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Involution products in Coxeter groups

S. B. Hart and P. J. Rowley

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Abstract. For W a Coxeter group, let

$$\mathscr{W} = \{w \in W \mid w = xy \text{ where } x, y \in W \text{ and } x^2 = 1 = y^2\}.$$

It is well known that if W is finite then $W = \mathscr{W}$. Suppose that $w \in \mathscr{W}$. Then the minimum value of $\ell(x) + \ell(y) - \ell(w)$, where $x, y \in W$ with $w = xy$ and $x^2 = 1 = y^2$, is called the *excess* of w (ℓ is the length function of W). The main result established here is that w is always W-conjugate to an element with excess equal to zero.

1 Introduction

The study of a Coxeter group W frequently weaves together features of its root system Φ and properties of its length function ℓ. The delicate interplay between $\ell(w_1w_2)$ and $\ell(w_1) + \ell(w_2)$ for various $w_1, w_2 \in W$ is often to be seen in investigations into the structure of W. Instances of the additivity of the length function, that is $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, are of particular interest. For example, if W_J is a standard parabolic subgroup of W, then there is a set X_J of so-called distinguished right coset representatives for W_J in W with the property that $\ell(wx) = \ell(w) + \ell(x)$ for all $w \in W_J$, $x \in X_J$ ([6, Proposition 1.10]). There is a parallel statement to this for double cosets of two standard parabolic subgroups of W ([5, Proposition 2.1.7]). Also, when W is finite it possesses an element w_0, the longest element of W, for which $\ell(w_0) = \ell(w) + \ell(ww_0)$ for all $w \in W$ ([5, Lemma 1.5.3]).

Of the involutions (elements of order 2) in W, the reflections, and particularly the fundamental reflections, more often than not play a major role in investigating W. This is due to there being a correspondence between the reflections in W and the roots in Φ. In general, involutions occupy a special position in a group and it is sometimes possible to say more about them than it is about other elements of the group.

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This is true in the case of Coxeter groups. For example, Richardson [7] gives an effective algorithm for parameterizing the involution conjugacy classes of a Coxeter group. In something of the same vein we have the fact that an involution can be expressed as a canonical product of reflections (see Deodhar [4] and Springer [8]).

Suppose that W is a Coxeter group (not necessarily finite or even of finite rank), and put

$$\mathscr{W} = \{w \in W \mid w = xy \text{ where } x, y \in W \text{ and } x^2 = 1 = y^2\}.$$

That is, $\mathscr{W}$ is the set of strongly real elements of W. For $w \in \mathscr{W}$ we define the *excess* of w, $e(w)$, by

$$e(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1\}.$$

Thus $e(w) = 0$ is equivalent to there existing $x, y \in W$ with $x^2 = 1 = y^2$, $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. We shall call (x, y), where x, y are involutions, a *spartan pair* for w if $w = xy$ with $\ell(x) + \ell(y) - \ell(w) = e(w)$. As a small example take $w = (1234)$ in $\mathrm{Sym}(4) \cong W(A_3)$ (the Coxeter group of type A_3). Then $\ell(w) = 3$ and w can be written in four ways as a product of involutions (see Table 1).

Table 1. $w = (1234) = xy$

x	y	$\ell(x) + \ell(y)$
(13)	(14)(23)	$3 + 6 = 9$
(14)(23)	(24)	$6 + 3 = 9$
(24)	(12)(34)	$3 + 2 = 5$
(12)(34)	(13)	$2 + 3 = 5$

Thus $e(w) = 2$ with $((24), (12)(34))$ and $((12)(34), (13))$ being spartan pairs for w. To give some idea of the distribution of excesses we briefly mention two other examples. The number of elements with excess 0, 2, 4, 6, 8 in $\mathrm{Sym}(6) \cong W(A_5)$ is, respectively, 489, 173, 46, 10, 2, the maximum excess being 8. For $\mathrm{Sym}(7) \cong W(A_6)$, the number of elements with excess 0, 2, 4, 6, 8, 10, 12 is, respectively, 2659, 1519, 574, 228, 50, 8, 2 and here the maximum excess is 12.

