The Cauchy Problem for a One Dimensional Nonlinear Elastic Peridynamic Model

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Abstract

This paper studies the Cauchy problem for a one-dimensional nonlinear peridynamic model describing the dynamic response of an infinitely long elastic bar. The issues of local well-posedness and smoothness of the solutions are discussed. The existence of a global solution is proved first in the sublinear case and then for nonlinearities of degree at most three. The conditions for finite-time blow-up of solutions are established.

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1. Introduction

In this study, we consider the one-dimensional nonlinear nonlocal partial differential equation, arising in the peridynamic modelling of an elastic bar,

$$u_{tt} = \int_{\mathbb{R}} \alpha(y-x)w(u(y,t)-u(x,t))dy, \quad x \in \mathbb{R}, \quad t > 0 \tag{1.1}$$

with initial data

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x). \tag{1.2}$$

In (1.1)-(1.2) the subscripts denote partial differentiation, $u = u(x,t)$ is a real-valued function, the kernel function α is an integrable function on $\mathbb{R}$ and w is a twice differentiable nonlinear function with $w(0) = 0$. We first establish the local well-posedness of the Cauchy problem (1.1)-(1.2), considering four different cases of initial data: *(i)* continuous and bounded functions, *(ii)* bounded L^p functions ($1 \leq p \leq \infty$), *(iii)* differentiable and bounded functions and *(iv)* L^p functions whose distributional derivatives are also in

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$L^p(\mathbb{R})$. We then extend the results to the case of L^2 Sobolev spaces of arbitrary (non-integer) order for the particular form $w(\eta) = \eta^3$. We prove global existence of solutions for two types of nonlinearities: when $w(\eta)$ is sublinear and when $w(\eta) = |\eta|^{\nu-1}\eta$ for $\nu \leq 3$. Lastly, for the general case we provide the conditions under which the solutions of the Cauchy problem blow-up in finite time.

Equation (1.1) is a model proposed to describe the dynamical response of an infinite homogeneous elastic bar within the context of the peridynamic formulation of elasticity theory. The peridynamic theory of solids, mainly proposed by Silling [1], is an alternative formulation for elastic materials and has attracted attention of a growing number of researchers. The most important feature of the peridynamic theory is that the force acting on a material particle, due to interaction with other particles, is written as a functional of the displacement field. This means the peridynamic theory is a nonlocal continuum theory and regarding nonlocality it bears a strong resemblance to more traditional theories of nonlocal elasticity, which are principally based on integral constitutive relations [2, 3, 4]. As in other nonlocal theories of elasticity, the main motivation is to propose a generalized elasticity theory that involves the effect of long-range internal forces of molecular dynamic, neglected in the conventional theory of elasticity. Another feature of the peridynamic theory is that the peridynamic equation of motion does not involve spatial derivatives of the displacement field. The absence of spatial derivatives of the displacement field in the equation of motion makes possible to use the peridynamic equations even at points of displacement discontinuity. Furthermore, in contrast to the conventional theory of elasticity, the peridynamic theory predicts dispersive wave propagation as a property of the medium even if the geometry does not define a length scale.

In the peridynamic theory, by assuming a uniform cross-section and the absence of body forces, the governing equation of an infinitely long elastic bar is given by

$$\rho_0 u_{tt} = \int_{\mathbb{R}} f(u(y,t) - u(x,t), y - x)dy, \tag{1.3}$$

where the axis of the bar coincides with the coordinate axis, a material point on the axis of the bar has coordinate x in the undeformed state, u and f may be interpreted as averages of the axial displacement and the axial force located at any x at time t, taken over a cross section of the bar, and ρ_0 is density of the bar material [5, 6]. The space integral in (1.3) implies that the displacement at a generic point is influenced by the displacements of all particles of the bar (As commonly known, in the conventional theory of elasticity, the equation governing the dynamic response of an infinitely long bar is a hyperbolic partial differential equation that does not involve such a space integral originating from the nonlocal character of the peridynamic theory). Equation (1.3) is obtained by integrating the equation of motion for the axial displacement over the cross-section and dividing through by the area of the cross-section. The bar is supposed to be composed of a homogeneous objective microelastic material [1, 5, 6] and its constitutive behavior is described by the function f. Newton's third law imposes the following restriction on the form of f:

$$f(\eta, \zeta) = -f(-\eta, -\zeta) \tag{1.4}$$

for all relative displacements $\eta = u(y,t) - u(x,t)$ and relative positions $\zeta = y - x$. For a linear peridynamic material the constitutive relation is given by

$$f(\eta, \zeta) = \alpha(\zeta)\eta$$

where α is called the micromodulus function [1, 5, 6]. It follows from (1.4) that α must be an even function. In [5, 6] the dynamic response of a linear peridynamic bar has been investigated and some striking observations that are not found in the classical theory of elastic bars have been made. Some results on the well-posedness of the Cauchy problem for the linear peridynamic model have been established in [7, 8, 9, 10]. In spite of its age, there is quite extensive literature on the linear peridynamic theory.

It is natural to think that more interesting behavior may be observed when the attention is confined to the nonlinear peridynamic materials. From this point of view, to the best of our knowledge, the present study appears to be the first study on mathematical analysis of nonlinear peridynamic equations. Techniques similar to those in [11, 12, 13] enable us to answer some basic questions, like local well-posedness and lifespan of solutions, as the groundwork of further analysis of the nonlinear peridynamic problem.

In this study we consider the case in which the constitutive behavior is described by a class of nonlinear peridynamic models in the separable form:

$$f(\eta, \zeta) = \alpha(\zeta)w(\eta) \tag{1.5}$$

where α and w are two functions satisfying the restriction imposed by (1.4). This separable form, while allowing us to exploit the properties of convolution-based techniques, is not a serious restriction and it just makes the proofs easier to follow. Our results can be carried over to the case of general $f(\eta, \zeta)$. We illustrate this in Theorem 2.8; by imposing certain differentiability and integrability conditions on f, we prove local well-posedness for the general nonlinear peridynamic problem. Throughout this study we assume that α is an integrable even function while w is a differentiable odd function so that (1.4) is satisfied.

Substitution of the separable form of (1.5) into (1.3) and non-dimensionalization of the resulting equation (or simply taking the mass density to be 1) gives the governing equation of the problem in its final form (1.1) (Henceforth we use non-dimensional quantities but for convenience use the same symbols). The aim of this study is three-fold: to establish the local well-posedness of the Cauchy problem, to investigate the existence of a global solution, and to present the conditions for finite-time blow-up of solutions.

