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How optimal synchronization of oscillators depends on the network structure and the individual dynamical properties of the oscillators

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Abstract. The problem of making a network of dynamical systems synchronize onto a common evolution is the subject of much ongoing research in several scientific disciplines. It is nowadays a well-known fact that the synchronization processes are gradually influenced by the interaction topology between the dynamically interacting units. A complex coupling configuration can significantly affect the synchronization abilities of a networked system. However, the question arises what is the optimal network topology that provides enhancement of the synchronization features under given circumstances. In order to address this issue we make use of a network model in which we can smoothly tune the topology from a highly heterogeneous and efficient scale-free network to a homogeneous and less efficient network. The network is then populated with Poincaré oscillators, a paradigmatic model for limit-cycle oscillations. This oscillator model exhibits a parameter that enables changes of the limit cycle attraction and is thus immediately related to flexibility/rigidity properties of the oscillator. Our results reveal that for weak attractions of the limit cycle, intermediate homogeneous topology ensures maximal synchronization, whereas highly heterogeneous scale-free topology ensures maximal synchronization for strong attractions of the limit cycle. We argue that the flexibility/rigidity of individual nodes of the networks defines the topology, where maximal global coherence is achieved.

Complex networks are nowadays used for the description of several natural and artificial systems. Since topological features of interactions between individual units characterize the global properties of a given system, the research of structural properties is increasingly gaining on attention. To qualitatively analyze the local and global structural properties of a network, numerous techniques have been developed, which have been utilized in various disciplines and diverse circumstances [1]. One of the basic measures that describes the properties of a network is its degree distribution. Scale-free networks, for instance, are known to have a highly heterogeneous power-law degree distribution [1], which can foster the synchronization abilities of a network of coupled oscillators [2], especially due to their hight global efficiency. Furthermore, it has been shown that a high level of heterogeneity can suppress synchronization, even thought it also reduces the average distance between the nodes. Another importing network measure is the average clustering coefficient C characterizing the cliquishness of a network. Remarkably, high clustering coefficients were detected in several real-life networks [1]. On the other hand, high clustering implies the existence of many transitive connections and can thus hinder global synchronization [3]. There are several other topological factors, such as for example the existence of a community structure [4], which also have an impact on synchronization processes. Nevertheless, even though the dynamical behavior of networks is inevitable and non-trivially connected with structural properties of the underlying networks, there are also several other important factors having an impact. With this in mind, in the present study we focus on the analysis of the role of dynamical properties of individual oscillators. In a previous theoretical study on cellular oscillators it has been argued that the local dissipation rate crucially determines the coupling ability of cellular oscillators [5]. Notably, the dissipation rate is directly correlated with flexibility/rigidity of an oscillator. A low dissipation rate stands for rigid oscillators whereas on the other hand, oscillators with a near zero dissipation rate have attractors that are very susceptible to external perturbations and are thus flexible. Therefore, the main goal of the present work is to explore how the relation between flexibility/rigidity properties of individual oscillators on one hand and the structure of the network topology that characterizes the connectivity patterns between them on the other hand impacts the synchronization of oscillators.

By applying the algorithm described previously [6, 7] we can generate networks whose topologies can smoothly be altered between a highly heterogeneous scale-free network and a more homogeneous network. First, N nodes are randomly distributed in a unit square and to each node a fitness value f_i is prescribed. Two nodes are connected if:

$$\theta < \frac{f_i f_j}{I_{ij}^{\delta}}, \tag{1}$$

where f_i and f_j are fitness values of the i-th and j-th node, I_{ij} is the Euclidean distance between them and the δ is used to alter the topology of the network. The parameter θ is used as a threshold to control the average node degree of the network $\langle k \rangle$. In order to quantify topological features of the network and its degree of heterogeneity, the standard deviation of the node degrees STD(k) (Fig. 1a), the standard deviation of the clustering coefficients STD(C) (Fig. 1b) and the average clustering coefficient C (Fig. 1c) are plotted as a function of δ. Both, STD (k) and STD(C) are decreasing monotonically with increasing values of the topology parameter δ. Higher values of STD(k) and STD(C) are indicators of diverse local topological properties indicating a high degree of network heterogeneity.

