How Hard Is Deciding Trivial Versus Nontrivial in the Dihedral Coset Problem?

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──── **Abstract** ────

We study the hardness of the dihedral hidden subgroup problem. It is known that lattice problems
reduce to it, and that it reduces to random subset sum with density > 1 and also to quantum
sampling subset sum solutions. We examine a decision version of the problem where the question
asks whether the hidden subgroup is trivial or order two. The decision problem essentially asks
if a given vector is in the span of all coset states. We approach this by first computing an explicit
basis for the coset space and the perpendicular space. We then look at the consequences of having
efficient unitaries that use this basis. We show that if a unitary maps the basis to the standard
basis in any way, then that unitary can be used to solve random subset sum with constant density
> 1. We also show that if a unitary can exactly decide membership in the coset subspace, then
the collision problem for subset sum can be solved for density > 1 but approaching 1 as the
problem size increases. This strengthens the previous hardness result that implementing the
optimal POVM in a specific way is as hard as quantum sampling subset sum solutions.

1   Introduction

The dihedral coset problem is an important open problem in quantum algorithms. It comes
from the hidden subgroup problem, which is defined as: given a function on a group G that
is constant and distinct on cosets a subgroup H, find H. Here we will focus on the case when
G is the dihedral group of order $2N$. It is known that this problem reduces to the case when
the subgroup is order two [4]. All known approaches for solving the hidden subgroup problem
over the dihedral group start by evaluating the function in superposition and measuring the
function value. The result is a random coset state $\frac{1}{\sqrt{2}}(|0, x\rangle + |1, x + d\rangle)$, where $d \in \mathbb{Z}_N$ is
a fixed label of the subgroup and x is a coset representative uniformly chosen in $\mathbb{Z}_N$. For

our purposes, it is more convenient to have the following quantum problem rather than the hidden subgroup problem.

The *dihedral coset problem* [13] is: given a tensor product of k coset states

$$|c_{x_1,x_2,\cdots,x_k}^{(d)}\rangle = \frac{1}{\sqrt{2}}(|0,x_1\rangle + |1,x_1+d\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0,x_k\rangle + |1,x_k+d\rangle),$$

where $x_1,\ldots,x_k$ are randomly chosen in $\mathbb{Z}_N$, compute d. The first register of each state is mod 2, and the second register is mod N.

This is a natural problem to consider after the successes with abelian groups such as $\mathbb{Z}_N$. The dihedral group with $2N$ elements has $\mathbb{Z}_N$ as a normal subgroup. The representations are mostly two dimensional, so it does not have obvious problems like the symmetric group, where we know large entangled measurements are required to get information from the states [7]. Furthermore, Regev [13] showed that the unique shortest vector problem reduces to the dihedral coset problem, so it could provide a pathway for finding a quantum algorithm for lattice problems.

Much is known about the dihedral coset problem, at least compared to most other nonabelian groups (although there are groups with efficient algorithms, e.g. [6, 9, 3]). Ettinger and Hoyer [4] showed that a polynomial number of measurements in the Fourier basis has enough classical information to determine d, but the best known algorithm takes exponential time to compute it. Kuperberg found subexponential time algorithms [10, 11] for the problem. He also showed that computing one bit of d was sufficient to compute all of d. This algorithm was a big step, although it should be noted that it seems difficult to combine this with Regev's uSVP to dihedral group HSP reduction to get a subexponential time algorithm for the uSVP, partly due to the fact that the coset states created in the reduction have errors with some probability.

The dihedral coset problem also has some connections to the subset sum problem. Bacon, Childs, and van Dam analyzed how well a "pretty good measurement" performs [1]. This type of measurement maximizes the probability of computing d correctly. It is unknown how to compute the measurement they find without quantum sampling subset sum solutions. A unitary implementing this can be used to solve the worst case subset sum, which is NP-complete. Regev showed how to reduce the dihedral coset problem to the random subset sum problem density $\rho > 1$ where ρ also approaches 1 as the problem size increases. Density 1 is the hardest case for the random subset sum problem as shown in Proposition 1.2 in [8]. But is solving the dihedral coset problem as hard as subset sum, and if so, for what parameters? The only connection we are aware of is to compose two known reductions. First, random subset sum with density $\rho = 1/\log k$ reduces to uSVP. Then uSVP reduces to the dihedral coset problem. It is open if an efficient quantum algorithm exists for random subset sum, and density $1/\log k$ may not be as hard to solve as constant density.

1.1   New approach

In this paper we focus on distinguishing trivial from order two subgroups. Instead of trying to compute d, we define a problem which asks if the state is an order two coset state, or is the trivial subgroup case. We define this problem as the *dihedral coset space problem (DCSP)*: either an order two coset state is given, or a random standard basis vector is given, decide which. The random standard basis vector corresponds to the trivial subgroup case in the hidden subgroup problem. This problem is a special case of the decision version of the HSP defined by Fenner and Zhou [5] since we are restricting to order two subgroups. In

their paper, they found a search to decision reduction when N is a power of two. So it turns out that the problem is not computationally easier in that case.