The paper is organized as follows. In Section 2, the existence and uniqueness of the local solution for the nonlinear Cauchy problem is proved by using the contraction mapping principle. For initial data in fractional Sobolev spaces the general case seems to involve technical difficulties and in Section 3 we consider the particular case $w(\eta) = \eta^3$ in the L^2 Sobolev space setting. We note that the cubic case can be easily generalized to an arbitrary polynomial of η. In Section 4, we consider the issue of global existence versus finite time blow-up of solutions. We first show that blow-up must necessarily occur in the L^∞-norm. We then prove two results on global existence and finally establish blow-up criteria.

Throughout this paper, C denotes a generic constant. We use $\|u\|_\infty$ and $\|u\|_p$ to denote the norms in $L^\infty(\mathbb{R})$ and $L^p(\mathbb{R})$ spaces, respectively. The notation $\langle u, v\rangle$ denotes the inner product in $L^2(\mathbb{R})$. Furthermore, $C_b(\mathbb{R})$ denotes the space of continuous bounded functions on $\mathbb{R}$, and $C_b^1(\mathbb{R})$ is the space of differentiable functions in $C_b(\mathbb{R})$

whose first-order derivatives also belong to $C_b(\mathbb{R})$. In the spaces $C_b(\mathbb{R})$ and $C_b^1(\mathbb{R})$ we have the norms $\|u\|_\infty$ and $\|u\|_{1,b} = \|u\|_\infty + \|u'\|_\infty$, respectively, where the symbol $'$ denotes the differentiation. The Sobolev space $W^{1,p}(\mathbb{R})$ is the space of L^p functions whose distributional derivatives are also in $L^p(\mathbb{R})$ with norm $\|u\|_{W^{1,p}} = \|u\|_p + \|u'\|_p$. Similarly, for integer $k \geq 1$, $C_b^k(\mathbb{R})$ denotes the space of functions whose derivatives up to order k are continuous and bounded; $W^{k,p}(\mathbb{R})$ denotes the space of L^p functions whose derivatives up to order k are in $L^p(\mathbb{R})$.

2. Local Well Posedness

Below we will give several versions of local well-posedness of the nonlinear Cauchy problem given by (1.1)-(1.2). This is achieved in Theorems 2.2-2.5 for four different cases of initial data spaces, namely $C_b(\mathbb{R})$, $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $C_b^1(\mathbb{R})$ and $W^{1,p}(\mathbb{R})$ $(1 \leq p \leq \infty)$. The proofs will follow the same scheme given below.

If (1.1) is integrated twice with respect to t, the solution of the Cauchy problem satisfies the integral equation $u = Su$ where

$$(Su)(x,t) = \varphi(x) + t\psi(x) + \int_0^t (t-\tau)(Ku)(x,\tau)d\tau, \tag{2.1}$$

with

$$(Ku)(x,t) = \int_{\mathbb{R}} \alpha(y-x)w(u(y,t) - u(x,t))dy. \tag{2.2}$$

Let X be the Banach space with norm $\|.\|_X$, where the initial data lie. We then define the Banach space $X(T) = C([0,T], X)$, endowed with the norm $\|u\|_{X(T)} = \max_{t\in[0,T]} \|u(t)\|_X$, and the closed R-ball $Y(T) = \{u \in X(T) : \|u\|_{X(T)} \leq R\}$. We will show that for suitably chosen R and sufficiently small T, the map S is a contraction on $Y(T)$. This will be achieved by estimating first Ku and then Su in appropriate norms.

In each of the four cases, for $u, v \in Y(T)$ we will get estimates of the form

$$\|Su\|_{X(T)} \leq \|\varphi\|_X + TJ_1(R,T) \tag{2.3}$$

$$\left\| \int_0^t (t-\tau)((Ku)(\tau) - (Kv)(\tau))d\tau \right\|_{X(T)} \leq TJ_2(R,T)\|u - v\|_{X(T)} \tag{2.4}$$

and hence

$$\|Su - Sv\|_{X(T)} \leq TJ_2(R,T)\|u - v\|_{X(T)} \tag{2.5}$$

with certain functions J_1 and J_2 nondecreasing in R and T. Taking $R \geq 2\|\varphi\|_X$ and then choosing T small enough to satisfy $TJ_1(R,T) \leq R/2$ will give $S : Y(T) \to Y(T)$; the further choice $TJ_2(R,T) \leq 1/2$ will show that S is a contraction. This implies that there is a unique $u \in Y(T)$ satisfying the integral equation $u = Su$. But, as Ku is clearly continuous in t, we can differentiate (2.1) to get

$$u_t(x,t) = \psi(x) + \int_0^t (Ku)(x,\tau)d\tau$$

and consequently $u_{tt}(x,t) = (Ku)(x,t)$. This shows that $u \in C^2([0,T],X)$ solves (1.1)-(1.2). Finally, if u_1 and u_2 satisfy (1.1)-(1.2) with initial data φ_i, ψ_i for $i = 1,2$ we get

$$u_1 - u_2 = \varphi_1 - \varphi_2 + t(\psi_1 - \psi_2) + \int_0^t (t-\tau)((Ku_1)(\tau) - (Ku_2)(\tau))d\tau.$$

Then the estimate (2.4) shows that

$$\|u_1 - u_2\|_{X(T)} \le \|\varphi_1 - \varphi_2\|_X + t\,\|\psi_1 - \psi_2\|_X + TJ_2(R,T)\,\|u_1 - u_2\|_{X(T)}.$$

When $TJ_2(R,T) \le 1/2$,

$$\|u_1 - u_2\|_{X(T)} \le 2\,\|\varphi_1 - \varphi_2\|_X + 2t\,\|\psi_1 - \psi_2\|_X$$

for $t \in [0,T]$. This shows that, locally, solutions of (1.1)-(1.2) depend continuously on initial data; thus the problem (1.1)-(1.2) is locally well posed.

The Mean Value Theorem for nonlinear estimates and the following lemma for convolution estimates will be our main tools:

Lemma 2.1. *Let $1 \le p \le \infty$ and $f \in L^1(\mathbb{R})$, $g \in L^p(\mathbb{R})$. The convolution $(f * g)(x) = \int_{\mathbb{R}} f(y-x)g(y)dy$ is well defined and $f * g \in L^p(\mathbb{R})$ with*

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

In the estimates below, we will often encounter the nondecreasing function $M(R)$ defined for $R > 0$ as

$$M(R) = \max_{|\eta| \le 2R} |w'(\eta)|. \tag{2.6}$$

We now state and prove (i.e. show that the estimates (2.3) and (2.5) hold) the four theorems of local well posedness.

