D_i Y_j + C_j S - D_j Y_i \Big]^T \\ \Xi_{13} &= \frac{1}{2} \Big[A_i S - B_i Y_j + A_j S - B_j Y_i \Big]^T \\ \Xi_{14} &= \frac{1}{2} \alpha^{1/2} \cdot \Big[C_i S - D_i Y_j + C_j S - D_j Y_i \Big]^T \\ \Xi_{15} &= \frac{1}{2} \Big[Y_i + Y_j \Big]^T \\ \Xi_{16} &= SQ^{T/2} \\ \Xi_{22} &= -\gamma I + \frac{\beta}{2} \Big(Z_i + Z_j \Big)^T \\ \Xi_{23} &= \frac{1}{2} \alpha^{1/2} \cdot \Big(F_i + F_j \Big)^T \\ \Xi_{24} &= \frac{1}{2} \alpha^{1/2} \cdot \Big[Z_i + Z_j \Big]^T \end{split}$$
(6.69)

where $S = P^{-1}$ and feedback gain $K = Y \cdot P$.

Hence, if the LMI (6.68) holds, performance criteria inequality (2.60) is satisfied. This concludes the proof of Theorem.

6.2.3 Application to the Inverted Pendulum on a Cart

The inverted pendulum on a cart problem is again used to test the proposed control algorithm. The system equation is given in (6.36). Due to the system nonlinearity, we approximate the system using the following two-rule fuzzy model:

RULE 1:

IF $|x_1(k)|$ is close to zero,

THEN
$$x(k+1) = A_1x(k) + B_1u(k) + F_1w(k)$$
.

RULE 2:

IF $|x_1(k)|$ is close to $\pi/2$,

THEN $x(k+1) = A_2 x(k) + B_2 u(k) + F_2 w(k)$.

where

$$A_{1} = \begin{bmatrix} 1 & T \\ gT \\ 4L/3 - amL & 1 \end{bmatrix}, B_{1} = \begin{bmatrix} 0 \\ -\frac{aT}{4L/3 - amL} \end{bmatrix}, F_{1} = \begin{bmatrix} \varepsilon_{1}T \\ \varepsilon_{2}T \end{bmatrix}$$
$$A_{2} = \begin{bmatrix} 1 & T \\ \frac{2gT}{\pi(4L/3 - amL\delta^{2})} & 1 \end{bmatrix}, B_{1} = \begin{bmatrix} 0 \\ -\frac{a\delta T}{4L/3 - amL\delta^{2}} \end{bmatrix}, F_{2} = \begin{bmatrix} \varepsilon_{1}T \\ \varepsilon_{2}T \end{bmatrix} \text{ with } \delta = \cos(80^{\circ})$$

The following values are used in our simulation:

$$M = 8.0kg, m = 2kg, L = 0.5m, g = 9.8m / s^2, \varepsilon_1 = 1, \varepsilon_2 = 0$$

with the sampling time T = 0.001 and initial conditions $x_1 = \pi / 6$, $x_2 = -\pi / 6$. The membership of Rule 1 and Rule 2 is shown in Fig.6.5.



Fig.6.5. Membership functions of Rule 1 and Rule 2

After the system discretization, the feedback control gain can be found from (6.55) by solving the LMI (6.52) at each time step. The following design parameters are

chosen to satisfy:

Mixed NLQR- H_{∞} criteria:

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = diag \{ \begin{bmatrix} 80 & 1 \end{bmatrix} \}, R = 1, \alpha = 1, \beta = 0, \gamma = -5$$

Mixed NLQR-passivity criteria:

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}, Q = diag\{\begin{bmatrix} 80 & 1 \end{bmatrix}\}, R = 1, \alpha = 1, \beta = 5, \gamma = 0$$

The mixed criteria control performance results are shown in the Figs.6.6-6.8. From these figures, we find that the novel fuzzy LMI control has satisfactory performance. The new technique controls the inverted pendulum well under the effect of finite energy disturbance.



Fig.6.6. Angle trajectory of the inverted pendulum



Fig.6.7. Angular velocity trajectory of the inverted pendulum



Fig.6.8. Control input applied to the inverted pendulum

6.3 Summary

This Chapter presents novel nonlinear system fuzzy control approaches based on the LMI solutions. We have first applied the Takagi-Sugeno fuzzy model to decompose the nonlinear system. Mixed performance criteria have been used to design the controller and the relative weighting matrices of these criteria can be achieved by choosing different coefficient matrices. The optimal control can be obtained by solving LMI at each time. The benchmark inverted pendulum on a cart problem has been used as an example to demonstrate its effectiveness. The simulation studies show that the proposed method provides a satisfactory alternative to the existing nonlinear control approaches.

CHAPTER 7 BACKGROUND THEORY FOR NONLINEAR ESTIMATION

In 1960, Rudolph E. Kalman published his famous paper describing a recursive solution to the discrete data linear filtering problem, which became a major milestone in the control science. The Kalman filter is a recursive unbiased filter to estimate the state of a process with a linear dynamic model in a way that minimizes the mean square error in estimation. The Kalman filter is one of the most widely used methods for linear system state estimation and tracking due to its simplicity, optimality and robustness. Traditional Kalman filtering proves to be the optimal solution for linear systems with additive noise; however, it does not work as well for nonlinear systems. Over the past 40 years, the EKF, which locally linearizes the nonlinear model so that Kalman Filter can be applied, has been the dominant tool for nonlinear state estimation. However, the EKF is also wellknown for its difficulty to implement, difficulty to tune and instability for severely nonlinear systems. Recently, researchers developed the UKF, which shows better estimation in some applications than EKF, especially for severely nonlinear models. It is based on the principle that "it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function or transformation." We also present another alternative to EKF, the discrete SDRE estimator, for nonlinear estimation. SDRE controllers have been widely deployed in recent advanced nonlinear control systems, and have shown to be far more robust than LQR based on standard linearization techniques. These nonlinear filters have extensive industrial applications from GPS navigation to military sensor networks, from autonomous vehicles to wireless communications. In this chapter, we discuss and provide a comparative study of these estimation techniques.

7.1 Nonlinear Estimation

7.1.1 Extended Kalman Filter

Nonlinear estimation can be a very difficult and complex problem. If we extend the Kalman filter to a form linearized about the current state estimates, this filter is called the Extended Kalman Filter (EKF). Over the past 40 years, EKF has undoubtedly been the most widely used nonlinear estimation technique. When it does not work quite well in severely nonlinear systems or when the noises have relatively large power, people often resort to higher order Extended Kalman Filter, by including the higher order Taylor series terms, to improve the performance of EKF at the price of much higher mathematical and computational complexity. Since there are many equivalent forms of the Kalman filter, we can obtain the corresponding forms of EKF equations. Two sets of commonly used discrete time EKF formulations are summarized as follows.

The two-step updated EKF equations

1. System and measurement model:

$$\begin{cases} x_{k+1} = f_k(x_k, u_k, v_k) \\ y_k = h_k(x_k, w_k) \\ v_k \sim (0, V_k) \\ w_k \sim (0, W_k) \end{cases}$$
(7.1)

At the k^{th} time step, x_k is the state vector and u_k is the input vector. The process noise v_k is AWGN with zero mean and V_k covariance; y_k is the measurement vector. The measurement noise w_k is AWGN with zero mean and W_k covariance; x_0 , v_k and w_k are mutually uncorrelated.