We start by finding a set of vectors that span C and $C^\perp$. Let $\vec{l} \in \mathbb{Z}_N^k$, and $p \in \mathbb{Z}_N$. The vectors have the form

$$|S_{\vec{l},p}^m\rangle = \frac{1}{\sqrt{|T_{\vec{l},p}|}} \sum_{j=0}^{|T_{\vec{l},p}|-1} \omega_{|T_{\vec{l},p}|}^{mj} |\vec{b}_{\vec{l},p}^{(j)}\rangle |\chi_{\vec{l}}\rangle,$$

where $T_{\vec{l},p}$ contains the subset sum solutions for $(\vec{l}, p)$, and the vectors $\vec{b}$ are an ordered set of the subset sum solutions. We call this set of orthonormal vectors the *subset sum* basis. We prove that the $m = 0$ subset of vectors span C and the remaining ones, which have $m \geq 1$, span $C^\perp$.

Ideally we would like to reduce subset sum to the DCSP. Since this is still out of reach, we prove a weaker relationship. Instead, we assume there is an algorithm that uses the subset sum basis to solve the DCSP and examine the consequences. Such an algorithm needs to decide if $m = 0$ or $m \geq 1$ to distinguish if the vector is in C or $C^\perp$. In this paper we consider two main types of unitaries that use this basis. We show that in one case such a unitary can be used to solve random subset sum and in the other case it can be used to solve the random collision problem. This may indicate that the unitaries are difficult to implement.

The first type of unitary we consider maps the subset sum basis to the standard basis. An example would be one that maps each vector $|S_{\vec{l},p}^m\rangle$ to the corresponding standard basis vector $|m, p, \vec{l}\rangle$, identifying the vector. This unitary can be used to solve a subset sum instance $(\vec{l}, p)$ by taking $|0, p, \vec{l}\rangle$, applying U^{-1} to get $|S_{\vec{l},p}^0\rangle$ and measuring, since $|S_{\vec{l},p}^0\rangle$ is a uniform superposition of solutions. The ability to identify the basis vector in this way is very strong because it can solve an NP-complete problem, but we show the connection for a wider range of unitaries. In particular, we show that any unitary that maps the subset sum basis to the standard basis in some way can be used to solve the random subset sum problem in the cryptographic range of constant density $\rho > 1$. This can be view as generalizing the connection to quantum sampling in [1].

The proof for this case works by showing that such a unitary can be used to solve worst case collision for the subset sum function. That is, given a subset sum instance $(\vec{l}, p)$ and a solution vector $\vec{b}$, the goal is to compute a second solution $\vec{b}'$ if one exists. Then we use the fact that random subset sum reduces to random collision for density a constant greater than one [8].

The second type of unitary we consider maps the subset sum basis to vectors where the first bit is zero if the vector is in C, and is one if the vector is from $C^\perp$. This type of unitary can be used to solve the DCSP by computing the unitary on the input vector and measuring the first bit. It is a relaxation of the first type of unitary because it could be followed by another unitary mapping to the standard basis. We show that this type of unitary can be used to solve the random collision problem for subset sum with density $\rho = 1 + c \log \log N / \log N$. This collision problem for this density appears to be less well understood than for constant density.

The proof for this case uses the unitary that can solve the DCSP to solve the random collision problem for subset sum. The problem in this case has an arbitrary solution vector $\vec{b}$ fixed, and then a vector $\vec{l}$ is chosen at random. The goal is again to find a second solution $\vec{b}' \neq \vec{b}$ such that $\vec{b}' \cdot \vec{l} = \vec{b} \cdot \vec{l} \bmod N$ on input $\vec{b}$ and $\vec{l}$.

In addition to these two main types of unitaries we show that a small generalization of the form of the subset sum bases has similar results.

The hardness of random subset sum depends on the density and the same is true for the collision problem. But for collision the definition is important also. There are four definitions of finding collisions of hash functions [14]. Our definition of random subset sum collision is based on the universal one-way hash-function family. That is, for any point in the domain, given the hash function uniformly at random from the family, the goal is to find another point in the domain having the same hash value. Impagliazzo and Naor have shown that random subset sum collision is at least as hard as the random subset sum problem when the density is a constant greater than 1 [8]. However, the density of the random subset sum collision problem we consider has density $\rho \leq 1 + c \log N \log N / \log N$. This density is between the density used for subset sum in [13] and the cryptographic one. The hardness of densities for collision in this range is not known, but it can be contrasted with random subset sum, where the problem gets harder as the density approaches one [8].

There are several open questions. Can the second type of unitary above also be used to solve random subset sum? Consider unitaries which decide membership of C with small error, e.g., $1/poly$. Can these unitaries be implemented efficiently or solve some hard problems? Is it possible to implement a unitary efficiently distinguishing C from $C^\perp$, with the subset sum basis, or some other basis? If a space has a basis that seems hard to be implemented for some reason, does that mean that no basis for that space is efficient? Is it possible that a larger space C' containing C exists where it is easier to test C' vs. $C'^\perp$? Deciding membership in a subspace or its complement is a generalization of classical languages to quantum languages. Are there other examples?