Theorem 2.2. *Assume that $\alpha \in L^1(\mathbb{R})$ and $w \in C^1(\mathbb{R})$ with $w(0) = 0$. Then there is some $T > 0$ such that the Cauchy problem (1.1)-(1.2) is well posed with solution in $C^2([0,T], C_b(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b(\mathbb{R})$.*

Proof. Take $X = C_b(\mathbb{R})$. For $u \in Y(T)$, clearly Ku is continuous in x and t and hence $Su \in C^2([0,T],X)$. Since $w(0) = 0$ and

$$|u(y,t) - u(x,t)| \le 2\|u(t)\|_\infty,$$

the Mean Value Theorem implies

$$|w(u(y) - u(x))| \le \sup_{|\eta| \le 2\|u\|_\infty} |w'(\eta)|\ |u(y) - u(x)| = M(\|u\|_\infty)(|u(y)| + |u(x)|),$$

where we have suppressed the t variable for convenience. Then

$$\begin{aligned} |(Ku)(x,t)| &\le M(\|u(t)\|_\infty) \int_{\mathbb{R}} |\alpha(y-x)|\,(|u(y,t)| + |u(x,t)|)dy \\ &= M(\|u(t)\|_\infty)\left[(|\alpha| * |u|)(x,t) + \|\alpha\|_1 |u(x,t)|\right], \end{aligned} \tag{2.7}$$

and

$$\|(Ku)(t)\|_{\infty} \leq 2M(\|u(t)\|_{\infty})\|\alpha\|_1\|u(t)\|_{\infty} \tag{2.8}$$

where we have used Lemma 2.1. Then

$$|(Su)(x,t)| \leq |\varphi(x)| + t\,|\psi(x)| + \int_0^t (t-\tau)\,|(Ku)(x,\tau)|\,d\tau,$$

and

$$\|(Su)(t)\|_{\infty} \leq \|\varphi\|_{\infty} + t\|\psi\|_{\infty} + 2\|\alpha\|_1 \int_0^t (t-\tau)M(\|u(\tau)\|_{\infty})\|u(\tau)\|_{\infty} d\tau. \tag{2.9}$$

As $u \in Y(T)$, this gives $M(\|u(\tau)\|_{\infty}) \leq M(R)$ and hence

$$\begin{aligned} \|Su\|_{X(T)} &\leq \|\varphi\|_{\infty} + T\|\psi\|_{\infty} + 2M(R)\|\alpha\|_1\|u\|_{X(T)} \sup_{t\in[0,T]} \int_0^t (t-\tau)d\tau \\ &\leq \|\varphi\|_{\infty} + T\|\psi\|_{\infty} + M(R)R\|\alpha\|_1 T^2. \end{aligned} \tag{2.10}$$

This proves (2.3) with $J_1(R,T) = \|\psi\|_{\infty} + M(R)R\|\alpha\|_1 T$. Now let $u, v \in Y(T)$. We start by estimating $Ku - Kv$. Again suppressing t,

$$|w(u(y)-u(x)) - w(v(y)-v(x))| \leq M(R)(|u(y)-v(y)| + |u(x)-v(x)|),$$

and

$$\begin{aligned} |(Ku)(x,t) - (Kv)(x,t)| &\leq M(R)\left(|\alpha| * |u-v|\right)(x,t) \\ &\quad + M(R)\|\alpha\|_1|u(x,t)-v(x,t)|. \end{aligned} \tag{2.11}$$

Similar to (2.10) we get

$$\|(Su)(t) - (Sv)(t)\|_{\infty} \leq 2M(R)\|\alpha\|_1 \int_0^t (t-\tau)\|u(\tau)-v(\tau)\|_{\infty}\, d\tau \tag{2.12}$$

and

$$\|Su - Sv\|_{X(T)} \leq M(R)\|\alpha\|_1 T^2 \|u-v\|_{X(T)} \tag{2.13}$$

which proves (2.5) with $J_2(R,T) = M(R)\|\alpha\|_1 T$. According to the scheme described above, this completes the proof. □

Theorem 2.3. *Let $1 \leq p \leq \infty$. Assume that $\alpha \in L^1(\mathbb{R})$ and $w \in C^1(\mathbb{R})$ with $w(0) = 0$. Then there is some $T > 0$ such that the Cauchy problem (1.1)-(1.2) is well posed with solution in $C^2([0,T], L^p(\mathbb{R}) \cap L^{\infty}(\mathbb{R}))$ for initial data $\varphi, \psi \in L^p(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$.*

Proof. Let $X = L^p(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ with norm $\|u\|_X = \|u\|_p + \|u\|_{\infty}$. As we already have the L^{∞} estimates given in (2.9) and (2.12), we now look for the corresponding L^p estimates. Lemma 2.1 implies $\|(|\alpha| * |u|)(t)\|_p \leq \|\alpha\|_1| \, \|u(t)\|_p$ so

$$\|(Ku)(t)\|_p \leq 2M(\|u(t)\|_{\infty})\|\alpha\|_1\|u(t)\|_p \tag{2.14}$$

and Minkowski's inequality for integrals will yield

$$\|(Su)(t)\|_p \leq \|\varphi\|_p + t\|\psi\|_p + 2\|\alpha\|_1 \int_0^t (t-\tau)M(\|u(\tau)\|_\infty)\|u(\tau)\|_p \, d\tau. \tag{2.15}$$

Adding this to the L^∞ estimate (2.9), we get

$$\|Su\|_{X(T)} \leq \|\varphi\|_X + T\|\psi\|_X + M(R)R\|\alpha\|_1 T^2.$$

Similarly we have

$$\|(Su)(t) - (Sv)(t)\|_p \leq 2M(R)\|\alpha\|_1 \int_0^t (t-\tau)\|u(\tau) - v(\tau)\|_p \, d\tau. \tag{2.16}$$

Adding this to (2.12) gives

$$\|Su - Sv\|_{X(T)} \leq M(R)\|\alpha\|_1 T^2 \|u - v\|_{X(T)}$$

and concludes the proofs of (2.3) and (2.5). □

Theorem 2.4. *Assume that $\alpha \in L^1(\mathbb{R})$ and $w \in C^2(\mathbb{R})$ with $w(0) = 0$. Then there is some $T > 0$ such that the Cauchy problem (1.1)-(1.2) is well posed with solution in $C^2([0,T], C_b^1(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b^1(\mathbb{R})$.*

Proof. We now take $X = C_b^1(\mathbb{R})$ for which the norm is $\|u\|_{1,b} = \|u\|_\infty + \|u'\|_\infty$. Since we have the sup norm estimates (2.9) and (2.12) all we need is estimates for their x derivatives. Throughout this proof we will suppress t (or τ) to keep the expressions shorter, whenever it is clear from the context. Differentiating (2.2) gives

$$\begin{aligned}
\frac{\partial}{\partial x}(Ku)(x) &= \frac{\partial}{\partial x}\int_{\mathbb{R}} \alpha(y-x)w(u(y)-u(x))dy \\
&= \frac{\partial}{\partial x}\int_{\mathbb{R}} \alpha(z)w(u(x+z)-u(x))dz \\
&= \int_{\mathbb{R}} \alpha(z)w'(u(x+z)-u(x))(u_x(x+z)-u_x(x))dz \\
&= \int_{\mathbb{R}} \alpha(y-x)w'(u(y)-u(x))(u_x(y)-u_x(x))dy.
\end{aligned}$$