2. Initialization:

$$\hat{x}_{0}^{+} = E[x_{0}]$$

$$P_{0}^{+} = E[(x_{0} - \hat{x}_{0}^{+}) \cdot (x_{0} - \hat{x}_{0}^{+})^{T}]$$
(7.2)

3. Compute the following partial derivative matrices:

$$A_{k} = \frac{\partial f_{k}}{\partial x}\Big|_{\hat{x}_{k}^{+}}$$

$$F_{k} = \frac{\partial f_{k}}{\partial v}\Big|_{\hat{x}_{k}^{+}}$$
(7.3)

4. The time update of state estimate and error covariance (effect of system dynamics):

The priori state and covariance:
$$\frac{\hat{x}_{k+1}^{-} = f_k(\hat{x}_k^{+}, u_k, 0)}{P_{k+1}^{-} = A_k P_k^{+} A_k^{T} + F_k V_k F_k^{T}}$$
(7.4)

5. Compute the following partial derivative matrices:

$$C_{k+1} = \frac{\partial h_{k+1}}{\partial x} \Big|_{\hat{x}_{k+1}^-}$$

$$G_{k+1} = \frac{\partial h_{k+1}}{\partial w} \Big|_{\hat{x}_{k+1}^-}$$
(7.5)

6. The measurement update of state estimate and error covariance (effect of measurement):

Kalman gain:
$$K_{k+1} = P_{k+1}^{-} C_{k+1}^{T} (C_{k+1} P_{k+1}^{-} C_{k+1}^{T} + G_{k+1} W_{k+1} G_{k+1}^{T})^{-1}$$
 (7.6)

The posteriori state and covariance:

$$\hat{x}_{k+1}^{+} = \hat{x}_{k+1}^{-} + K_{k+1} \Big(y_{k+1} - h_k (\hat{x}_{k+1}^{-}, 0) \Big) \\
P_{k+1}^{+} = (I - K_{k+1} C_{k+1}) P_{k+1}^{-}$$
(7.7)

Equivalently, the second set of EKF equations is shown as follows:

One-step updated EKF equations (propagation from time step k to (k+1))

$$\begin{cases} \hat{x}_{k+1} = f_k(\hat{x}_k, u_k, 0) + K_k(y_k - h_k(\hat{x}_k, u_k, 0)) \\ K_k = A_k P_k C_k^T (C_k P_k C_k^T + G_k W_k G_k^T)^{-1} \\ P_{k+1} = A_k P_k A_k^T - A_k P_k C_k^T (C_k P_k C_k^T + G_k W_k G_k^T)^{-1} C_k P_k A_k^T + F_k V_k F_k^T \end{cases}$$
(7.8)

The second set of EKF equations is equivalent to the first set of equations and has the same initial condition as the first set of equations shown previously. Notice that, in this form, we have actually combined both time and measurement updates into one step update equations. Notice that $P_k \cong E\left[(x_k - \hat{x}_k) \cdot (x_k - \hat{x}_k)^T\right]$ is approximately the error covariance.

EKF extends the capability of Kalman filter for nonlinear estimation, but EKF has the following drawbacks:

- 1. Linearization can produce unstable filter performance if the time sampling step is not sufficiently small (the local linearity is not valid).
- 2. The computation of Jacobian matrices is nontrivial and in some applications lead to unstable performance.
- 3. Sufficiently small time step intervals require high sampling rate and high computational complexity.
- 4. Hardware implementation is difficult and tuning is not possible.
- 5. EKF using higher order approximations can be very computationally complicated.

The SDRE and UKF generally show better performance compared with EKF in applications, and provides important tools for nonlinear estimation.

7.1.2 State Dependent Riccati Equation Estimation

As previously mentioned, control and estimation are dual problems. The Linear Quadratic Regulator and Kalman Filtering are dual problems, both of which satisfy H_2 performance. H_{∞} control and H_{∞} estimation are dual problems, which maximize the capability in rejecting extraneous noises and disturbances. Similarly, we have dual problem: SDRE controller and SDRE estimator for nonlinear systems.

Consider the same discrete time nonlinear system as before:

$$\begin{cases} x_{k+1} = f(x_k, u_k, v_k) \\ y_k = h(x_k, w_k) \\ v_k \sim (0, V_k) \\ w_k \sim (0, W_k) \end{cases}$$
(7.9)

The procedure for setting up the discrete SDRE estimator can be summarized as below. The nonlinear dynamics can be expressed in the following form:

$$x_{k+1} = A_k(x_k)x_k + B_k(x_k)u_k + F_k v_k$$

$$y_k = C_k(x_k)x_k + G_k w_k$$
(7.10)

Denote $A_k(x_k)$ as A_k , $B_k(x_k)$ as B_k and $C_k(x_k)$ as C_k . The covariance matrix P_k is solved from the discrete time algebraic Riccati equation:

$$P_{k} = A_{k}P_{k}A_{k}^{T} - A_{k}P_{k}C_{k}^{T}(C_{k}P_{k}C_{k}^{T} + G_{k}W_{k}G_{k}^{T})^{-1}C_{k}P_{k}A_{k}^{T} + F_{k}V_{k}F_{k}^{T}$$
(7.11)

The SDRE filter gain is updated by:

$$K_{k} = A_{k} P_{k} C_{k}^{T} (C_{k} P_{k} C_{k}^{T} + G_{k} W_{k} G_{k}^{T})^{-1}.$$
(7.12)

The state estimate is updated by:

$$\hat{x}_{k+1} = A_k(\hat{x}_k)\hat{x}_k + B_k(\hat{x}_k)u_k + K_k(y_k - h_k(\hat{x}_k)).$$
(7.13)

In order to get a reliable estimation result, the following major stability issues of SDRE filter must be considered:

- The coefficients A_k , B_k , C_k , F_k and G_k must be bounded in magnitude.
- In order to get the P_k at every time step, the discrete time algebraic Riccati equation must have a unique positive definite solution at every step.

Notice that essentially, there is no linearization effect and no Jacobian matrices need to be calculated in SDRE estimator, which explains the better estimation performance of SDRE compared with EKF.

7.1.3 Unscented Kalman Filter

Nonlinear filtering requires a complete description of the conditional probability density maintained; therefore, it requires an unbounded number of sample points to approximate the conditional probability density function. The particle filter, essentially an application of the Monte Carlo method, can use tens of thousands of points to accurately fulfill this process. Therefore, the particle filter is rarely practical in use due to its computational burden.

As mentioned earlier, the EKF is a linearized version using a Kalman filter and is based on approximating the nonlinear function or transformation (up to the first order term of Taylor series) instead of the probability density function (PDF); therefore, it sometimes produces poor filtering performance if the higher order terms of the Taylor series are dominant.

The recently proposed Unscented Kalman Filter (UKF) exhibits less estimation error in many applications. The UKF is based on two important principles. First, it is easy to perform a nonlinear transformation on a single point. Second, it is possible to select a set of single points in state space (called sigma points) whose sample PDF can represent the true PDF of the state vector. Therefore, based on the information from these sample points, we can approximate the statistical properties of the true nonlinear transformation. There are many different forms of unscented transformations for UKF. The commonly used forms are summarized as follows. For more detailed information about UKF, please refer to references.

The general UKF equations

1. System model and measurement model:

$$\begin{cases} x_{k+1} = f(x_k, u_k) + v_k \\ y_k = h(x_k) + w_k \\ v_k \sim (0, V_k) \\ w_k \sim (0, W_k) \end{cases}$$
(7.14)

At the k^{th} time step, x_k is the n-dimensional state vector and u_k is the input vector. The process noise v_k is AWGN with zero mean and V_k covariance; y_k is the measurement vector. The measurement noise, w_k , is AWGN with zero mean and W_k covariance. Denote κ as a tuning factor, satisfying $n + \kappa = 3$.