In the case when W is finite we have $W = \mathscr{W}$, and so excess is defined for every element of W. (Since W is a direct product of irreducible Coxeter groups, it suffices to check this for W an irreducible Coxeter group. If W is a Weyl group see Carter [3]. The case when W is a dihedral group is straightforward to verify while types H_3 or H_4 may be checked using [2].) However if W is infinite we can have $W \neq \mathscr{W}$. This can be seen when W is of type $\widetilde{A}_2$. Then $W = HN$, the semidirect product of $N = \{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i \in \mathbb{Z}, \lambda_1 + \lambda_2 + \lambda_3 = 0\} \cong \mathbb{Z} \times \mathbb{Z}$ and $H \cong W(A_2) \cong \mathrm{Sym}(3)$ with H acting on N by permuting the co-ordinates of $(\lambda_1, \lambda_2, \lambda_3)$. Let $g = (12) \in H$

and $0 \neq \lambda \in \mathbb{Z}$, and set $w = g(\lambda, \lambda, -2\lambda)$. Clearly g and $(\lambda, \lambda, -2\lambda)$ commute and $(\lambda, \lambda, -2\lambda)$ has infinite order. Therefore w has infinite order. If w can be written as a product of two involutions, then there exist $hm, kn \in W$ with $h, k \in H$, $m, n \in N$ and $(hm)(kn) = w$. Therefore h, k are self-inverse elements of H with $hk = (12)$. So one of h or k is (12) and the other is the identity. But N has no elements of order 2, so either hm or kn is the identity, contradicting the fact that w is not an involution. So we conclude that w cannot be expressed as a product of two involutions and hence $W \neq \mathscr{W}$.

As we have observed, a Coxeter group may have many elements with non-zero excess. Nevertheless our main theorem shows the zero excess elements are ubiquitous from a conjugacy class viewpoint.

Theorem 1.1. *Suppose that W is a Coxeter group, and let $w \in \mathscr{W}$. Let X denote the W-conjugacy class of w. Then there exists $w_* \in X$ such that $e(w_*) = 0$.*

We prove Theorem 1.1 in Section 3, after gathering together a number of preparatory results about Coxeter groups in Section 2. Also some easy properties of excess are noted and, in Proposition 2.7, we demonstrate that there are Coxeter groups in which elements can have arbitrarily large excess.

2 Background results and notation

Assume, for this section, that W is a finite rank Coxeter group. So, by its very definition, W has a presentation of the form

$$W = \langle R \mid (rs)^{m_{rs}} = 1, r, s \in R \rangle$$

where R is finite, $m_{rr} = 1$, $m_{rs} = m_{sr} \in \mathbb{Z}^+ \cup \{\infty\}$ and $m_{rs} \geqslant 2$ for $r, s \in R$ with $r \neq s$. The elements of R are called the fundamental reflections of W and the rank of W is the cardinality of R. The length of an element w of W, denoted by $\ell(w)$, is defined to be

$$\ell(w) = \begin{cases} \min\{l \mid w = r_1 r_2 \dots r_l : r_i \in R\} & \text{if } w \neq 1, \\ 0 & \text{if } w = 1. \end{cases}$$

Now let V be a real vector space with basis $\Pi = \{\alpha_r \mid r \in R\}$. Define a symmetric bilinear form $\langle\, , \rangle$ on V by

$$\langle \alpha_r, \alpha_s \rangle = -\cos\left(\frac{\pi}{m_{rs}}\right),$$

where $r, s \in R$ and the m_{rs} are as in the above presentation of W. Letting $r, s \in R$ we define

$$r \cdot \alpha_s = \alpha_s - 2\langle \alpha_r, \alpha_s \rangle \alpha_r.$$

This then extends to an action of W on V which is both faithful and respects the bilinear form $\langle\,,\rangle$ (see [6, (5.6)]) and the elements of R act as reflections upon V. The module V is called a reflection module for W and the subset

$$\Phi = \{w \cdot \alpha_r \mid r \in R, w \in W\}$$

of V is the all important root system of W. Setting

$$\Phi^+ = \left\{\sum_{r \in R} \lambda_r \alpha_r \in \Phi \mid \lambda_r \geqslant 0 \text{ for all } r\right\} \quad \text{and} \quad \Phi^- = -\Phi^+$$

we have the basic fact that Φ is the disjoint union $\Phi^+ \dot\cup \Phi^-$ (see [6, (5.4)–(5.6)]). Elements of Φ^+ and Φ^- are referred to, respectively, as positive and negative roots of Φ.

For $w \in W$ we define

$$N(w) = \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^-\}.$$

The connection between $\ell(w)$ and the root system of W is contained in our next lemma.