Recall that $|w'(u(y)-u(x))| \leq M(\|u(t)\|_\infty)$ due to (2.6). Then

$$\begin{aligned}
|(Ku)_x(x)| &\leq M(\|u\|_\infty)\int_{\mathbb{R}} |\alpha(y-x)|\,(|u_x(y)| + |u_x(x)|)dy \\
&\leq M(\|u\|_\infty)[(|\alpha| * |u_x|)(x) + \|\alpha\|_1|u_x(x)|]. \qquad (2.17)
\end{aligned}$$

Since

$$|(Su)_x(x,t)| \leq |\varphi'(x)| + t\,|\psi'(x)| + \int_0^t (t-\tau)\,|(Ku)_x(x,\tau)|d\tau, \tag{2.18}$$

we have

$$\|(Su)_x(t)\|_\infty \leq \|\varphi'\|_\infty + t\|\psi'\|_\infty + 2\|\alpha\|_1 \int_0^t (t-\tau)M(\|u(\tau)\|_\infty)\|u_x(\tau)\|_\infty d\tau.$$

But $M(\|u(\tau)\|_\infty) \le M(R)$ so adding up with the estimate (2.9) proves (2.3)

$$\begin{aligned}\|Su\|_{X(T)} &= \max_{t\in[0,T]}(\|(Su)(t)\|_\infty + \|(Su)_x(t)\|_\infty) \\ &\le \|\varphi\|_{1,b} + T\|\psi\|_{1,b} + M(R)R\|\alpha\|_1 T^2.\end{aligned}$$

Next, for $|\eta_i| \le 2R$ and $|\mu_i| \le 2R$ for $(i=1,2)$, we estimate

$$\begin{aligned}\left|w'(\eta_1)\mu_1 - w'(\eta_2)\mu_2\right| &\le \left|w'(\eta_1)\right|\left|\mu_1-\mu_2\right| + \left|w'(\eta_1) - w'(\eta_2)\right|\left|\mu_2\right| \\ &\le M(R)\left|\mu_1-\mu_2\right| + 2R\max_{\eta\le 2R}\left|w''(\eta)\right|\left|\eta_1-\eta_2\right| \\ &\le M(R)\left|\mu_1-\mu_2\right| + 2RN(R)\left|\eta_1-\eta_2\right|\end{aligned}$$

where $N(R) = \max_{\eta\le 2R}\left|w''(\eta)\right|$. Then

$$\begin{aligned}|(Ku-Kv)_x(x)| &\le M(R)\int_{\mathbb{R}}|\alpha(y-x)|\left(|u_x(y)-v_x(y)| + |u_x(x)-v_x(x)|\right)dy \\ &\quad +2RN(R)\int_{\mathbb{R}}|\alpha(y-x)|\left(|u(y)-v(y)| + |u(x)-v(x)|\right)dy \\ &\le M(R)((|\alpha| * |u_x-v_x|)(x) + \|\alpha\|_1|u_x(x)-v_x(x)|) \\ &\quad +2RN(R)((|\alpha| * |u-v|)(x) + \|\alpha\|_1|u(x)-v(x)|) \qquad (2.19)\end{aligned}$$

and

$$\begin{aligned}\|(Su-Sv)_x(t)\|_\infty &\le 2M(R)\|\alpha\|_1\int_0^t (t-\tau)\|u_x(\tau)-v_x(\tau)\|_\infty d\tau \\ &\quad +4RN(R)\|\alpha\|_1\int_0^t (t-\tau)\|u(\tau)-v(\tau)\|_\infty d\tau \\ &\le (M(R)+2RN(R))\|\alpha\|_1 T^2\|u-v\|_{X(T)} \qquad (2.20)\end{aligned}$$

Finally, adding this to (2.12) we get (2.5) in the form

$$\begin{aligned}\|Su-Sv\|_{X(T)} &\le \max_{t\in[0,T]}(\|(Su-Sv)(t)\|_\infty + \|(Su-Sv)_x(t)\|_\infty) \\ &\le 2(M(R)+RN(R))\|\alpha\|_1 T^2\|u-v\|_{X(T)}.\end{aligned}$$

□

Theorem 2.5. *Let $1 \le p \le \infty$. Assume that $\alpha \in L^1(\mathbb{R})$ and $w \in C^2(\mathbb{R})$ with $w(0)=0$. Then there is some $T>0$ such that the Cauchy problem (1.1)-(1.2) is well posed with solution in $C^2([0,T], W^{1,p}(\mathbb{R}))$ for initial data $\varphi,\psi \in W^{1,p}(\mathbb{R})$.*

Proof. Let $X = W^{1,p}(\mathbb{R}) \subset L^\infty(\mathbb{R})$. Since $\|u\|_{W^{1,p}} = \|u\|_p + \|u'\|_p$, we need derivative estimates only in addition to the L^p estimates (2.15) and (2.16). For $u,v \in Y(T)$, from (2.17)-(2.18) and Minkowski's inequality we have

$$\|(Su)_x(t)\|_p \le \|\varphi'\|_p + t\|\psi'\|_p + 2\|\alpha\|_1\int_0^t (t-\tau)M(\|u(\tau)\|_\infty)\|u_x(\tau)\|_p d\tau.$$

We note that the term $\|u\|_\infty$ can be eliminated by using $\|u\|_\infty \leq C\|u\|_{W^{1,p}}$ due to the Sobolev Embedding Theorem. So $M(\|u(\tau)\|_\infty) \leq M(CR)$ and adding up the above estimate with (2.15) proves (2.3);

$$\begin{aligned}\|Su\|_{X(T)} &= \max_{t\in[0,T]}(\|(Su)(t)\|_p + \|(Su)_x(t)\|_p) \\ &\leq \|\varphi\|_{W^{1,p}} + T\|\psi\|_{W^{1,p}} + M(CR)R\|\alpha\|_1 T^2.\end{aligned}$$

Again from (2.19) we get

$$\begin{aligned}\|(Su - Sv)_x(t)\|_p &\leq 2M(CR)\|\alpha\|_1 \int_0^t (t-\tau)\|u_x(\tau) - v_x(\tau)\|_p d\tau \\ &\quad +4RN(CR)\|\alpha\|_1 \int_0^t (t-\tau)\|u(\tau) - v(\tau)\|_p d\tau.\end{aligned}$$

Together with (2.16), we conclude the proof:

$$\|Su - Sv\|_{X(T)} \leq 2\left(M(CR) + RN(CR)\right)\|\alpha\|_1 T^2 \|u - v\|_{X(T)}.$$