2. Initialization:

$$\hat{x}_{0}^{+} = E[x_{0}]$$

$$P_{0}^{+} = E[(x_{0} - \hat{x}_{0}^{+}) \cdot (x_{0} - \hat{x}_{0}^{+})^{T}]$$
(7.15)

3. The time update equations of state estimate and covariance:

(a) To propagate from step k to (k+1), the set of translated sigma points need to be computed from the $n \times n$ covariance matrix P_k^+ . The n-dimensional state vector x_k with mean \hat{x}_k^+ and covariance P_k^+ is approximated by (2n+1) weighted samples or sigma points selected by:

$$\begin{cases} x_{k}^{(0)} = \hat{x}_{k}^{+} \\ x_{k}^{(i)} = \hat{x}_{k}^{+} + \tilde{x}^{(i)}, i = 1, \dots, 2n \\ \tilde{x}^{(i)} = \left(\sqrt{(n+\kappa)}P_{k}^{+}\right)_{i}^{T}, i = 1, \dots, n \\ \tilde{x}^{(n+i)} = -\left(\sqrt{(n+\kappa)}P_{k}^{+}\right)_{i}^{T}, i = 1, \dots, n \end{cases}$$

$$(7.16)$$

 $\left(\sqrt{(n+\kappa)P_k^+}\right)_i$ is the *i*th row or column of the matrix square root of $(n+\kappa)P_k^+$, and the (2n+1) weighting coefficients are given as:

$$\begin{cases} W^{(0)} = \frac{\kappa}{n+\kappa} \\ W^{(i)} = \frac{1}{2(n+\kappa)}, i = 1,...,2n \end{cases}$$
(7.17)

(b) Each sigma point is initiated through the process model:

$$x_{k+1}^{(i)} = f(x_k^{(i)}, u_k)$$
(7.18)

(c) The predicted mean is computed by:

$$\hat{x}_{k+1}^{-} = \sum_{i=0}^{2n} W^{(i)} \cdot x_{k+1}^{(i)} = \frac{1}{n+\kappa} \left\{ \kappa \cdot x_{k+1}^{(0)} + \frac{1}{2} \sum_{i=1}^{2n} x_{k+1}^{(i)} \right\}$$
(7.19)

(d)The predicted covariance, which is the a priori error covariance, is computed as:

$$P_{k+1}^{-} = V_{k} + \sum_{i=0}^{2n} W^{(i)} \cdot (x_{k+1}^{(i)} - \hat{x}_{k+1}^{-}) \cdot (x_{k+1}^{(i)} - \hat{x}_{k+1}^{-})^{T}$$

$$= V_{k} + \frac{1}{n+\kappa} \left\{ \frac{\kappa \cdot (x_{k+1}^{(0)} - \hat{x}_{k+1}^{-}) \cdot (x_{k+1}^{(0)} - \hat{x}_{k+1}^{-})^{T} + \frac{1}{2} \sum_{i=1}^{2n} (x_{k+1}^{(i)} - \hat{x}_{k+1}^{-}) \cdot (x_{k+1}^{(i)} - \hat{x}_{k+1}^{-})^{T} \right\}$$
(7.20)

4. The measurement update equations of state estimate and covariance:

(a) Each predicted observation point is initialized using the given observation model. The state variables are the sigma points from the time update part shown above.

$$y_{k+1}^{(i)} = h(x_{k+1}^{(i)}) \tag{7.21}$$

(b) The predicted observation is calculated by:

$$\hat{y}_{k+1} = \sum_{i=0}^{2n} W^{(i)} \cdot y_{k+1}^{(i)} = \frac{1}{n+\kappa} \left\{ \kappa \cdot y_{k+1}^{(0)} + \frac{1}{2} \sum_{i=1}^{2n} y_{k+1}^{(i)} \right\}$$
(7.22)

(c) Since the measurement noise is AWGN with covariance W_{k+1} , the covariance of the predicted measurement is:

$$P_{y} = W_{k+1} + \sum_{i=0}^{2n} W^{(i)} \cdot (y_{k+1}^{(i)} - \hat{y}_{k+1}) \cdot (y_{k+1}^{(i)} - \hat{y}_{k+1})^{T}$$

$$= W_{k+1} + \frac{1}{n+\kappa} \left\{ \frac{\kappa \cdot (y_{k+1}^{(0)} - \hat{y}_{k+1}) \cdot (y_{k+1}^{(0)} - \hat{y}_{k+1})^{T}}{\frac{1}{2} \sum_{i=1}^{2n} (y_{k+1}^{(i)} - \hat{y}_{k+1}) \cdot (y_{k+1}^{(i)} - \hat{y}_{k+1})^{T}} \right\}$$
(7.23)

(d)The cross correlation is determined by:

$$P_{xy} = \sum_{i=0}^{2n} W^{(i)} \cdot (x_{k+1}^{(i)} - \hat{x}_{k+1}^{-}) \cdot (y_{k+1}^{(i)} - \hat{y}_{k+1})^{T}$$

$$= \frac{1}{n+\kappa} \begin{cases} \kappa \cdot (x_{k+1}^{(0)} - \hat{x}_{k+1}^{-}) \cdot (y_{k+1}^{(0)} - \hat{y}_{k+1})^{T} + \\ \frac{1}{2} \sum_{i=1}^{2n} (x_{k+1}^{(i)} - \hat{x}_{k+1}^{-}) \cdot (y_{k+1}^{(i)} - \hat{y}_{k+1})^{T} \end{cases}$$
(7.24)

(e)The final measurement update can be performed using normal Kalman filter equations as:

$$\begin{cases} K_{k+1} = P_{xy} \cdot P_{y}^{-1} \\ \hat{x}_{k+1}^{+} = \hat{x}_{k+1}^{-} + K_{k+1} (y_{k+1} - \hat{y}_{k+1}) \\ P_{k+1}^{-} = P_{k+1}^{-} - K_{k+1} P_{y} K_{k+1}^{T} \end{cases}$$
(7.25)

7.2 Summary

Chapter 7 briefly goes over the nonlinear estimation techniques, including the Extended Kalman Filter, State Dependent Riccati Equation estimator, Unscented Kalman Filter. It should be noticed that nonlinear estimation techniques have been applied in chaotic communication applications in recent research studies. A novel nonlinear estimation based chaotic communication scheme, proposed in [77, 79, 80, 81], have shown significant improvement over traditional chaotic communication techniques. Since nonlinear estimators are the dual problems of nonlinear controllers, Chapter 7 serves as an intermediate chapter for Chapter 8, in which we propose a novel resilient estimator for nonlinear stochastic systems.

CHAPTER 8 RESILIENT NONLINEAR FILTERING FOR STOCHASTIC SYSTEMS WITH SENSOR FAILURES

It is well-known that the sensor measurements do not always contain a true signal but corrupted signal caused by various reasons such as attenuation, distortion, sensor failures, delays, multipath and strong noise interference. The estimation with missing measurements was first pointed out in [21, 48], where the missing data were modeled by a binary switching sequence specified by a conditional probability distribution. Such a problem has recently stirred renewed research attention, due to the extensive deployment of sensor networks in various applications. Missing data phenomena have become more and more common in today's data-intensive control engineering, and severely influence the overall system performance in applications such as networked control computing systems, wireless communication systems, navigation systems, etc.