Lemma 2.1. *Let $w \in W$ and $r \in R$.*

(i) *If $\ell(wr) > \ell(w)$ then $w \cdot \alpha_r \in \Phi^+$ and if $\ell(wr) < \ell(w)$ then $w \cdot \alpha_r \in \Phi^-$. In particular, $\ell(wr) < \ell(w)$ if and only if $\alpha_r \in N(w)$.*

(ii) $\ell(w) = |N(w)|$.

Proof. See [6, §§5.4, 5.6]. □

Lemma 2.2. *Let $g, h \in W$. Then*

$$N(gh) = N(h)\backslash(-h^{-1}N(g)) \cup h^{-1}(N(g)\backslash N(h^{-1})).$$

Hence $\ell(gh) = \ell(g) + \ell(h) - 2|N(g) \cap N(h^{-1})|$.

Proof. Let $\alpha \in \Phi^+$. Suppose that $\alpha \in N(h)$. Then $gh \cdot \alpha = g \cdot (h \cdot \alpha)$ is negative if and only if $-h \cdot \alpha \notin N(g)$. That is, $\alpha \notin -h^{-1}N(g)$. Thus

$$N(gh) \cap N(h) = N(h)\backslash -h^{-1}N(g).$$

If on the other hand $\alpha \notin N(h)$, then $gh \cdot \alpha \in \Phi^-$ if and only if $h \cdot \alpha \in N(g)$. That is, $\alpha \in \Phi^+ \cap h^{-1}N(g)$. Thus

$$N(gh)\backslash N(h) = h^{-1}[(N(g)\backslash N(h^{-1})).$$

Combining the two equations gives the expression for $N(gh)$ stated in Lemma 2.2. Since $N(gh) \cap N(h)$ and $N(gh) \backslash N(h)$ are clearly disjoint, using Lemma 2.1 (ii) we deduce that

$$\ell(gh) = \ell(g) + \ell(h) - 2|N(g) \cap N(h^{-1})|,$$

which completes the proof of the lemma. □

Proposition 2.3. *Let $w \in W$ and $r \in R$. If $\ell(rw) < \ell(w)$ and $\ell(wr) < \ell(w)$, then either $rwr = w$ or $\ell(rwr) = \ell(w) - 2$.*

Proof. See [6, §5.8, Exercise 3]. □

For $J \subseteq R$ define W_J to be the subgroup of W generated by J. Such a subgroup of W is referred to as a standard parabolic subgroup. Standard parabolic subgroups are Coxeter groups in their own right with root system

$$\Phi_J = \{w \cdot \alpha_r \mid r \in J, w \in W_J\}$$

(see [6, §5.5] for more on this). A conjugate of a standard parabolic subgroup is called a parabolic subgroup of W. Finally, a *cuspidal* element of W is an element which is not contained in any proper parabolic subgroup of W. Equivalently, an element is cuspidal if its W-conjugacy class has empty intersection with all the proper standard parabolic subgroups of W.

Theorem 2.4. *Let $0 \neq v \in V$. Then the stabilizer of v in W is a parabolic subgroup of W. Furthermore, if $v \in \Phi$, then the stabilizer of v in W is a proper parabolic subgroup of W.*

Proof. The fact that the stabilizer of v is a parabolic subgroup is proved in [1, Chapter V, §3.3]. If $v \in \Phi$, then $v = w \cdot \alpha_r$ for some $r \in R$, $w \in W$ and hence $(wrw^{-1}) \cdot v = -v$. Thus the stabilizer of v cannot be the whole of W, so is a proper parabolic subgroup of W. □

Theorem 2.5. *Suppose that w is an involution in W. Then there exists $J \subseteq R$ such that w is W-conjugate to w_J, an element of W_J which acts as -1 upon Φ_J.*

Proof. See Richardson [7]. □

Next we give some easy properties of excess.

Lemma 2.6. *Let $w \in \mathscr{W}$. Then the following hold.*

(i) *If w is an involution or the identity element, then $e(w) = 0$.*

(ii) *$e(w)$ is non-negative and even.*

(iii) *If* $w = xy$ *where* x *and* y *are involutions and* $2|N(x) \cap N(y)| = e(w)$, *then* (x, y) *is a spartan pair for* w.