□

Remark 2.6. *We remark that the investigation can also continue for smoother data in along the same lines. That is, for initial data in $C_b^k(\mathbb{R})$ or $W^{k,p}(\mathbb{R})$ with integer k we can prove higher-order versions of Theorems 2.4-2.5. Also, the proofs clearly indicate that in Theorems 2.2 and 2.3 we can replace the assumption $w \in C^1(\mathbb{R})$ with its weaker form: w is locally Lipschitz. Similarly, in Theorems 2.4 and 2.5 the assumption $w \in C^2(\mathbb{R})$ can be weakened to the condition: w' is locally Lipschitz.*

Remark 2.7. *The above theorems of local well-posedness can be easily adapted to the general peridynamic equation (1.3). Theorem 2.8 below extends Theorem 2.2 to the general peridynamic equation (1.3). Clearly, similar extensions are also possible in the cases of Theorems 2.3-2.5.*

Theorem 2.8. *Assume that $f(\zeta, 0) = 0$ and $f(\zeta, \eta)$ is continuously differentiable in η for almost all ζ. Moreover, suppose that for each $R > 0$, there are integrable functions Λ_1^R, Λ_2^R satisfying*

$$|f(\zeta,\eta)| \leq \Lambda_1^R(\zeta), \qquad |f_\eta(\zeta,\eta)| \leq \Lambda_2^R(\zeta)$$

for almost all ζ and for all $|\eta| \leq 2R$. Then there is some $T > 0$ such that the Cauchy problem (1.3)-(1.2) is well posed with solution in $C^2([0,T], C_b(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b(\mathbb{R})$.

Proof. We proceed as in the proof of Theorem 2.2. By the Dominated Convergence Theorem, the condition $|f(\zeta,\eta)| \leq \Lambda_1^R(\zeta)$ implies that Ku is continuous in x so that $S : X(T) \to X(T)$. Using the second inequality $|f_\eta(\zeta,\eta)| \leq \Lambda_2^R(\zeta)$ the estimates for $\|(Su)(t)\|_\infty$ and $\|(Su)(t) - (Sv)(t)\|_\infty$ follow as in (2.10) and (2.13), just replacing the term $M(R)\|\alpha\|_1$ by $\|\Lambda_1^R\|_1$ and $\|\Lambda_2^R\|_1$ respectively, completing the proof. □

Remark 2.9. *To finish this section let us briefly mention the issue of multidimensional case in the general three-dimensional peridynamic theory. Although our analysis in this section has been presented for the one-dimensional case of the peridynamic formulation, the techniques used can be extended to the case of a system of three peridynamic equations in three space variables without any additional complication. Namely, if we replace the scalars x, y, u, w and α in (1.1)-(1.2) by the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{w}(\mathbf{u})$ and the matrix $\alpha(\mathbf{x})$, respectively, the local existence theorems given above will still be valid.*

3. The Cubic Nonlinear Case in $H^s(\mathbb{R})$

We now want to consider the Cauchy problem (1.1)-(1.2) in the L^2 Sobolev space setting. We will denote the L^2 Sobolev space of order s on $\mathbb{R}$ by $H^s(\mathbb{R})$ with norm

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{u}(\xi)|^2 \mathrm{d}\xi$$

where $\widehat{u}$ denotes the Fourier transform of u. For integer $k \geq 0$, $H^k(\mathbb{R}) = W^{k,2}(\mathbb{R})$.

As mentioned in Remark 2.6, the proof in the case of $H^1(\mathbb{R})$ can be extended to $H^k(\mathbb{R})$. On the other hand, for non-integer s, H^s estimates of the nonlinear term $w(u(y)-u(x))$ involve technical difficulties. Nevertheless, the case of polynomial nonlinearities can be handled in a straightforward manner. We illustrate this in the typical case $w(\eta) = \eta^3$. Then, the integral on the right-hand side of (1.1) can be computed explicitly in terms of convolutions and the Cauchy problem (1.1)-(1.2) becomes

$$u_{tt} = \alpha * u^3 - 3u(\alpha * u^2) + 3u^2(\alpha * u) - Au^3 \tag{3.1}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \tag{3.2}$$

where $A = \int_{\mathbb{R}} \alpha(y)dy$.

For the estimates below we need the following lemmas.

Lemma 3.1. *Let $\alpha \in L^1(\mathbb{R})$ and $u \in H^s(\mathbb{R})$ for $s \geq 0$. Then $\alpha * u \in H^s(\mathbb{R})$ and*

$$\|\alpha * u\|_{H^s} \leq \|\alpha\|_1 \|u\|_{H^s}.$$

Lemma 3.2. *[14] Let $s \geq 0$ and $u, v \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then $uv \in H^s(\mathbb{R})$ and for some constant C (independent of u and v)*

$$\|uv\|_{H^s} \leq C(\|u\|_\infty \|v\|_{H^s} + \|v\|_\infty \|u\|_{H^s}).$$

For the space $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we use the norm $\|u\|_{s,\infty} = \|u\|_{H^s} + \|u\|_\infty$. In general, Lemma 3.2 implies that $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is an algebra

$$\|uv\|_{s,\infty} \leq C \|u\|_{s,\infty} \|v\|_{s,\infty}, \tag{3.3}$$

and, by Lemmas 2.1 and 3.1, for $\alpha \in L^1(\mathbb{R})$

$$\|\alpha * u\|_{s,\infty} \leq \|\alpha\|_1 \|u\|_{s,\infty}. \tag{3.4}$$

We are now ready to prove the following theorem.

Theorem 3.3. *Let* $s > 0$. *Assume that* $\varphi, \psi \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Then there is some* $T > 0$ *such that the Cauchy problem (3.1)-(3.2) is well posed with solution in* $C^2([0,T], H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

Proof. We follow the scheme summarized at the beginning of Section 2 for $X = H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Explicitly,

$$Ku = \alpha * u^3 - 3u(\alpha * u^2) + 3u^2(\alpha * u) - Au^3.$$

We start by estimating the terms of the form $u^i(\alpha * u^j)$ for $i + j = 3$. Clearly from (3.3) and (3.4), $\left\|u^i\left(\alpha * u^j\right)\right\|_{s,\infty} \le C\left\|\alpha\right\|_1 \left\|u\right\|_{s,\infty}^3$. Nevertheless, for later use we derive a more precise estimate. By repeated use of Lemma 3.2 we have $\left\|u^j\right\|_{H^s} \le C_j \left\|u\right\|_\infty^{j-1} \left\|u\right\|_{H^s}$. Again, by Lemmas 3.1 and 3.2

$$\begin{aligned}
\left\|u^i\left(\alpha * u^j\right)\right\|_{H^s} &\le C(\left\|u^i\right\|_{H^s}\left\|\alpha * u^j\right\|_\infty + \left\|u^i\right\|_\infty \left\|\alpha * u^j\right\|_{H^s}) \\
&\le C\left(C_i + C_j\right)\left\|\alpha\right\|_1 \left\|u\right\|_\infty^2 \left\|u\right\|_{H^s},
\end{aligned}$$

so that

$$\left\|Ku\right\|_{s,\infty} \le C\left\|\alpha\right\|_1 \left\|u\right\|_\infty^2 \left\|u\right\|_{s,\infty}.$$