In order to address this important issue, several solutions for estimation of linear systems with missing measurements have been investigated using different approaches. For example, in [55]-[57], the filtering problem with missing data has been investigated by using a jump Riccati equation approach. The variance-constrained filtering problem has been considered in [91] for discrete-time stochastic systems with probabilistic missing measurements subject to norm-bounded parameter uncertainties. The statistical convergence properties of Kalman filter with intermittent missing measurements have been studied in [59], where a critical value has been shown to exist for the arrival rate of the observations. Robust finite-horizon estimator for linear system with missing measurements was proposed in [92]. Another alternative approach is proposed in [45] for

the problem of state estimation via asynchronous communication with irregular transmission times.

Due to the inherent nonlinearity of physical communication transmitters, receivers and channels, the study of nonlinear filtering for discrete time stochastic systems is of practical importance. Recent development involving the case of a nonlinear system with multiple sensors, which may fail independently, has been investigated by Hounkpevi and Yaz in [28, 29], in which the measurements are assumed to be linear functions of the state variables. An alternative stochastic observer design for nonlinear models with intermittent measurements has been studied by NaNacara and Yaz in [49], requiring the upper bounds on the estimation error covariance to be assigned.

In this chapter, a novel resilient filtering technique is proposed for discrete time nonlinear systems with multiple sensors having different failure rates. It should be noted that the local unbiased resilient minimum variance estimator is designed for state estimation, while both system and measurement model are assumed to be nonlinear.

8. 1 Resilient Nonlinear Filtering for Stochastic Systems with Sensor Failures

Missing sensor data is a common problem which severely influences the overall performance of today's data-intensive applications. In order to address this important issue, a resilient filtering technique is proposed for discrete-time nonlinear stochastic system and measurement equations with sensor failures and random gain perturbations. The failure mechanisms of multiple sensors are assumed to be independent of each other with different failure rates. The local unbiased minimum variance estimation is designed for state estimation under these conditions. Lorenz oscillator, a benchmark nonlinear

chaotic system, is used to demonstrate the effectiveness and resilience of the proposed approach [90].

8.1.1 Problem Formulation and Preliminaries

Consider the discrete-time nonlinear stochastic system dynamics and measurement equation:

$$\begin{aligned}
x_{k+1} &= f(x_k) + v_k \\
y_k &= \begin{bmatrix} \gamma_k^1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_k^p \end{bmatrix} \begin{bmatrix} h^1(x_k) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & h^p(x_k) \end{bmatrix} + \begin{bmatrix} w_k^1 \\ \vdots \\ w_k^p \\ w_k^p \end{bmatrix}
\end{aligned}$$
(8.1)

where

$x_k \in \Re^n$	state vector
$V_k \in \Re^n$	system noise
$y_k \in \Re^p$	measurement vector
$W_k \in \Re^p$	measurement noise
f,h	differentiable nonlinear vector functions

The mean of initial state x_0 is $E[x_0] = \overline{x}_0$ and covariance $X_0 = E[(x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T]$. The noise process $\{v_k\}$ and $\{w_k\}$ are white, zero mean,

uncorrelated with each other and with x_0 , and have covariance V_k and W_k respectively:

$$v_{k} \sim (0, V_{k}), \ w_{k} \sim (0, W_{k})$$

$$E\left[v_{k}v_{j}^{T}\right] = V_{k}\delta_{k-j}, \ E\left[w_{k}w_{j}^{T}\right] = W_{k}\delta_{k-j}, \ E\left[v_{k}w_{j}^{T}\right] = 0$$

$$E\left[v_{k}x_{0}^{T}\right] = 0, \ E\left[w_{k}x_{0}^{T}\right] = 0$$
(8.2)

The scalar binary Bernoulli distributed random variables γ_k^i are with mean π_i and variance $\pi_i (1 - \pi_i)$ whose possible outcomes $\{1, 0\}$ are defined as $P(\gamma_k^i = 1) = \pi_i$ and $P(\gamma_k^i = 0) = 1 - \pi_i$. The formulation involves hard sensor failures, where $\gamma^i = 1$ means the i^{th} sensor is properly working and $\gamma^i = 0$ means it is failing, i.e. either the sensor works or it fails. There is no other alternative considered in this work.

$$\Gamma_{k} = diag\left(\left[\gamma_{k}^{1}, ..., \gamma_{k}^{p}\right]\right)$$
(8.3)

$$h(x_k) = diag\left(\left[h^1(x_k), ..., h^p(x_k)\right]\right)$$
(8.4)

$$w_k = \left[w_k^1, \dots, w_k^p \right]^T \tag{8.5}$$

the measurement equation can be written as

By denoting

$$y_k = \Gamma_k h(x_k) + w_k \tag{8.6}$$

Our goal is to estimate the state vector x_k based on our knowledge of system dynamics and the availability of the noisy measurement y_k under the effect of sensor failures. The following discrete-time nonlinear Luenberger observer is considered in this work.

$$\hat{x}_{k+1} = f\left(\hat{x}_{k}\right) + \left(K_{k} + \Delta_{k}\right) \left(y_{k} - \overline{\Gamma}_{k} \cdot h\left(\hat{x}_{k}\right)\right)$$

$$(8.7)$$

The predictor part of (8.7) is a replica of the nonlinear plant dynamics. The correction term corrects future state estimates based on the present error in estimation of the measured value. K_k is the feedback gain with additive uncertainty Δ_k . The uncertainty Δ_k , which arises due to computational or tuning errors, is assumed to have zero mean, bounded second moment and be uncorrelated with initial state, process and

$$E\left\{\Delta_{k}\Delta_{k}^{T}\right\} \leq \delta I$$

$$E\left[\Delta_{k}x_{0}^{T}\right] = 0, \ E\left[\Delta_{k}v_{k}^{T}\right] = 0, \ E\left[\Delta_{k}w_{k}^{T}\right] = 0$$
(8.8)

The term $\overline{\Gamma}_k$ is defined as

$$\overline{\Gamma}_{k} = E\left\{\Gamma_{k}\right\} = diag\left\{\left[\pi_{1},...,\pi_{p}\right]\right\}$$
(8.9)

8.1.2 Resilient Filtering for Nonlinear Stochastic Systems with Random Sensor Failures In this section, we derive the locally unbiased, resilient and minimum variance state estimator for the nonlinear system and measurement model based on the structure of the estimator (8.7). This means for small error values, the estimator is unbiased, has robustness against gain perturbations (8.8) and has minimum estimation error covariance. The following theorem summarizes the main result of this chapter.