2   Background

In this section, we give the background of the dihedral coset problem and the random subset sum problem.

The dihedral coset problem comes from the dihedral hidden subgroup problem which is:

▶ **Definition 2.1** (Dihedral Hidden Subgroup Problem). Given the dihedral group D_{2N} and a function f that maps D_{2N} to some finite set such that f hides a subgroup H (f takes same value within each coset of H and takes distinct value on different cosets), the problem is to find a set of generators for H.

Ettinger and Hoyer showed that the problem reduces to the case when the subgroup is order two [4]. Hence, we can assume H is an order two subgroup, which can be represented as $\{1, d\}$ for $d \in \mathbb{Z}_N$. All known approaches for solving this problem start by evaluating the function in superposition to get $\sum_{g \in D_{2N}} |g, f(g)\rangle$, and then measuring the function value. This results in an order two coset state $\frac{|0,x\rangle + |1,x+d\rangle}{\sqrt{2}}$, where $x \in \mathbb{Z}_N$ is a random coset representative. Then the problem becomes to find d when given many random order two coset states. This problem is defined as follows:

▶ **Definition 2.2** (Dihedral Coset Problem (DCP)). Given a random k-register order two coset state

$$|c^{(d)}_{x_1,x_2,...,x_k}\rangle = \frac{1}{\sqrt{2}}(|0,x_1\rangle + |1,x_1+d\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0,x_k\rangle + |1,x_k+d\rangle).$$

The problem is to find d.

The hardness of the DCP has been studied by reducing to the random subset sum problem [13] which is defined as follows:

▶ **Definition 2.3** (Random Subset Sum Problem). Given a vector of positive integers $\vec{l} = [l_1, l_2, \ldots, l_k]^T$ uniformly distributed in $\mathbb{Z}_N^k$ and $s = \vec{b} \cdot \vec{l} \pmod{N}$ where $\vec{b} \in \mathbb{Z}_2^k$ is chosen uniformly at random, find a vector $\vec{b}' \in \mathbb{Z}_2^k$ such that $\vec{l} \cdot \vec{b}' = s \pmod{N}$. The density is defined as $\rho = \frac{k}{\log(N)}$.

Although the worst-case subset sum problem is NP-hard, the random subset sum problem can be solved in polynomial time when the density is in a certain range. There is no known polynomial-time algorithm for solving the case when ρ is $\Omega(1/k)$ and $O(\frac{k}{\log^2(k)})$. Regev [13] showed that a solution to the random subset sum problem with $\rho > 1$ implies an efficient quantum algorithm for solving the DCP. Moreover, we note that one can reduce the random subset sum problem with $\rho = O(1/\log k)$ to a lattice problem [12, 2], and then to the DCP [13]. Since these two ranges are generally believed not equivalent, it is still not clear if the DCP is equivalent to random subset sum with ρ in a hard range.

In the rest of this section, we define one more problem which will be used in the section 5.

▶ **Definition 2.4** (Random Subset Sum Collision Problem). Let $\vec{b} \in \mathbb{Z}_2^k$ be an arbitrary fixed vector. Given $\vec{b}$, and a vector $\vec{l} \in \mathbb{Z}_N^k$ chosen uniformly at random, the problem is to find a solution $\vec{b}' \in \mathbb{Z}_2^k$ such that $\vec{b} \cdot \vec{l} \equiv \vec{b}' \cdot \vec{l} \pmod{N}$ and $\vec{b}' \neq \vec{b}$.

The worst-case version of this problem is to find $\vec{b}'$ for arbitrary $\vec{b}$ and $\vec{l}$ which are given. For simplicity, we will call this problem the random collision problem and the worst-case version as the worst-case collision problem in the rest of the paper.

Impagliazzo and Naor showed a relationship between random collision problem and the random subset sum problem. The input in their notation has n numbers modulo $2^{\ell(n)}$ plus the target value.

▶ **Theorem 2.5** (Theorem 3.1 in [8]). *Let $\ell(n) = (1-c)n$ for $c > 0$. If the subset sum function for length $\ell(n)$ is one-way, then it is also a family of universal one-way hash functions.*

The subset sum function for length $\ell(n)$ can be represented by n integers each of which is $\ell(n)$-bits long. The input is an n-bit binary string $\vec{b}$ which indicates a subset of the n integers and the function outputs an integer s which is the sum of the subsets of integers indexed by $\vec{b}$. A family of universal one-way hash functions is the set of functions $\mathcal{F} = \{f\}$ which satisfies the property that if for all x, when f is chosen randomly from $\mathcal{F}$, then finding a collision (i.e., $y \neq x$ and $f(x) = f(y)$) is hard. Note that the random subset sum problem can be viewed as inverting a random subset sum function and the random collision problem is as finding a collision for a random subset sum function.

In the proof of Theorem 2.5 [8], Impagliazzo and Naor showed that finding a collision for a random subset sum function is at least as hard as inverting a random subset sum function. Therefore, we can give the following corollary:

▶ **Corollary 2.6.** *The random subset sum problem with N a power of 2 and ρ a constant > 1 reduces to the random collision problem with the same N and ρ.*

This corollary will be used in the Section 5.