Proof. If w is an involution or $w = 1$, then $w = w1$ with $w^2 = 1 = 1^2$, whence $e(w) = 0$. For (ii), suppose that $x^2 = y^2 = 1$ and $xy = w$. Then, using Lemma 2.2, we have $\ell(w) = \ell(x) + \ell(y) - 2|N(x) \cap N(y)|$ and hence $\ell(x) + \ell(y) - \ell(w)$ is even and (ii) follows. Part (iii) is also immediate from Lemma 2.2. □

We now have the tools needed for the proof of Theorem 1.1, but before continuing with this we calculate the excess of the element $(12\dots n)$ of $\mathrm{Sym}(n)$. The aim of this is to show that there are Coxeter groups in which elements may have arbitrarily large excess. Before stating our next result we require some notation. For q a rational number, $\lfloor q \rfloor$ denotes the floor of q (that is, the largest integer less than or equal to q), and $\lceil q \rceil$ denotes the ceiling of q (that is, the smallest integer greater than or equal to q).

Let $n \geqslant 2$. Then $\mathrm{Sym}(n)$ is isomorphic to the Coxeter group $W(A_{n-1})$ of type A. If $W = W(A_{n-1}) \cong \mathrm{Sym}(n)$, then we set $R = \{(12), (23), \dots, ((n-1)\ n)\}$ and the set of positive roots is $\Phi^+ = \{e_i - e_j \mid 1 \leqslant i < j \leqslant n\}$. An alternative description of $\ell(w)$ for $w \in W$ in this case is

$$\ell(w) = |\{(i, j) \mid 1 \leqslant i < j \leqslant n, w(i) > w(j)\}|.$$

For $0 \leqslant k \leqslant n-1$, let s_k be the longest element of $\mathrm{Sym}(\{1, 2, \dots, k\})$ and let t_k be the longest element of $\mathrm{Sym}(\{k+1, k+2, \dots, n\})$. A straightforward calculation shows that

$$s_k = (1 \quad k)(2 \quad k-1)\dots\left(\left\lfloor \frac{k}{2} \right\rfloor \quad \left\lceil \frac{k}{2} \right\rceil + 1\right);$$

and

$$t_k = (k+1 \quad n)(k+2 \quad n-1)\dots\left(\left\lfloor \frac{n-k}{2} \right\rfloor + k \quad \left\lceil \frac{n-k}{2} \right\rceil + k + 1\right).$$

Note that $s_0 = s_1 = t_{n-1} = 1$. Finally, for $0 \leqslant k \leqslant n-1$, set $y_k = s_k t_k$.

Proposition 2.7. *Let* $w = (12\dots n) \in W = W(A_{n-1}) \cong \mathrm{Sym}(n)$. *Put*

$$\mathscr{I}_w = \{x \in W \mid x^2 = 1, w^x = w^{-1}\}.$$

Then

(i) $\mathscr{I}_w = \{y_k \mid 0 \leqslant k \leqslant n-1\}$.

(ii) *If* n *is odd, then* $(wy_{(n-1)/2}, y_{(n-1)/2})$ *is a spartan pair for* w.

(iii) *If* n *is even, then* $(wy_{n/2}, y_{n/2})$ *and* $(wy_{n/2-1}, y_{n/2-1})$ *are both spartan pairs for* w.

(iv) $e(w) = \lfloor (n-2)^2/2 \rfloor$.

Proof. It is easy to check that $y_k \in \mathscr{I}_w$ whenever $0 \leqslant k \leqslant n-1$. Since $|I_w| \leqslant |C_W(w)| = n$, part (i) follows immediately.

Suppose that $y \in \mathscr{I}_w$. Write

$$\varepsilon_y(w) = \ell(wy) + \ell(y) - \ell(w), \quad \text{so that} \quad e(w) = \min\{\varepsilon_y(w) \mid y \in \mathscr{I}_w\}.$$

We have

$$\begin{aligned}\varepsilon_y(w) &= \ell(wy) + \ell(y) - \ell(w) \\ &= \ell(w) + \ell(y) - 2|N(y) \cap N(w)| + \ell(y) - \ell(w) \\ &= 2(\ell(y) - |N(y) \cap N(w)|).\end{aligned}$$

Now $N(w) = \{e_i - e_n \mid 1 \leqslant i < n\}$. Therefore

$$|N(y) \cap N(w)| = |\{i \mid y(i) > y(n)\}| = n - y(n).$$

Let $y = y_k$ for some $0 \leqslant k \leqslant n-1$. We have

$$\ell(y_k) = \ell(s_k) + \ell(t_k) = \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2}.$$