Theorem 8.1: Consider the discrete-time nonlinear stochastic system and measurement equations given by (8.1) and the Luenberger observer type nonlinear estimator given by (8.7), the local unbiased resilient minimum variance estimator is defined as follows:

1. Initialization

$$\hat{x}_{0} = E[x_{0}]$$

$$P_{0} = E[(x_{0} - \hat{x}_{0})(x_{0} - \hat{x}_{0})^{T}]$$
(8.10)

2. Computation of Jacobian matrices

$$A_{k} = \frac{\partial f}{\partial x}\Big|_{x=\hat{x}_{k}}, \ C_{k} = \frac{\partial h}{\partial x}\Big|_{x=\hat{x}_{k}}$$
(8.11)

3. For time steps k = 1, 2, 3, ..., the finite horizon filter propagates by calculating the feedback gain

$$K_k^o = \left(\overline{\Gamma}_k C_k P_k A_k\right)^T \left(\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T + Z_k + W_k\right)^{-1}$$
(8.12)

from the local estimation error covariance

$$P_{k+1} = A_k P_k A_k^T + V_k + \delta \cdot \left(\lambda_{\max} \left(\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T\right) + \lambda_{\max} \left(W_k\right) + \lambda_{\max} \left(Z_k\right)\right) I - \left(\overline{\Gamma}_k C_k P_k A_k\right)^T \left(\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T + Z_k + W_k\right)^{-1} \left(\overline{\Gamma}_k C_k P_k A_k\right)$$

$$(8.13)$$

to be used in updating the state estimate as

$$\hat{x}_{k+1} = f\left(\hat{x}_{k}\right) + \left(K_{k} + \Delta_{k}\right)\left(y_{k} - \overline{\Gamma}_{k} \cdot h\left(\hat{x}_{k}\right)\right)$$

$$(8.14)$$

where

$$Z_{k} = E\left\{ \left[\tilde{\Gamma}_{k} h(\hat{x}_{k}) \right] \left[\tilde{\Gamma}_{k} h(\hat{x}_{k}) \right]^{T} + \left[\tilde{\Gamma}_{k} C_{k} \right] P_{k} \left[\tilde{\Gamma}_{k} C_{k} \right]^{T} \right\}$$

$$= \Upsilon \otimes \left(h(\hat{x}_{k}) h^{T}(\hat{x}_{k}) + C_{k} P_{k} C_{k}^{T} \right)$$
(8.15)

$$\Upsilon = diag \left\{ \begin{bmatrix} \pi_1 (1 - \pi_1), ..., \pi_p (1 - \pi_p) \end{bmatrix} \right\}$$
$$= \begin{bmatrix} \pi_1 (1 - \pi_1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \pi_p (1 - \pi_p) \end{bmatrix}$$
(8.16)

Proof

The estimation error $e_k = x_k - \hat{x}_k$ dynamics is given by:

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1} = f(x_k) + v_k - \left\{ f(\hat{x}_k) + (K_k + \Delta_k) (y_k - \overline{\Gamma}_k \cdot h(\hat{x}_k)) \right\}$$
(8.17)

Expanding $f(x_k)$ and $h(x_k)$ into Taylor series at \hat{x}_k , we have

$$f(x_k) = f(\hat{x}_k) + \frac{\partial f}{\partial x}\Big|_{x=\hat{x}_k} (x_k - \hat{x}_k) + \dots$$
(8.18)

$$h(x_k) = h(\hat{x}_k) + \frac{\partial h}{\partial x}\Big|_{x=\hat{x}_k} (x_k - \hat{x}_k) + \dots$$
(8.19)

Neglecting higher order terms, and denoting the following Jacobian matrices

$$A_{k} = \frac{\partial f}{\partial x}\Big|_{x=\hat{x}_{k}}, \ C_{k} = \frac{\partial h}{\partial x}\Big|_{x=\hat{x}_{k}}$$
(8.20)

we have the following approximations

$$f(x_k) \cong f(\hat{x}_k) + A_k e_k \tag{8.21}$$

$$h(x_k) \cong h(\hat{x}_k) + C_k e_k \tag{8.22}$$

Using (8.18)-(8.22), (8.17) can be approximated by

$$e_{k+1} \cong \left\{ f\left(\hat{x}_{k}\right) + A_{k}e_{k} + v_{k} \right\} - \left\{ f\left(\hat{x}_{k}\right) + \left(K_{k} + \Delta_{k}\right)\left(y_{k} - \overline{\Gamma}_{k} \cdot h\left(\hat{x}_{k}\right)\right)\right\}$$

$$\cong A_{k}e_{k} + v_{k} - \left(K_{k} + \Delta_{k}\right)\left\{\Gamma_{k}\left[h\left(\hat{x}_{k}\right) + C_{k}e_{k}\right] + w_{k} - \overline{\Gamma}_{k} \cdot h\left(\hat{x}_{k}\right)\right\}$$

$$= \left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k} + v_{k} - \left(K_{k} + \Delta_{k}\right)w_{k} - \left(K_{k} + \Delta_{k}\right)\widetilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)$$
(8.23)

where $\tilde{\Gamma}_k = \Gamma_k - \overline{\Gamma}_k$.

In order to find the optimal estimator gain K_k , we need to consider the estimation error covariance matrix $P_k = E\left\{e_k e_k^T\right\}$. After applying (8.23), we find the error covariance matrix evolves as

(8.24)
$$P_{k+1} = E\left[e_{k+1}e_{k+1}^{T}\right] = E\left[\left(x_{k+1} - \hat{x}_{k+1}\right)\left(x_{k} - \hat{x}_{k+1}\right)^{T}\right]$$

$$= E\left\{\left[\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k} + v_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k} + v_{k} - \left(K_{k} + \Delta_{k}\right)W_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}h(\hat{x}_{k})\right]^{T}\right\}$$

$$= E\left[\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}e_{k}^{T}\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]^{T}\right]$$

$$+ E\left[\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}w_{k}^{T}\left(K_{k} + \Delta_{k}\right)^{T} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]^{T}\right]$$

$$+ E\left[-\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}w_{k}^{T}\left(K_{k} + \Delta_{k}\right)^{T} - \left(K_{k} + \Delta_{k}\right)w_{k}e_{k}^{T}\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]^{T}\right]$$

$$+ E\left[-\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}h^{T}(\hat{x}_{k})\tilde{\Gamma}_{k}^{T}\left(K_{k} + \Delta_{k}\right)^{T}\right]$$

$$+ E\left[-\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}h^{T}(\hat{x}_{k})\tilde{\Gamma}_{k}^{T}\left(K_{k} + \Delta_{k}\right)^{T}\right]$$

$$+ E\left[-\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}h^{T}(\hat{x}_{k})\tilde{\Gamma}_{k}^{T}\left(K_{k} + \Delta_{k}\right)^{T}\right]$$

$$+ E\left[\left(K_{k} + \Delta_{k}\right)\tilde{\Gamma}_{k}h(\hat{x}_{k})e_{k}^{T}\left[A_{k} - \left(K_{k} + \Delta_{k}\right)\Gamma_{k}C_{k}\right]^{T}\right]$$

$$+ E\left[\left(K_{k} + \Delta_{k}\right)\tilde{\Gamma}_{k}h(\hat{x}_{k})e_{k}^{T}\left(K_{k} + \Delta_{k}\right)^{T}\right] + E\left[\left(K_{k} + \Delta_{k}\right)\tilde{\Gamma}_{k}h(\hat{x}_{k})e_{k}^{T}\right]$$

$$+ E\left[\left(K_{k} + \Delta_{k}\right)\tilde{\Gamma}_{k}h(\hat{x}_{k})\tilde{\Gamma}_{k}^{T}\left(K_{k} + \Delta_{k}\right)^{T}\right]$$

$$+ E\left[\left(K_{k} + \Delta_{k}\right)\tilde{\Gamma}_{k}h(\hat{x}_{k})h^{T}(\hat{x}_{k})\tilde{\Gamma}_{k}^{T}\left(K_{k} + \Delta_{k}\right)^{T}\right]$$

$$(8.25)$$

By applying (8.2) and (8.8), (8.25) can be simplified as follows

By applying (8.8) and Rayleigh's matrix inequality, the term

$$E\left[\left[A_{k}-\left(K_{k}+\Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}e_{k}^{T}\left[A_{k}-\left(K_{k}+\Delta_{k}\right)\Gamma_{k}C_{k}\right]^{T}\right]\right]$$