3 The Dihedral Coset Space Problem

In this section we set up our approach. We first define the dihedral coset space problem and show how to use it to solve the dihedral coset problem. Then we define the coset space which we wish to understand.

▶ **Definition 3.1** (Dihedral Coset Space Problem (DCSP))**.** Given a state $|\tau\rangle$ which is promised to be a random order-two coset state $|c_{x_1,x_2,\ldots,x_k}^{(d)}\rangle$ or a random standard basis state $|\vec{b},x\rangle$ where $\vec{b} \in \mathbb{Z}_2^k$ and $x \in \mathbb{Z}_N^k$, the problem is to decide if $|\tau\rangle$ is a k-register order two coset state or not.

A solution to the DCSP implies a polynomial-time algorithm for solving the DCP with N a power of 2 as shown in [5]. We include a proof of our special case here.

▶ **Claim 3.2.** *The dihedral coset problem (DCP) with N a power of 2 reduces to the dihedral coset space problem (DCSP).*

Proof. Suppose we are given the input of the DCP with subgroup d, we first show how to get the least significant bit of d.

Since N is a power of 2, the least significant bit of x and $x + d$ (mod N) are equal for $x \in \mathbb{Z}_N$ if and only if d is even. Therefore by measuring the least significant bit of the state $\frac{|0,x\rangle + |1,x+d\rangle}{\sqrt{2}}$, we get the same state if d is even and get either $|0,x\rangle$ or $|1,x+d\rangle$ (which are standard-basis states) otherwise.

According to the observation above, the least significant bit of d can be computed by the following algorithm. First, we measure the least significant bit of each register. Then all the registers do not change or collapse to a standard-basis state. Finally, apply the algorithm for the DCSP. If the result is an order-two coset state, the least significant bit is 0; otherwise, the least significant bit is 1.

To get bit $(i + 1)$, one subtracts d by the least significant i bits computed and measure the $I + 1$-th least significant bit of the state. Repeat the process above until all bits of d are known.   ◀

It is worth noting that this fact also implies that the lattice problem can be reduced to the DCSP due to the known reduction from the lattice problem to the DCP with N a power of 2 [13].

The main objects we want to understand are the coset space and its complement.

▶ **Definition 3.3.** The coset space $C = \mathrm{span}(\{|c_{x_1,\ldots,x_k}^{(d)}\rangle : d, x_1, \ldots, x_k \in \mathbb{Z}_N\})$ and the orthogonal complement of C is $C^\perp$.

Note that a test for a vector being in C or $C^\perp$ is sufficient to solve the DCSP if k is big enough. This follows from counting the number of k-register order two coset states. There are at most N subgroups, and at most N^k coset representatives, so the number of k-register order two coset states is at most $N(N)^k$. The dimension of the whole space is $(2N)^k$. Hence, the subspace spanned by k-register order-two coset states is at most $1/2$ of the whole space when $k \geq \log 2N$.

▶ **Claim 3.4.** *Let $k = \log 2N + k'$. Let Π_C be a projector onto C and $\Pi_{C^\perp}$ be a projector onto $C^\perp$. If the input is an order two coset state, the measurement $\{\Pi_C, \Pi_{C^\perp}\}$ outputs C always. Otherwise, if the input is a random standard basis state, then this measurement outputs $C^\perp$ with probability at least $1 - 1/2^{k'+1}$.*

4   The Subset Sum Basis

In this section, we start by finding an orthonormal basis for C and one for $C^\perp$. Note that if we can give a unitary which distinguishes which of the two subspaces we are in (C or $C^\perp$) efficiently, we can solve the DCSP efficiently as in Claim 3.4.

In order to make the basis easier to understand, we permute the subsystems so that the first bit of all registers are on the left, and the integers mod N are on the right. That is, write the original basis state $|b_1, x_1, b_2, x_2, \ldots, b_k, x_k\rangle$ as $|b_1, b_2, \cdots, b_k, x_1, x_2, \cdots, x_k\rangle$. In this notation the coset state is written as

$$|c^{(d)}_{x_1,x_2,\ldots,x_k}\rangle = \frac{1}{2^{k/2}} \sum_{b_1,\ldots,b_k=0}^{1} |b_1, \ldots, b_k, x_1 + b_1 d, \ldots, x_k + b_k d\rangle = \frac{1}{2^{k/2}} \sum_{\vec{b} \in \{0,1\}^k} |\vec{b}, \vec{x} + \vec{b}d\rangle.$$

The subset sum basis is defined as follows:

▶ **Definition 4.1** (The Subset Sum Basis). Let $\vec{l} = (l_1, l_2, \cdots, l_k)^T \in \mathbb{Z}_N^k$, and $p \in \mathbb{Z}_N$. Let $T_{\vec{l},p} = \{\vec{b} : \vec{b} \cdot \vec{l} = p, \vec{b} \in \mathbb{Z}_2^k\}$ contain subset sum solutions for input $\vec{l}, p$. If $|T_{\vec{l},p}| = 0$ then define $|S^m_{\vec{l},p}\rangle = 0$. If $|T_{\vec{l},p}| \geq 1$, then let $m \in \{0, \ldots, |T_{\vec{l},p}| - 1\}$ and pick an ordering $\{\vec{b}^{(j)}_{\vec{l},p}\}$ of the solutions in $T_{\vec{l},p}$. Define the vector

$$|S^m_{\vec{l},p}\rangle = \frac{1}{\sqrt{|T_{\vec{l},p}|}} \sum_{j=0}^{|T_{\vec{l},p}|-1} \omega^{mj}_{|T_{\vec{l},p}|} |\vec{b}^{(j)}_{\vec{l},p}\rangle. \tag{1}$$

For $N, k \in \mathbb{Z}$, define two sets

$$\mathcal{B}^\perp = \mathcal{B}^\perp_{k,N} = \{|S^m_{\vec{l},p}\rangle|\chi_{\vec{l}}\rangle : \vec{l} \in \mathbb{Z}_N^k, p \in \mathbb{Z}_N, m \in \{1, \ldots, |T_{\vec{l},p}| - 1\}, |T_{\vec{l},p}| \geq 2\} \tag{2}$$

and

$$\mathcal{B}^0 = \mathcal{B}^0_{k,N} = \{|S^m_{\vec{l},p}\rangle|\chi_{\vec{l}}\rangle : \vec{l} \in \mathbb{Z}_N^k, p \in \mathbb{Z}_N, m = 0, |T_{\vec{l},p}| \geq 1\}. \tag{3}$$

The set $\mathcal{B} = \mathcal{B}^0 \bigcup \mathcal{B}^\perp$ is called the *subset sum basis* of $\mathbb{C}^{(2N)^k}$.

In this definition, $|\chi_j\rangle$ is the Fourier basis state $|\chi_j\rangle = \frac{1}{\sqrt{N}} \sum_i \omega^{ij}_N |i\rangle$, and $|\chi_{\vec{l}}\rangle = |\chi_{l_1}\rangle \cdots |\chi_{l_k}\rangle$. Note that $\mathcal{B}^0 \cup \mathcal{B}^\perp$ is an orthonormal basis for the whole space and the two sets are disjoint. The vector $|S^m_{\vec{l},p}\rangle$ is a superposition of solution vectors $\vec{b}$ to the equation $\vec{l} \cdot \vec{b} = p$. If no such $\vec{b}$ exists then there is no corresponding $|S^m_{\vec{l},p}\rangle$. If at least one solution $\vec{b}$ exists then $|S^0_{\vec{l},p}\rangle$ is in $\mathcal{B}^0$. If at least two solutions $\vec{b}$ exist then vectors appear in $\mathcal{B}^\perp$. Varying m gives orthogonal superpositions of the solutions. Ranging over all $\vec{l} \in \mathbb{Z}_N^k$ and $p \in \mathbb{Z}$ covers all possible bit vectors. Furthermore, these vectors are tensored with every possible Fourier basis state over $\mathbb{Z}_N$.

Next we show that $\mathcal{B}^\perp$ forms an orthonormal basis for $C^\perp$.

▶ **Claim 4.2.** *The vectors in the set $\mathcal{B}^\perp$ form an orthonormal basis of a space that is orthogonal to the k-register order two coset space.*

Proof. As noted, the vectors form an orthonormal basis of the whole space. We will show that an arbitrary state in $\mathcal{B}^\perp$ is orthogonal to all k-register order two coset states. Fix $\vec{l}$ and p, and let

$$|\psi\rangle = |S^m_{\vec{l},p}\rangle|\chi_{\vec{l}}\rangle = \frac{1}{\sqrt{|T|}} \sum_{j=0}^{|T|-1} \omega^{mj}_{|T|} |\vec{b}^{(j)}\rangle|\chi_{\vec{l}}\rangle$$

be a state in $\mathcal{B}^\perp$ where $T = T_{\vec{l},p}$ and $\vec{b}^{(j)} = \vec{b}_{\vec{l},p}^{(j)}$ to simplify notation. Then for an arbitrary order-two coset state $|c_{x_1,x_2,\cdots,x_k}^d\rangle$, the inner product $\langle c_{x_1,x_2,\cdots,x_k}^{(d)}|\psi\rangle$ is

$$
\frac{1}{\sqrt{2^k|T|}} \sum_{\vec{b}\in\{0,1\}^k} \langle\vec{b}|\langle\vec{x}+\vec{b}d| \sum_{j=0}^{|T|-1} \omega_{|T|}^{mj}|\vec{b}^{(j)}\rangle|\chi_{\vec{l}}\rangle \quad = \quad \frac{1}{\sqrt{2^k N^k |T|}} \sum_{j=0}^{|T|-1} \omega_{|T|}^{mj} \omega_N^{\vec{l}\cdot(\vec{x}+d\vec{b}^{(j)})}
$$

$$
= \quad \frac{\omega_N^{\vec{l}\cdot\vec{x}}}{\sqrt{2^k N^k |T|}} \sum_{j=0}^{|T|-1} \omega_{|T|}^{mj}\omega_N^{dp} \tag{4}
$$

$$
= \quad \frac{\omega_N^{\vec{l}\cdot\vec{x}+dp}}{\sqrt{2^k N^k |T|}} \sum_{j=0}^{|T|-1} \omega_{|T|}^{mj} = 0. \tag{5}
$$