Similarly

$$\begin{aligned}
\left\|u^i(\alpha * u^j) - v^i(\alpha * v^j)\right\|_{s,\infty} &\le \left\|u^i(\alpha * (u^j - v^j))\right\|_{s,\infty} + \left\|(u^i - v^i)(\alpha * v^j)\right\|_{s,\infty} \\
&\le C\left(\left\|u^i\right\|_{s,\infty}\left\|\alpha * (u^j - v^j)\right\|_{s,\infty} + \left\|u^i - v^i\right\|_{s,\infty}\left\|\alpha * v^j\right\|_{s,\infty}\right) \\
&\le C\left\|\alpha\right\|_1\left(\left\|u^i\right\|_{s,\infty}\left\|u^j - v^j\right\|_{s,\infty} + \left\|v^j\right\|_{s,\infty}\left\|u^i - v^i\right\|_{s,\infty}\right) \\
&\le \left\|\alpha\right\|_1 P\left(\left\|u\right\|_{s,\infty}, \left\|v\right\|_{s,\infty}\right)\left\|u - v\right\|_{s,\infty}
\end{aligned}$$

where P is some quadratic polynomial of two variables with nonnegative coefficients. The above results yield the following estimates for $u, v \in Y(T)$

$$\|Su\|_{X(T)} \le \|\varphi\|_{s,\infty} + T\|\psi\|_{s,\infty} + C\|\alpha\|_1 R^3 T^2,$$

and

$$\|Su - Sv\|_{X(T)} \le P\left(R, R\right)\|\alpha\|_1 T^2 \|u - v\|_{X(T)}$$

concluding the proofs of (2.3) and (2.5). □

4. Global Existence and Blow Up in Finite Time

In this section,we will first show that the maximal time of existence for the solution of the Cauchy problem (1.1)-(1.2) depends only on the L^∞ norm of the initial data. Then we will prove the existence of a global solution for two classes of nonlinearities and finally investigate blow-up for general nonlinearities.

4.1. Global Existence

By repeatedly applying local existence theorems (Theorems 2.2-2.5 and 3.3) the solution can be continued to the maximal time interval $[0, T_{\max})$ where either $T_{\max} = \infty$, i.e. we have a global solution, or

$$\lim_{t \to T_{\max}^-} \sup \left(\| u(t) \|_X + \| u_t(t) \|_X \right) = \infty,$$

where $\| \ \|_X$ denotes either one of the norms in $C_b(\mathbb{R})$, $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $C_b^1(\mathbb{R})$, $W^{1,p}(\mathbb{R})$ or $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Theorem 4.1. *Assume that the conditions in either one of Theorems 2.2-2.5 or 3.3 hold. Then either there is a global solution or maximal time is finite, where $T_{\max}$ is characterized by the L^∞ blow-up condition*

$$\lim_{t \to T_{\max}^-} \sup \| u(t) \|_\infty = \infty.$$

Proof. Clearly in each case the norm $\| \ \|_\infty$ is smaller than $\| \ \|_X$. Hence it suffices to prove that if $\limsup_{t \to T^-} \| u(t) \|_\infty = M < \infty$, then $\limsup_{t \to T^-} (\| u(t) \|_X + \| u_t(t) \|_X) < \infty$. So assume that the solution exists in some interval $[0, T)$ and satisfies $\| u(t) \|_\infty \le R$ for all $0 \le t < T$. The solution satisfies

$$\begin{aligned} u(x,t) &= \varphi(x) + t\psi(x) + \int_0^t (t - \tau)(Ku)(x, \tau) d\tau, \\ u_t(x,t) &= \psi(x) + \int_0^t (Ku)(x, \tau) d\tau. \end{aligned}$$

In all cases the estimate for Ku is of the form

$$\|Ku\|_X \le \mathcal{M}(\|u\|_\infty) \|u\|_X$$

with a nondecreasing function $\mathcal{M}$ of $\|u\|_\infty$. Since $\| u(t) \|_\infty \le R$ for all $t \in [0, T)$,

$$\|u(t)\|_X + \|u_t(t)\|_X \le \|\varphi\|_X + (1+T) \|\psi\|_X + (1+T) \mathcal{M}(R) \int_0^t \|u(\tau)\|_X \, d\tau,$$

so that Gronwall's Lemma implies

$$\|u(t)\|_X + \|u_t(t)\|_X \le (\|\varphi\|_X + (1+T) \|\psi\|_X) \, e^{(1+T)\mathcal{M}(R)t}$$

for all $t \in [0, T)$. So $\limsup_{t \to T^-} (\| u(t) \|_X + \| u_t(t) \|_X) < \infty$. □

Theorem 4.2. *Assume that the conditions in either one of Theorems 2.2-2.5 hold. If the nonlinear term w in (1.1) satisfies $|w(\eta)| \le a|\eta| + b$ for all $\eta \in \mathbb{R}$, then there is a global solution.*

Proof. Assume the solution exists on $[0, T)$. Then

$$\begin{aligned} |(Ku)(x,t)| &\le \int_{\mathbb{R}} |\alpha(y - x)| \left(a \, |u(y, \tau) - u(x, \tau)| + b \right) dy \\ &\le a \left(|\alpha| * |u| \right)(x,t) + a \|\alpha\|_1 |u(x,t)| + b \|\alpha\|_1 , \end{aligned}$$

and by (2.1)

$$\begin{aligned} \|u(t)\|_\infty &\le \|\varphi\|_\infty + t\|\psi\|_\infty + \int_0^t (t-\tau)\left(a\left\|(|\alpha|*|u|)(\tau)\right\|_\infty + a\|\alpha\|_1\|u(\tau)\|_\infty + b\|\alpha\|_1\right)d\tau \\ &\le \|\varphi\|_\infty + T\|\psi\|_\infty + bT\|\alpha\|_1 + 2aT\|\alpha\|_1\int_0^t \|u(\tau)\|_\infty d\tau, \end{aligned}$$

and Gronwall's lemma shows that $\limsup_{t\to T^-}\|u(t)\|_\infty < \infty$. □

Lemma 4.3. *(The Energy Identity) Assume that $\alpha \in L^1(\mathbb{R})$ is even and $w \in C^1(\mathbb{R})$ is odd with $w(0) = 0$. If u satisfies the Cauchy problem (1.1)-(1.2) on $[0, T)$ with initial data $\varphi, \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then the energy*

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^2} \alpha(y-x)\ W(u(y,t) - u(x,t))\,dydx,$$

is constant for $t \in [0, T)$, where $W(\eta) = \int_0^\eta w(\rho)d\rho$.