$$=E\left[\left[A_{k}-K_{k}\Gamma_{k}C_{k}\right]P_{k}\left[A_{k}-K_{k}\Gamma_{k}C_{k}\right]^{T}\right]+$$

$$E\left[\Delta_{k}\Gamma_{k}C_{k}P_{k}C_{k}^{T}\Gamma_{k}^{T}\Delta_{k}^{T}\right]$$

$$=\left[A_{k}-K_{k}\overline{\Gamma}_{k}C_{k}\right]P_{k}\left[A_{k}-K_{k}\overline{\Gamma}_{k}C_{k}\right]^{T}+E\left[K_{k}\widetilde{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\widetilde{\Gamma}_{k}^{T}K_{k}^{T}\right]$$

$$+E\left[\Delta_{k}\overline{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\overline{\Gamma}_{k}^{T}\Delta_{k}^{T}\right]+E\left[\Delta_{k}\widetilde{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\widetilde{\Gamma}_{k}^{T}\Delta_{k}^{T}\right]$$

$$\leq\left[A_{k}-K_{k}\overline{\Gamma}_{k}C_{k}\right]P_{k}\left[A_{k}-K_{k}\overline{\Gamma}_{k}C_{k}\right]^{T}$$

$$+K_{k}E\left[\widetilde{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\widetilde{\Gamma}_{k}^{T}\right]K_{k}^{T}$$

$$+\lambda_{\max}\left(\overline{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\overline{\Gamma}_{k}^{T}\right)\cdot\delta I+\lambda_{\max}\left\{E\left[\widetilde{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\widetilde{\Gamma}_{k}^{T}\right]\right\}\cdot\delta I$$
(8.26)

Since e_k, v_k are uncorrelated, the term

$$E\left[\left[A_{k}-\left(K_{k}+\Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}v_{k}^{T}+v_{k}e_{k}^{T}\left[A_{k}-\left(K_{k}+\Delta_{k}\right)\Gamma_{k}C_{k}\right]^{T}\right]=0$$
(8.27)

Since e_k, w_k are uncorrelated, the term

$$E\left[-\left[A_{k}-\left(K_{k}+\Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}w_{k}^{T}\left(K_{k}+\Delta_{k}\right)^{T}-\left(K_{k}+\Delta_{k}\right)w_{k}e_{k}^{T}\left[A_{k}-\left(K_{k}+\Delta_{k}\right)\Gamma_{k}C_{k}\right]^{T}\right]=0$$
(8.28)

Since $e_k, \Delta_k, \tilde{\Gamma}_k$ are mutually uncorrelated and $E[e_k] = 0, E[\tilde{\Gamma}_k] = 0$, the term

$$E\begin{bmatrix} -\left[A_{k}-\left(K_{k}+\Delta_{k}\right)\Gamma_{k}C_{k}\right]e_{k}h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\left(K_{k}+\Delta_{k}\right)^{T}\\ -\left(K_{k}+\Delta_{k}\right)\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)e_{k}^{T}\left[A_{k}-\left(K_{k}+\Delta_{k}\right)\Gamma_{k}C_{k}\right]^{T}\end{bmatrix}=0$$
(8.29)

By applying (8.8), the term

$$V_{k} + E\left[\left(K_{k} + \Delta_{k}\right)W_{k}\left(K_{k} + \Delta_{k}\right)^{T}\right] = V_{k} + K_{k}W_{k}K_{k}^{T} + E\left[\Delta_{k}W_{k}\Delta_{k}^{T}\right]$$

$$\leq V_{k} + K_{k}W_{k}K_{k}^{T} + \lambda_{\max}\left(W_{k}\right)E\left[\Delta_{k}\Delta_{k}^{T}\right] \leq V_{k} + K_{k}W_{k}K_{k}^{T} + \lambda_{\max}\left(W_{k}\right)\delta I$$
(8.30)

Since v_k, w_k are uncorrelated, the term

$$E\left[-v_{k}w_{k}^{T}\left(K_{k}+\Delta_{k}\right)^{T}-\left(K_{k}+\Delta_{k}\right)w_{k}v_{k}^{T}\right]=0$$
(8.31)

Since v_k , Δ_k are uncorrelated and $E[v_k] = 0$, the term

$$E\left[-v_{k}h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\left(K_{k}+\Delta_{k}\right)^{T}-\left(K_{k}+\Delta_{k}\right)\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)v_{k}^{T}\right]=0$$
(8.32)

Since $w_k, \Delta_k, \tilde{\Gamma}_k$ are mutually uncorrelated and $E[w_k] = 0, E[\tilde{\Gamma}_k] = 0$, the term

$$E\left[\left(K_{k}+\Delta_{k}\right)w_{k}h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\left(K_{k}+\Delta_{k}\right)^{T}+\left(K_{k}+\Delta_{k}\right)\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)w_{k}^{T}\left(K_{k}+\Delta_{k}\right)^{T}\right]=0$$
(8.33)

By applying (8.8), the term

$$E\left[\left(K_{k}+\Delta_{k}\right)\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\left(K_{k}+\Delta_{k}\right)^{T}\right]$$

$$=K_{k}E\left[\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\right]K_{k}^{T}+E\left[\Delta_{k}\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\Delta_{k}^{T}\right]$$

$$\leq K_{k}E\left[\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\right]K_{k}^{T}+\lambda_{\max}\left\{E\left[\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\right]\right\}\cdot\delta I$$

(8.34)

Based on the results of terms (8.26)-(8.34), (8.25) can be reduced to

$$\begin{split} P_{k+1} &\leq \left[A_{k} - K_{k}\overline{\Gamma}_{k}C_{k}\right]P_{k}\left[A_{k} - K_{k}\overline{\Gamma}_{k}C_{k}\right]^{T} + K_{k}E\left[\tilde{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\tilde{\Gamma}_{k}^{T}\right]K_{k}^{T} \\ &+ \lambda_{\max}\left(\overline{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\overline{\Gamma}_{k}^{T}\right)\cdot\delta I + \lambda_{\max}\left\{E\left[\tilde{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\tilde{\Gamma}_{k}\right]\right\}\cdot\delta I + V_{k} + K_{k}W_{k}K_{k}^{T} + \lambda_{\max}\left(W_{k}\right)\delta I \\ &+ K_{k}E\left[\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\right]K_{k}^{T} + \lambda_{\max}\left\{E\left[\tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\right]\right\}\cdot\delta I \\ &= \left[A_{k} - K_{k}\overline{\Gamma}_{k}C_{k}\right]P_{k}\left[A_{k} - K_{k}\overline{\Gamma}_{k}C_{k}\right]^{T} + K_{k}E\left[\tilde{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\tilde{\Gamma}_{k} + \tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\right]K_{k}^{T} \\ &+ \lambda_{\max}\left(\overline{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\overline{\Gamma}_{k}^{T}\right)\cdot\delta I + \lambda_{\max}\left\{E\left[\tilde{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\tilde{\Gamma}_{k} + \tilde{\Gamma}_{k}h\left(\hat{x}_{k}\right)h^{T}\left(\hat{x}_{k}\right)\tilde{\Gamma}_{k}^{T}\right]\right\}\cdot\delta I \\ &+ V_{k} + K_{k}W_{k}K_{k}^{T} + \lambda_{\max}\left(W_{k}\right)\delta I \end{split}$$