Eq. 4 is true because $\vec{b}^{(j)}\cdot\vec{l} = p$ iff $\vec{b}^{(j)}\in T$ by the definition of T. Then since $m \geq 1$ and $|T| \geq 2$ by the definition of $\mathcal{B}_{k,N}^\perp$, Eq. 5 is true.   ◀

According to Claim 4.2, $\text{span}(\mathcal{B}^\perp) \subseteq C^\perp$. Next we show that $\mathcal{B}^0$ exactly spans the subspace C (and thus $\text{span}(\mathcal{B}^\perp) = C^\perp$).

▶ **Lemma 4.3.** *The set $\mathcal{B}^0$ is an orthonormal basis for the subspace spanned by the order-two coset states.*

Proof. Because C is orthogonal to $\text{span}(\mathcal{B}^\perp)$ by Claim 4.2, $C \subseteq \text{span}(\mathcal{B}^0)$. We want to show equality. Suppose for contradiction that $C \subset \text{span}(\mathcal{B}^0)$. Then there is a vector $|\alpha\rangle \in C^\perp$ that is orthogonal to $\text{span}(\mathcal{B}^\perp)$, so $|\alpha\rangle \in C^\perp \cap \text{span}(\mathcal{B}^0)$. We show that there is no non-zero linear combination of states in $\mathcal{B}^0$ whose inner product with all order-two coset states is zero.
Suppose the state

$$
|\alpha\rangle = \sum_{\vec{l}\in\mathbb{Z}_N^k,p\in\mathbb{Z}_N} \alpha_{\vec{l},p}|S_{\vec{l},p}^0\rangle|\chi_{l_1},\ldots,\chi_{l_k}\rangle
$$

is orthogonal to all order-two coset states, i.e., $\langle c_{x_1,x_2,\ldots,x_k}^{(d)}|\alpha\rangle = 0$ for $x_1,\ldots,x_k,d\in\mathbb{Z}_N$, for some nonzero vector $|\alpha\rangle \in \text{span}(B^0)$. This inner product is

$$
\frac{1}{\sqrt{2^k}} \sum_{\vec{b}\in\{0,1\}^k} \langle\vec{b},\vec{x}+\vec{b}d| \sum_{\vec{l}\in\mathbb{Z}_N^k,p\in\mathbb{Z}_N} \alpha_{\vec{l},p}|S_{\vec{l},p}^0\rangle|\chi_{l_1},\ldots,\chi_{l_k}\rangle
$$

$$
= \frac{1}{\sqrt{2^k}} \sum_{\vec{b}\in\{0,1\}^k} \langle\vec{b},\vec{x}+\vec{b}d| \sum_{\vec{l}\in\mathbb{Z}_N^k,p\in\mathbb{Z}_N} \alpha_{\vec{l},p}\frac{1}{\sqrt{|T_{\vec{l},p}|}} \sum_{j=0}^{|T_{\vec{l},p}|-1} |\vec{b}_{\vec{l},p}^{(j)}\rangle|\chi_{l_1},\ldots,\chi_{l_k}\rangle
$$

$$
= \frac{1}{\sqrt{2^k N^k}} \sum_{\vec{l}\in\mathbb{Z}_N^k,p\in\mathbb{Z}_N} \alpha_{\vec{l},p}\frac{1}{\sqrt{|T_{\vec{l},p}|}} \sum_{\vec{b}\in\{0,1\}^k} \sum_{j=0}^{|T_{\vec{l},p}|-1} \langle\vec{b}|b_{\vec{l},p}^{(j)}\rangle\omega_N^{\vec{l}\cdot(\vec{x}+\vec{b}d)}
$$

$$
= \frac{1}{\sqrt{2^k N^k}} \sum_{\vec{l}\in\mathbb{Z}_N^k,p\in\mathbb{Z}_N} \alpha_{\vec{l},p}\frac{\omega_N^{\vec{l}\cdot\vec{x}+pd}}{\sqrt{|T_{\vec{l},p}|}} \sum_{\vec{b}:\vec{b}\cdot\vec{l}=p} \sum_{j=0}^{|T_{\vec{l},p}|-1} \langle\vec{b}|b_{\vec{l},p}^{(j)}\rangle
$$

$$
= \frac{1}{\sqrt{2^k N^k}} \sum_{\vec{l}\in\mathbb{Z}_N^k,p\in\mathbb{Z}_N} \alpha_{\vec{l},p}\omega_N^{\vec{l}\cdot\vec{x}+pd}\sqrt{|T_{\vec{l},p}|}.
$$