Characteristic examples of generated networks are shown in Fig. 1d-f. It can be observed that for low values of δ indeed very heterogeneous networks are generated, in which mostly long-range connections exist. On the other hand, if $\delta >> 1$ mostly nearby nodes are connected and, in addition, there are no expressive differences in individual node degrees. Finally, intermediate values of δ, i.e. $\delta = 1.1$, result in an intermediate heterogeneous network with both long- and short-range connections. In other words, Fig. 1d-f provide a visual assessment that the network heterogeneity indeed decreases with increasing values of δ.

The dynamics of individual nodes is driven by the paradigmatic Poincaré oscillator:

$$\dot{x}_i = -\gamma \left(r_i - A\right) x_i - \omega_i y_i - \epsilon \sum_{i=1}^{N} d_{ij} \left(x_i - x_j\right), \tag{2}$$

$$\dot{y}_i = -\gamma \left(r_i - A\right) y_i + \omega_i x_i - \epsilon \sum_{i=1}^{N} d_{ij} \left(y_i - y_j\right), \tag{3}$$

where x_i and y_i are the phase space coordinates of the i-th oscillator, A is the limit cycle radius, $r_i = \sqrt{x_i^2 + y_i^2}$ is the distance between the origin of the phase space and the position of the i-th

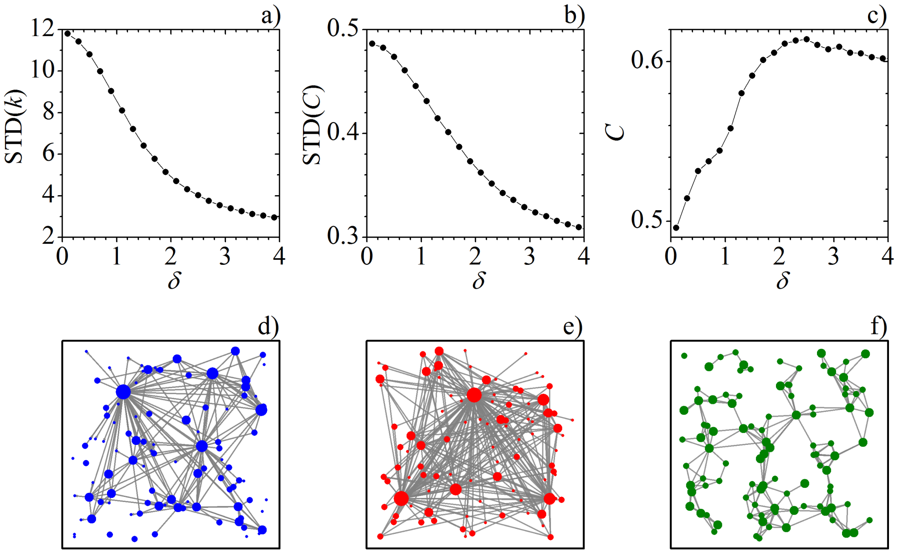

Figure 1. Standard deviation of node degrees (a), standard deviation of the local clustering coefficient (b), the average clustering coefficient (c) and characteristic network structures obtained for $\delta = 0.1$ (d), $\delta = 1.1$ (e) and $\delta = 4.0$ (f). Note that in lower panels the size of the nodes is proportional to their degrees. The number of nodes is $N = 100$ and the average node degree is $\langle k \rangle = 5.0$

node in the phase space, ω_i is the angular velocity of the i-th oscillator and ϵ is the coupling strength and d_{ij} is the connectivity matrix with values of 1 if the i-th and j-th node are connected and 0 otherwise. Angular velocities ω_i were distributed according to a normal distribution with mean angular velocity $\overline{\omega} = 1.0$ with standard deviation of 0.2. The limit cycle radius was set to $A = 1.0$ and the coupling strength to $\epsilon = 0.25$.