Denote

$$Z_{k} = E\left\{ \left[\tilde{\Gamma}_{k} h(\hat{x}_{k}) \right] \left[\tilde{\Gamma}_{k} h(\hat{x}_{k}) \right]^{T} + \left[\tilde{\Gamma}_{k} C_{k} \right] P_{k} \left[\tilde{\Gamma}_{k} C_{k} \right]^{T} \right\}$$

= $\Upsilon \otimes \left(h(\hat{x}_{k}) h^{T}(\hat{x}_{k}) + C_{k} P_{k} C_{k}^{T} \right)$ (8.36)

where

$$\Upsilon = diag \left\{ \begin{bmatrix} \pi_1 (1 - \pi_1), ..., \pi_p (1 - \pi_p) \end{bmatrix} \right\}
= \begin{bmatrix} \pi_1 (1 - \pi_1) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \pi_p (1 - \pi_p) \end{bmatrix}$$
(8.37)

Therefore, the error covariance equation can be obtained as

$$P_{k+1} = \left[A_k - K_k \overline{\Gamma}_k C_k\right] P_k \left[A_k - K_k \overline{\Gamma}_k C_k\right]^T + V_k + K_k W_k K_k^T + K_k Z_k K_k^T + \delta \cdot \left(\lambda_{\max} \left(\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T\right) + \lambda_{\max} \left(W_k\right) + \lambda_{\max} \left(Z_k\right)\right) I$$
(8.38)

Equivalently, we have

$$P_{k+1} = \Omega_k + K_k \Xi_k + \Xi_k^T K_k^T + K_k \Phi_k K_k^T$$
(8.39)

where

$$\Phi_{k} = \overline{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\overline{\Gamma}_{k}^{T} + Z_{k} + W_{k}$$

$$\Xi_{k} = -\overline{\Gamma}_{k}C_{k}P_{k}A_{k}$$

$$\Omega_{k} = A_{k}P_{k}A_{k}^{T} + V_{k} + \delta \cdot \left(\lambda_{\max}\left(\overline{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\overline{\Gamma}_{k}^{T}\right) + \lambda_{\max}\left(W_{k}\right) + \lambda_{\max}\left(Z_{k}\right)\right)I$$
(8.40)

By completing the square in observer gain K_k , we have

$$P_{k+1} = \Omega_k + (K_k - K_k^o) \Phi_k (K_k - K_k^o)^T - K_k^o \Phi_k K_k^{oT}$$
(8.41)

For (8.41) to be equal to (8.39), we must have

$$K_k \Xi_k = -K_k \Phi_k K_k^{oT} \tag{8.42}$$

Therefore, the optimal feedback gain

$$K_{k}^{o} = -\Xi_{k}^{T} \Phi_{k}^{-1}$$

$$= \left(\overline{\Gamma}_{k} C_{k} P_{k} A_{k}\right)^{T} \left(\overline{\Gamma}_{k} C_{k} P_{k} C_{k}^{T} \overline{\Gamma}_{k}^{T} + Z_{k} + W_{k}\right)^{-1}$$
(8.43)

When $K_k = K_k^o$, the resulting matrix difference equation for the minimum error covariance is:

$$P_{k+1} = \Omega_{k} - K_{k}^{o} \Phi_{k} K_{k}^{oT}$$

$$= A_{k} P_{k} A_{k}^{T} + V_{k} + \delta \cdot \left(\lambda_{\max} \left(\overline{\Gamma}_{k} C_{k} P_{k} C_{k}^{T} \overline{\Gamma}_{k}^{T}\right) + \lambda_{\max} \left(W_{k}\right) + \lambda_{\max} \left(Z_{k}\right)\right) I \qquad (8.44)$$

$$- \left(\overline{\Gamma}_{k} C_{k} P_{k} A_{k}\right)^{T} \left(\overline{\Gamma}_{k} C_{k} P_{k} C_{k}^{T} \overline{\Gamma}_{k}^{T} + Z_{k} + W_{k}\right)^{-1} \left(\overline{\Gamma}_{k} C_{k} P_{k} A_{k}\right)$$

This concludes the proof of the theorem.

Remark 8.1: As a limiting case, if we have no perturbations on the estimator gain, i.e. $\delta = 0$, then the following estimator can be derived following a similar procedure to the previously given. In this case, the optimal feedback gain is

$$K_{k}^{o} = \left(\overline{\Gamma}_{k}C_{k}P_{k}A_{k}\right)^{T}\left(\overline{\Gamma}_{k}C_{k}P_{k}C_{k}^{T}\overline{\Gamma}_{k}^{T} + Z_{k} + W_{k}\right)^{-1}$$

$$(8.45)$$

The minimum error covariance equation is

$$P_{k+1} = A_k P_k A_k^T + V_k - \left(\overline{\Gamma}_k C_k P_k A_k\right)^T \left(\overline{\Gamma}_k C_k P_k C_k^T \overline{\Gamma}_k^T + Z_k + W_k\right)^{-1} \left(\overline{\Gamma}_k C_k P_k A_k\right)$$

and the state estimate is updated as

$$\hat{x}_{k+1} = f\left(\hat{x}_{k}\right) + K_{k}\left(y_{k} - \overline{\Gamma}_{k} \cdot h\left(\hat{x}_{k}\right)\right)$$

$$(8.47)$$

8.1.3 Resilient Estimation for Lorenz Systems with Random Measurement Failures The Lorenz oscillator, introduced by Edward Lorenz in 1963, is a 3-dimensional

dynamical system corresponding to the behavior of chaotic flow. The air flow dynamics can be characterized by Lorenz oscillator equation as follows:

$$\begin{cases} \frac{dx_{1}}{dt} = \sigma(x_{2} - x_{1}) \\ \frac{dx_{2}}{dt} = x_{1}(\rho - x_{3}) - x_{2} \\ \frac{dx_{3}}{dt} = x_{1}x_{2} - \beta x_{3} \end{cases}$$
(8.48)

The physical meanings of the non-dimensional variables are as follows:

- x_1 : The dimensionless velocity
- x_2 : The dimensionless temperature difference between up and down currents
- x_3 : The dimensionless departure from the conductive equilibrium
- σ : The Prandtl number to characterize the momentum diffusivity.
- β : The parameter related to the horizontal wave-number of convective motion
- ρ : The Rayleigh number

It is assumed that noise corrupted nonlinear measurements x_1^3, x_2^3, x_3^3 are available. Bernoulli distributed sensor failure rates are $(1-\gamma_1), (1-\gamma_2), (1-\gamma_3)$ respectively. The measurement equation is

$$\begin{cases} y_1 = \gamma_1 x_1^3 + w_1 \\ y_2 = \gamma_2 x_2^3 + w_2 \\ y_3 = \gamma_3 x_3^3 + w_3 \end{cases}$$
(8.49)

The following parameter values are chosen for simulation:

Sampling time T = 0.001

Lorenz map parameters $\rho = 28, \sigma = 10, \beta = \frac{8}{3}$

Mean values of $\gamma_1, \gamma_2, \gamma_3$ are $\pi_1 = 0.95, \pi_2 = 0.98, \pi_3 = 0.9$

Measurement noise covariance matrix $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The uncertainty bound is taken as $\delta = 0.0001$

Lorenz oscillator system and the proposed estimator are initialized as follows.