Proof. By Theorem 2.3 with $p = 1$ we know $u \in C^2([0, T], L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. Since $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$, we have $u_t(t) \in L^2(\mathbb{R})$. Moreover, an estimate similar to (2.7) where w is replaced by W shows that the term $\alpha(y-x)W(u(y,t)-u(x,t))$ is integrable on $\mathbb{R}^2$. Hence $E(t)$ is defined for all $t \in [0, T)$. Multiplying (1.1) by $u_t(x,t)$ and integrating in x we obtain

$$\begin{aligned} \int_{\mathbb{R}} u_{tt}(x)\,u_t(x)\,dx &= \int_{\mathbb{R}^2} \alpha(y-x)\,w(u(y)-u(x))\,u_t(x)\,dydx \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \alpha(y-x)\,w(u(y)-u(x))\,u_t(x)\,dydx \\ &\quad +\frac{1}{2}\int_{\mathbb{R}^2} \alpha(y-x)\,w(u(y)-u(x))\,u_t(x)\,dydx, \end{aligned}$$

where we have again suppressed t. We now change the order of integration and switch the variables x, y in the last integral to obtain

$$\frac{1}{2}\int_{\mathbb{R}^2} \alpha(x-y)\,w(u(x)-u(y))\,u_t(y)\,dydx.$$

Since α is even while w is odd, this gives

$$-\frac{1}{2}\int_{\mathbb{R}^2} \alpha(y-x)\,w(u(y)-u(x))\,u_t(y)\,dydx,$$

so that

$$\int_{\mathbb{R}} u_{tt}(x)\,u_t(x)\,dx = -\frac{1}{2}\int_{\mathbb{R}^2} \alpha(y-x)\,w(u(y)-u(x))\,(u_t(y)-u_t(x))\,dydx.$$

But since $W' = w$; we have

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} (u_t(x))^2\,dx = -\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^2} \alpha(y-x)\,W(u(y)-u(x))\,dydx$$

so that $\frac{dE}{dt} = 0$. □

Theorem 4.4. *Assume that $\alpha \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is even with $\alpha \geq 0$ almost everywhere; $w \in C^1(\mathbb{R})$ is odd with $w(0) = 0$ and $W \geq 0$. If there is some $q \geq \frac{4}{3}$ and $C > 0$ so that*

$$|w(\eta)|^q \leq CW(\eta) \tag{4.1}$$

for all $\eta \in \mathbb{R}$, then there is a global solution for initial data $\varphi, \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Proof. Assume that the solution exists in $[0, T)$. By Lemma 4.3 the energy is finite and the energy identity $E(t) = E(0)$ holds for all $t \in [0, T)$. Consider the energy density function

$$e(x,t) = \frac{1}{2}(u_t(x,t))^2 + \int_{\mathbb{R}} \alpha(y-x)\ W\left(u\left(y,t\right) - u\left(x,t\right)\right) dy.$$

Differentiating with respect to t

$$\begin{aligned} e_t(x,t) &= u_t(x,t)u_{tt}(x,t) + \int_{\mathbb{R}} \alpha(y-x)\ w(u(y,t) - u(x,t))(u_t(y,t) - u_t(x,t))dy \\ &= \int_{\mathbb{R}} \alpha(y-x)\ w(u(y,t) - u(x,t))u_t(y,t)dy. \end{aligned}$$

Note that by the assumptions of the theorem $e(x,t)$ and $e_t(x,t)$ are in $L^\infty(\mathbb{R})$ for each fixed t. Letting p be the dual index to q; i.e. $1/p + 1/q = 1$, we have

$$\begin{aligned} |e_t\left(x,t\right)| &\leq \int_{\mathbb{R}} \alpha(y-x)\ \ |w(u(y,t) - u(x,t))|\, |u_t(y,t)|\, dy \\ &\leq \|\alpha\|_\infty^{1/p} \|u_t(t)\|_\infty^{1-2/p} \int_{\mathbb{R}} |u_t(y,t)|^{2/p} \left(\alpha(y-x)\right)^{1/q}\ \ |w(u(y,t) - u(x,t))|\, dy, \end{aligned}$$

and by Hölder's inequality

$$|e_t(x,t)| \leq \|\alpha\|_\infty^{1/p} \|u_t(t)\|_\infty^{1-2/p} \left(\int_{\mathbb{R}} |u_t(y,t)|^2\, dy\right)^{1/p} \left(\int_{\mathbb{R}} \alpha(y-x)\ \ |w(u(y,t) - u(x,t))|^q\, dy\right)^{1/q}.$$

Using the condition (4.1) we have

$$|e_t(x,t)| \leq \|\alpha\|_\infty^{1/p} \|u_t(t)\|_\infty^{1-2/p} \|u_t(t)\|_2^{2/p} \left(C \int_{\mathbb{R}} \alpha(y-x)\ W\left(u\left(y,t\right) - u\left(x,t\right)\right) dy\right)^{1/q}.$$

Since $\alpha \geq 0$ and $W \geq 0$, by the energy identity we have $\|u_t\left(t\right)\|_2^2 \leq 2E\left(0\right)$. Also, both terms in $e(x,t)$ are nonnegative so that taking essential supremum over $x \in \mathbb{R}$,

$$\begin{aligned} \|e_t\left(t\right)\|_\infty &\leq \|\alpha\|_\infty^{1/p} \left(2E(0)\right)^{1/p} \left(2\left\|e\left(t\right)\right\|_\infty\right)^{1/2-1/p} \left(C\left\|e\left(t\right)\right\|_\infty\right)^{1/q} \\ &\leq C\left\|e\left(t\right)\right\|_\infty^r \end{aligned}$$

with $r = 1/2 - 1/p + 1/q = 2/q - 1/2$ and some other constant C in the last line. Note that when $q \geq 4/3$, $r = 2/q - 1/2 \leq 1$. Since

$$e\left(x,t\right) = e\left(x,0\right) + \int_0^t e_t(x,\tau)d\tau$$

we have

$$\begin{aligned} \|e(t)\|_\infty &\leq \|e(0)\|_\infty + \int_0^t \|e_t(\tau)\|_\infty \, d\tau \\ &\leq \|e(0)\|_\infty + C\int_0^t \|e(\tau)\|_\infty^r \, d\tau, \end{aligned}$$

for all $t \in [0, T)$. As $r \leq 1$, we have $\|e(t)\|_\infty^r \leq \|e(t)\|_\infty + 1$. By Gronwall's lemma $\|e(t)\|_\infty$ and thus $\|u_t(t)\|_\infty$ stay bounded in $[0, T)$. Integration again gives

$$\|u(t)\|_\infty \leq \|\varphi\|_\infty + \int_0^t \|u_t(\tau)\|_\infty \, d\tau$$

so that $\|u(t)\|_\infty$ does not blow up in finite time. □

Remark 4.5. *Considering the typical nonlinearity* $w(\eta) = |\eta|^{\nu-1}\eta$ *we have* $W(\eta) = \frac{1}{\nu+1}|\eta|^{\nu+1}$. *Then the exponent* q *of Theorem 4.4 equals* $(\nu+1)/\nu$ *and* $q \geq \frac{4}{3}$ *if and only if* $\nu \leq 3$. *In other words Theorem 4.4 applies to at most cubic nonlinearities.*