Then we have the following equations:

$$\sum_{\vec{l}\in\mathbb{Z}_N^k, p\in\mathbb{Z}_N} \sqrt{\frac{|T_{\vec{l},p}|}{2^k N^k}} \alpha_{\vec{l},p} \cdot \omega_N^{x_1\cdot l_1+\cdots+x_k\cdot l_k+d\cdot p} = 0, \; \forall x_1,\ldots,x_k, d \in \mathbb{Z}_N.$$

Define $\vec{v}$ as an $N^{k+1} \times 1$ vector such that $(\vec{v})_{\vec{l},p} = \sqrt{\frac{|T_{\vec{l},p}|}{2^k N^k}} \alpha_{\vec{l},p}$. The sums above can be represented as follows:

$$A^{\otimes(k+1)} \cdot \vec{v} = \vec{0}, \tag{6}$$

where A is an $N \times N$ Fourier matrices with the (i,j)-th entry as $A_{i,j} = \omega_N^{ij}$ and $\vec{0}$ is an $N^{k+1} \times 1$ vector with all entries as 0. Note that the column of $A^{\otimes(k+1)}$ is indexed by $\vec{l}$ and p and the the row is indexed by $\vec{x}$ and d.

The determinant of $A^{\otimes(k+1)}$ is not zero, so the only vector $\vec{v}$ satisfying Equation 6 is $\vec{v} = \vec{0}$. When $|T_{\vec{l},p}| \geq 1$ this forces $\alpha_{\vec{l},p} = 0$ for every coefficient used in $|\alpha\rangle$. When $|T_{\vec{l},p}| = 0$, $\alpha_{\vec{l},p}$ is not used in the sum because $|S_{\vec{l},p}^m\rangle = 0$. Therefore, these facts contradict the hypothesis that there exists a nonzero vector $|\alpha\rangle \in \text{span}(\mathcal{B}^0)$ which is orthogonal to all order-two coset states. ◄

Now, it is easy to see that a unitary which can efficiently distinguish $\text{span}(\mathcal{B}^0)$ from $\text{span}(\mathcal{B}^\perp)$ also distinguishes C from $C^\perp$ by Claim 3.4 and Lemma 4.3. The next question we address is whether any unitaries that use this basis can be implemented efficiently or not.

5 The hardness results

In general we would like to understand unitaries that can be used to decide if a state is in the coset space C or in $C^\perp$. In this section we look at two types of unitaries using the subset sum basis, plus an extension of each one:

1. A unitary U_S that maps every basis vector $|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle$ to a standard basis state. Note that if these standard basis states specify p and $\vec{l}$, then this can be used to solve the worst case subset sum, but we are allowing a more general type of unitary here.

2. A unitary U_C that maps every basis vector $|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle$ to $|m=0?\rangle|\phi_{\vec{l},p}^m\rangle$, indicating whether or not the state is in the coset space.

3. A unitary $U = \tilde{U}_S$ that satisfies condition (1) or $U = \tilde{U}_C$ that satisfies (2), but U uses a slightly more general basis, where any basis can be chosen for each $(\vec{l}, p)$ subspace $\text{span}\{|S_{\vec{l},p}^m, \chi_{\vec{l}}\rangle : m \geq 1\}$.

For the last type we use any basis satisfying the following definition.

▶ **Definition 5.1.** Let $\tilde{\mathcal{B}}^0 = \mathcal{B}^0 = \{|S_{\vec{l},p}^0\rangle|\chi_{\vec{l}}\rangle : \vec{l} \in \mathbb{Z}_N^k, p \in \mathbb{Z}_N, m = 0, |T_{\vec{l},p}| \geq 1\}$ be as in Definition 4.1, and let $\tilde{\mathcal{B}}^\perp = \{|\tilde{S}_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle : \vec{l} \in \mathbb{Z}_N^k, p \in \mathbb{Z}_N, m \in \{1,\ldots,|T_{\vec{l},p}|-1\}, |T_{\vec{l},p}| \geq 2\}$ be an orthogonal basis such that $\text{span}(\{|\tilde{S}_{\vec{l},p}^m\rangle : m \in \{1,\ldots,|T_{\vec{l},p}|-1\}\}) = \text{span}(\{|S_{\vec{l},p}^m\rangle : m \in \{1,\ldots,|T_{\vec{l},p}|-1\}\})$ for all $\vec{l}, p$.

We show that unitaries of type 1 above can be used to solve random subset sum for the cryptographic density ρ a constant greater than 1, indicating that such a unitary may be hard to implement. This strengthens the result in [1] which is a special case where the unitary must perform quantum sampling, i.e., map an input $|\vec{l},p\rangle$ to a superposition of solutions $|\vec{l}, S_{\vec{l},p}^0\rangle$. Such a unitary implementing quantum sampling can used to solve worst-case subset

sum by taking an input $|0, p, \vec{l}\rangle$, applying U inverse to get $|S_{\vec{l},p}^0\rangle|\vec{l}\rangle$ and measuring, since this is a uniform superposition of solutions.

An algorithm that uses the subset sum basis to solve the DCSP needs to decide if $m = 0$ (for C), or $m > 0$ (for $C^\perp$). The second type of unitary above allows an arbitrary unitary that writes the answer in the first bit. We show that such a unitary can solve random collision for density $\rho = 1 + c \log \log N / \log N$. This may indicate that no such unitary can be efficiently implemented, although we are less clear on the difficulty of the random collision problem.