	10		30	,	2	0	0
$x_0 =$	10	$, \hat{x}_0 =$	20	$, X_0 =$	0	1	0
	10		15		0	0	1

The 3-dimensional phase plot of Lorenz oscillator is shown in Fig.8.1, noted for its lemniscates shape.



Fig.8.1. 3-dimensional phase plot of Lorenz system

If the sensor failure rate is zero, i.e. all measurements are the actual values contaminated by the additive white noise; the phase plane of the measurement y is shown as the blue dash-dotted line in Fig. 8.2. However, when the independent sensor failures are included in the simulation, the measurement y is shown as the red solid line in Fig.8.2 with a lot of "spikes". The spikes are actually the effect of binary Bernoulli distributed sensor failures, since any sensor failure will force the corresponding component of the measurement vector to zero.



Fig.8.2. 3-Dimensional phase plot of measurement with and without considering the effect of sensor failure

Fig.8.3–8.5 shows the phase plot trajectories of each individual measurement in Fig.8.2. Notice that the red dashed line is the measurement without sensor failure, and the blue dash-dotted line is the measurement with sensor failure. Since the sensor failures obey the scalar binary Bernoulli distribution, the curve will suddenly "jump" to zero, when the corresponding sensor fails to work, therefore, "spikes" appear in Fig.8.3-8.5.



Fig.8.3. Phase plot of measurement y1 without sensor failure in comparison with its phase plot with sensor failure



Fig.8.4. Phase plot of measurement y2 without sensor failure in comparison with its phase plot with sensor failure



Fig.8.5. Phase plot of measurement y3 without sensor failure in comparison with its phase plot with sensor failure

By applying the proposed nonlinear stochastic estimator, the phase plot of the estimated state of Lorenz oscillator system is shown as red dashed line in Fig.8.6 in comparison with the state phase plot in blue dash-dotted line. Notice that the state estimates starting from different initial positions than the actual state, eventually converge to the actual state trajectories.

Figs.8.7-8.9 shows the plots of trajectories of the individual state values and the state estimates. As the time increases, the estimated state in blue dash-dotted line effectively converges to the state trajectory in red dashed line under the effect of additive white noise and sensor failures. It is apparent from these graphs that the newly proposed discrete-time nonlinear stochastic estimator shows satisfactory performance in the case of system and measurement nonlinearity, noise interference and independent sensor failures.

For comparison purposes, the Extended Kalman Filter for the discretized form of the nonlinear model (8.46) and (8.47) with no failure information was designed and simulated. However, EKF was unstable and failed to tract the trajectories in the presence of sensor failures.



Fig.8.6. 3-dimensional estimated state phase plot in comparison with the state phase plot



Fig.8.7. Estimated state phase plot of x1 in comparison with its state phase plot



Fig.8.8. Estimated state phase plot of x2 in comparison with its state phase plot



Fig.8.9. Estimated state phase plot of x3 in comparison with its state phase plot

8.2 Summary

Resilient state estimator design for nonlinear stochastic systems with nonlinear measurement equations and sensor failures is proposed in this chapter. The analytic solution is derived for local minimum variance unbiased estimator for nonlinear multi-sensor systems with independent sensor measurement failures. A benchmark chaotic nonlinear system, Lorenz oscillator, is used as a simulation example to demonstrate the effectiveness and robustness of the proposed approach. Simulation results show that the novel resilient state estimator provides better performance than the traditional Extended Kalman Filter in the presence of sensor failures.

CHAPTER 9 CONCLUSIONS AND FUTURE WORK

The last chapter concludes the dissertation with a brief summary of the results and a discussion of related future research directions.

9.1 Summary

The central issues and challenges of control and estimation problems are to satisfy the desired performance objectives in the presence of noises, disturbances, parameter perturbations, unmodeled dynamics, sensor failures, actuator failures, time delays, etc. The focus of this dissertation is to address the following outstanding issues in robust and optimal nonlinear control and estimation:

State Dependent Linear Matrix Inequality Control (Chapter 3, 4)

Two powerful alternatives for Hamilton Jacobi Equations and Hamilton Jacobi Inequality are State Dependent Linear Matrix Inequality control and State Dependent Riccati Equation control approach. The valuable insights we have gained about the State Dependent Linear Matrix Inequality control are summarized in Chapter 3 and Chapter 4, where both continuous and discrete systems are discussed. We further propose general performance criteria to design the controller guaranteeing the quadratic sub-optimality with inherent stability property in combination with dissipativity type of disturbance attenuation. By solving the linear matrix inequality at each time, the optimal control solution can be found to satisfy the desired performance objectives. Moreover, any type of controller is subject to the actuator failures. The control of nonlinear stochastic systems under actuator failures has also been proposed at the end of Chapter 3, which shows significant improvement over the traditional nonlinear control techniques. Chapter 4 presents a more general solution of the optimal, robust and resilient control State Dependent Linear Matrix Inequality control of nonlinear systems with general performance criteria. And the controller is robust for model uncertainties and resilient for control gain perturbations. The benchmark underactuated system: inverted pendulum on a cart is used to demonstrate the effectiveness and robustness of the proposed control techniques.

$H_2 - H_{\infty}$ State Dependent Riccati Equation control (Chapter 5)

The traditional SDRE approach to nonlinear quadratic regulator problem has been applied in a wide variety of military and industrial control applications. Chapter 6 has proposed novel $H_2 - H_{\infty}$ State Dependent Riccati Equation control approaches with the purpose of providing a generalized control framework to nonlinear systems. By solving the generalized State Dependent Riccati Equation, the optimal control solution is found to satisfy mixed performance criteria guaranteeing quadratic optimality with inherent stability property in combination with H_{∞} type of disturbance reduction. The effectiveness of the proposed technique is also demonstrated by simulations involving the control of benchmark inverted pendulum on a cart system.

Multi-criteria Optimal Fuzzy Control of Nonlinear Systems (Chapter 6)

Fuzzy control has shown growing popularity in both industry and academia. To improve the optimality and robustness, we have proposed optimal fuzzy control for nonlinear systems with general performance criteria. Takagi-Sugeno fuzzy model provides us an effective tool to control nonlinear systems through the fuzzy rule models. Mixed performance criteria have been used to design the controller and the relative weighting matrices of these criteria can be achieved by choosing different coefficient matrices. The optimal control can be obtained by solving LMI at each time. The benchmark inverted pendulum on a cart problem has been used as an example to demonstrate its effectiveness. The simulation studies show that the proposed method provides a satisfactory alternative to the existing nonlinear control approaches.

Resilient Nonlinear Filtering for Stochastic Systems with Sensor Failures (Chapter 8) Any type of observer is subject to the sensor failures. Because of this, resilient nonlinear filtering for nonlinear stochastic systems with sensor failures is proposed in Chapter 8, which shows significant improvement over the traditional nonlinear estimation techniques.

9.2 Future Work

We have shown that State Dependent LMI, $H_2 - H_{\infty}$ State Dependent Riccati Equation and Muticriteria Optimal Fuzzy Control with LMI provide tractable ways to synthesize controllers for a large class of nonlinear systems in achieving general performance criteria. Since controllers and estimators are dual problems, some of our proposed control techniques results can be extended to the applications of nonlinear estimation in the future, such as optimal fuzzy estimator with general performance criteria. At the same time, we feel that the other nonlinear control approaches can be further investigated targeting nonlinear systems with special structures. Another natural followup of all this work is to investigate the real time implementation of these proposed control and estimation techniques using the latest embedded system platforms such as Digital Signal Processors and Field Programmable Gate Arrays.

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