4.2. Blow-up

In this section, we will consider the blow-up of the solution for the Cauchy problem (1.1)-(1.2) by the concavity method. For this purpose, we will use the following lemma to prove blow up in finite time.

Lemma 4.6. *[15] Suppose* $H(t)$, $t \geq 0$ *is a positive, twice differentiable function satisfying* $H''(t)H(t) - (1+\nu)(H'(t))^2 \geq 0$ *where* $\nu > 0$. *If* $H(0) > 0$ *and* $H'(0) > 0$, *then* $H(t) \to \infty$ *as* $t \to t_1$ *for some* $t_1 \leq H(0)/\nu H'(0)$.

Theorem 4.7. *Suppose that* α *is even,* w *is odd, the conditions of Theorem 2.3 hold for* $p = 1$ *and* $\alpha \geq 0$ *almost everywhere. If there is some* $\nu > 0$ *such that*

$$\eta w(\eta) \leq 2(1+2\nu) W(\eta) \quad \textit{for all} \quad \eta \in \mathbb{R},$$

and

$$E(0) = \frac{1}{2}\|\psi\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^2} \alpha(y-x)\ W(\varphi(y) - \varphi(x))\, dydx < 0,$$

then the solution u *of the Cauchy problem (1.1)-(1.2) blows up in finite time.*

Proof. Assume that there is a global solution. Then $u(t), u_t(t) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ for all $t > 0$. Let $H(t) = \|u(t)\|_2^2 + b(t+t_0)^2$ for some positive constants b and t_0 to be determined later. Suppressing the t variable throughout the computations

$$\begin{aligned} H'(t) &= 2\langle u, u_t \rangle + 2b(t+t_0) \\ H''(t) &= 2\|u_t\|_2^2 + 2\langle u, u_{tt}\rangle + 2b. \end{aligned}$$

Using (1.1)

$$\begin{aligned} 2\langle u, u_{tt}\rangle &= 2\int_{\mathbb{R}^2} \alpha(y-x)\ w(u(y) - u(x))\, u(x) dydx \\ &= \int_{\mathbb{R}^2} \alpha(y-x)\, w(u(y) - u(x))\, u(x)\, dydx \\ &\quad + \int_{\mathbb{R}^2} \alpha(y-x)\, w(u(y) - u(x))\, u(x)\, dydx. \end{aligned}$$

Interchanging the variables x and y in the second integral and noting that α is even and w is odd we get

$$\begin{aligned} 2\langle u, u_{tt}\rangle &= \int_{\mathbb{R}^2} \alpha(y-x)\ w\left(u(y)-u(x)\right) u(x) dy dx \\ &\quad - \int_{\mathbb{R}^2} \alpha(y-x)\ w\left(u(y)-u(x)\right) u(y) dy dx \\ &= -\int_{\mathbb{R}^2} \alpha(y-x)\ w\left(u(y)-u(x)\right) (u(y)-u(x)) dy dx. \end{aligned}$$

So that

$$\begin{aligned} 2\langle u, u_{tt}\rangle &\geq -2\left(1+2\nu\right) \int_{\mathbb{R}^2} \alpha(y-x)\ W\left(u(y)-u(x)\right) dy dx \\ &= 4\left(1+2\nu\right)\left(\frac{1}{2}\left\|u_t\right\|_2^2 - E(0)\right). \end{aligned}$$

Hence we get

$$H''\left(t\right) \geq 4\left(1+\nu\right)\left\|u_t\right\|_2^2 - 4\left(1+2\nu\right) E\left(0\right) + 2b.$$

On the other hand, we have

$$\begin{aligned} \left(H'\left(t\right)\right)^2 &= 4\left[\langle u, u_t\rangle + b\left(t+t_0\right)\right]^2 \\ &\leq 4\left[\left\|u\right\|_2 \left\|u_t\right\|_2 + b\left(t+t_0\right)\right]^2 \\ &= 4\left[\left\|u\right\|_2^2 \left\|u_t\right\|_2^2 + 2\left\|u\right\|_2 \left\|u_t\right\|_2 b\left(t+t_0\right) + b^2\left(t+t_0\right)^2\right)] \\ &\leq 4\left[\left\|u\right\|_2^2 \left\|u_t\right\|_2^2 + b\left\|u\right\|_2^2 + b\left\|u_t\right\|_2^2\left(t+t_0\right)^2 + b^2\left(t+t_0\right)^2\right]. \end{aligned}$$

Thus

$$\begin{aligned} H''\left(t\right) H\left(t\right) &- \left(1+\nu\right)\left(H'\left(t\right)\right)^2 \\ &\geq \left[4\left(1+\nu\right)\left\|u_t\right\|_2^2 - 4\left(1+2\nu\right) E(0) + 2b\right]\left[\left\|u\right\|_2^2 + b\left(t+t_0\right)^2\right] \\ &\quad -4\left(1+\nu\right)\left[\left\|u\right\|_2^2 \left\|u_t\right\|_2^2 + b\left\|u\right\|_2^2 + b\left\|u_t\right\|_2^2\left(t+t_0\right)^2 + b^2\left(t+t_0\right)^2\right] \\ &= \left[-4\left(1+2\nu\right) E(0) + 2b - 4b\left(1+\nu\right)\right]\left[\left\|u\right\|_2^2 + b\left(t+t_0\right)^2\right] \\ &= -2\left(1+2\nu\right)\left(b+2E(0)\right) H(t). \end{aligned}$$

Now if we choose $b \leq -2E(0)$, this gives

$$H''\left(t\right) H\left(t\right) - \left(1+\nu\right)\left(H'\left(t\right)\right)^2 \geq 0.$$

Moreover

$$H'\left(0\right) = 2\langle \varphi, \psi\rangle + 2bt_0 > 0$$

for sufficiently large t_0. According to Lemma 4.6, this implies that $H\left(t\right)$, and thus $\left\|u(t)\right\|_2^2$ blows up in finite time contradicting the assumption that the global solution exists. □

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