The parameter γ determines the relaxation towards the limit cycle and is immediately related with the dissipation rate of the oscillators [5, 8]. In order to investigate the interplay between structural and dynamical features of the network, we simulated how the overall synchronization in the systems changes according to the initial rigidity of the oscillators and structural properties of the network. For that purpose we analyze the synchronization behavior as a function of the relaxation rate γ and network topology parameter δ. For the quantification of the global network synchronization we calculate the average correlation coefficient $\overline{R}$. Accordingly, we have to construct the $N \times N$ correlation matrix, whose ij-th element is defined as:

$$R_{ij} = \frac{1}{M} \sum_{t=1}^{M} \frac{[x_i(t) - \overline{x_i}][x_j(t) - \overline{x_j}]}{S_i S_j}, \tag{4}$$

where M is the number of integration steps, $x_i(t)$ and $x_j(t)$ are the time series of the i-th and j-th oscillator, $\overline{x_i}$ and $\overline{x_j}$ are the average values of the time series and S_i and S_j the corresponding standard deviations. The average correlation coefficient $\overline{R}$ is then obtained as the average over all non-diagonal elements of the matrix. Its values range between 0 (uncorrelated dynamics) and 1 (complete synchronized motion).

Numerical simulations were carried out on a network of $N = 100$ Poincaré oscillators, with an average node degree $\langle k \rangle = 5.0$. Fig. 2a features the results showing the average correlation as a function of the dissipation rate γ and the network structure δ. It can be observed that both parameters mutually affect the synchronization behavior. For rigid oscillations where $\gamma > 100$, maximal synchronization is achieved in the highly heterogeneous scale-free network.

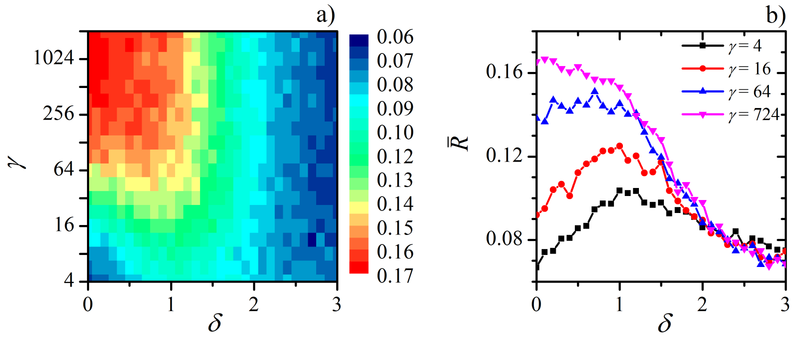

Figure 2. a) The average correlation $\overline{R}$ as a function of the topology parameter δ and flexibility parameter γ. b) Average correlation values for four different values of γ as a function of δ.

On the other hand, in case the network is populated with flexible oscillators ($\gamma < 100$), maximal synchronization is achieved for intermediate values of the topology parameter δ, where the network is less heterogeneous as the scale-free network and is constituted by long- as well as short-range links (see Fig. 1). Notably, for this intermediate values of δ the network exhibits a higher average clustering coefficient. Furthermore, in the homogeneous network ($\delta > 2$) with mostly short-range interactions, the degree of synchronization does not change by changing γ, thus indicating that in this case the synchronization does not depend on flexibility/rigidity properties of oscillators. In order to provide a better inside into the reported phenomena, we additionally plotted characteristic cross-sections of the color-contour plots for different values of γ. The results presented in Fig. 2b additionally confirm the existence of a resonant response due to changes in network topology for flexible oscillators, whereas for rigid oscillators the level of synchronization decreases monotonically with increasing δ.

In sum, we have shown that the network topology ensuring most synchronized response in an ensemble of coupled oscillators depends on individual oscillator properties. While rigid oscillators synchronize best in heterogeneous scale-free networks, flexible oscillator exhibit the most coherent collective response when they are connected in a less heterogeneous network. Our findings provide novel insights into synchronization behavior of coupled oscillators, which may be of importance especially from biological point of view. Signal transduction systems have to respond sensitively to weak external stimuli, thus indicating that cellular oscillators in general should behave like flexible oscillators. On the other hand it is known that different biological oscillators differ in their rigidity and, moreover, that the coupling influences the flexibility/rigidity properties of an oscillator [8]. Apparently, the optimal structural organization of the intercellular communication networks is determined also by characteristics of individual cells. Interestingly, the intermediate network structure that ensures best synchronizability of flexible oscillators is very economic, since it represents a good compromise between efficiency, wiring economy and robustness - a desirable attribute of several real-life systems.

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