Moreover $|N(y_k) \cap N(w)| = n - y_k(n) = n - (k+1)$. Therefore

$$\begin{aligned}\varepsilon_{y_k}(w) &= 2(\ell(y_k) - |N(y_k) \cap N(w)|) \\ &= k(k-1) + (n-k)(n-k-1) - 2(n-k-1) \\ &= 2k^2 - 2k(n-1) + n^2 - 3n + 2 \\ &= 2\left(k - \frac{1}{2}(n-1)\right)^2 + \frac{1}{2}(n^2 - 4n + 3) \\ &= 2\left(k - \frac{1}{2}(n-1)\right)^2 + \frac{1}{2}(n-2)^2 - \frac{1}{2}.\end{aligned}$$

If n is odd, then this quantity is minimal when $k = \frac{1}{2}(n-1)$. Hence part (ii) holds. In this case,

$$e(w) = \varepsilon_{y_{(n-1)/2}}(w) = \frac{1}{2}(n-2)^2 - \frac{1}{2} = \left\lfloor \frac{1}{2}(n-2)^2 \right\rfloor.$$

If n is even, then ε_{y_k} is minimal when $k = n/2$ or $k = n/2 - 1$. Hence we obtain part (iii), and in either case, $e(w) = \frac{1}{2}(n-2)^2$. Combining the odd and even cases we see that $e(w) = \lfloor (n-2)^2/2 \rfloor$. $\square$

3 Zero excess in conjugacy classes

Proof of Theorem 1.1. Suppose that W is a Coxeter group, $w \in \mathcal{W}$ and X is the W-conjugacy class of w. Now $w = r_1 \dots r_\ell$ for certain $r_i \in R$ and some finite $\ell = \ell(w)$. So $w \in \langle r_1, \dots, r_\ell \rangle$. Thus it suffices to establish the theorem for W of finite rank. Accordingly we argue by induction on $|R|$. Suppose that $K \subsetneq R$. If $X \cap W_K \neq \varnothing$, then by induction there exists $w' \in X \cap W_K$ with $e_{W_K}(w') = 0$, whence $e(w') = 0$ and we are done. So we may suppose that $X \cap W_K = \varnothing$ for all $K \subsetneq R$. That is, X is a cuspidal class of W.

Choose $w \in X$. If $w = 1$ or w is an involution, then $e(w) = 0$ by Lemma 2.6 (i). Thus we may suppose that $w = xy$ where x and y are involutions. By Theorem 2.5 we may conjugate w so that $y \in W_J$ for some $J \subseteq R$, with y acting as -1 on Φ_J. Thus $y \in Z(W_J)$. Now choose z to have minimal length in $\{g^{-1}xg \mid g \in W_J\}$. So we have $z = g^{-1}xg$ for some $g \in W_J$.

Suppose for a contradiction that there exists $r \in J$ with $\ell(zr) < \ell(z)$. Since z and r are involutions,

$$\ell(rz) = \ell((rz)^{-1}) = \ell(zr) < \ell(z).$$

Applying Proposition 2.3 yields that either $rzr = z$ or $\ell(rzr) = \ell(z) - 2 < \ell(z)$. Since $r \in W_J$, the latter possibility would contradict the minimal choice of z. Hence $rzr = z$. So $r \cdot (z \cdot a_r) = z \cdot (r \cdot a_r) = -z \cdot a_r$. It is well known that the only roots β for which $r \cdot \beta = -\beta$ are α_r and $-\alpha_r$. Thus $z \cdot \alpha_r = \pm\alpha_r$. By assumption $\ell(zr) < \ell(z)$ and therefore $z \cdot \alpha_r \in \Phi^-$ by Lemma 2.1 (i), whence $z \cdot \alpha_r = -\alpha_r$. Combining this with $y \cdot \alpha_r = -\alpha_r$ we then deduce that $zy \cdot \alpha_r = \alpha_r$. Then Theorem 2.4 gives that zy is in a proper parabolic subgroup of W. Noting that $zy = g^{-1}xgy = g^{-1}xyg = g^{-1}wg$, as $g \in W_J$, we infer that X is not a cuspidal class, a contradiction. We conclude therefore that $\ell(zr) > \ell(z)$ for all $r \in J$. Consequently $N(z) \cap \Phi_J^+ = \varnothing$. Since $N(y) = \Phi_J^+$ we deduce that $N(z) \cap N(y) = \varnothing$ and hence, using Lemma 2.2, that $\ell(zy) = \ell(z) + \ell(y)$. Setting $w_* = zy = g^{-1}wg$ we have $w_* \in X$ and $e(w_*) = 0$, so completing the proof of Theorem 1.1. □

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