The third type of unitary allows an arbitrary basis within each subspace of solutions, but does not mix solutions of different inputs $\vec{l}, p$. Note that $|S_{\vec{l},p}^0\rangle$ cannot change in this case, since it is one dimension in $\mathcal{B}^0$. Let $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}^0 \cup \tilde{\mathcal{B}}^\perp$ be the basis used by the unitary.

The proofs work by using U_S to solve the worst-case collision problem, or U_C, $\tilde{U}_S$, or $\tilde{U}_C$ to solve the random collision problem.

5.1   Unitary mapping to a standard basis

First we give an algorithm that finds a solution to the worst-case collision problem when given a unitary U_S that maps the subset sum basis $\mathcal{B}$ to the standard basis in an arbitrary way. Given $\vec{b}$ and $\vec{l}$ where $\vec{b} \in \mathbb{Z}_2^k$ and $\vec{l} \in \mathbb{Z}_N^k$, the task in the worst-case collision problem is to find $\vec{b}' \neq \vec{b}$ such that $\vec{b}' \cdot \vec{l} = \vec{b} \cdot \vec{l}$.

▶ **Algorithm 1.** *On input $\vec{l} \in \mathbb{Z}_N^k$ and $\vec{b} \in \mathbb{Z}_2^k$:*
1. *Prepare the quantum state $|\vec{b}, \vec{l}\rangle$.*
2. *Apply QFT_N^k on $\vec{l}$, then the state becomes $|\vec{b}\rangle|\chi_{\vec{l}}\rangle$.*
3. *Apply U_S to $|\vec{b}\rangle|\chi_{\vec{l}}\rangle$.*
4. *Measure $U_S(|\vec{b}\rangle|\chi_{\vec{l}}\rangle)$ in the standard basis.*
5. *Apply $U_S^\dagger$.*
6. *Measure value $\vec{b}'$ in the first register.*

Here QFT_N^k is the quantum Fourier transform over $\mathbb{Z}_N^k$.

▶ **Theorem 5.2.** *If there exists an efficient unitary operator U_S, where U_S is a bijection between the subset sum basis and the standard basis, then the worst-case collision problem can be solved efficiently by a quantum algorithm. Therefore random subset sum with density a constant greater than 1 can also be solved.*

Proof. Given $\vec{l}$ and $\vec{b}$ as input, let $p = \vec{l} \cdot \vec{b}$ and $T = T_{\vec{l},p}$. For $\vec{b} = \vec{b}^{(j_0)} \in T$, after computing the Fourier transform of the second register, the resulting state $|\vec{b}^{(j_0)}, \chi_{\vec{l}}\rangle$ can be wrtten in the subset sum basis as

$$|\vec{b}^{(j_0)}, \chi_{\vec{l}}\rangle = \frac{1}{\sqrt{|T|}} \sum_{m=0}^{|T|-1} \omega^{-j_0 m} |S_{\vec{l},p}^m\rangle |\chi_{\vec{l}}\rangle.$$

Applying U_S to this state gives the state

$$\frac{1}{\sqrt{|T|}} \sum_{m=0}^{|T|-1} \omega^{-j_0 m} |D_{\vec{l},p}^m\rangle, \tag{7}$$

where $|D_{\vec{l},p}^m\rangle := U_S(|S_{\vec{l},p}^m\rangle |\chi_{\vec{l}}\rangle)$ is a standard basis vector by assumption on U_S. Measuring the state in Equation (7) in the standard basis gives $|D_{\vec{l},p}^m\rangle$ for some $m \in [0 : |T| - 1]$. Applying

$U_S^\dagger$ to $|D_{\vec{l},p}^m\rangle$ gives $|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle$, where the first register is $|S_{\vec{l},p}^m\rangle = \frac{1}{\sqrt{|T|}}\sum_{j=0}^{|T|-1}\omega^{jm}|\vec{b}^{(j)}\rangle$ in the standard basis. Measuring this gives a vector $\vec{b}' \neq \vec{b}$ with probability $\frac{|T|-1}{|T|}$.

Theorem 2.5 reduces random subset sum to solving the random collision problem for constant density greater than one, so random subset sum also reduces to the worst case collision problem.   ◀

The proof that Algorithm 1 works used a special property of the subset sum basis, which is that every basis vector $|S_{\vec{l},p}^m\rangle$ spreads the solutions with equal magnitude. When this is not the case then the algorithm does not work for the worst case collision problem. However, we will later show that it solves the random collision problem, as long as the number of solutions is not too large.

First we describe an example basis where the algorithm fails. The idea is that the unitary can map a solution vector $|\vec{b},\chi_\ell\rangle$ to a vector that is very close to itself. In that case the algorithm will measure the same value $\vec{b}$ that it started with and not solve the collision problem, which can be seen as follows. Let $\vec{b} = \vec{b}^{(0)}$, let

$$|\hat{S}^1\rangle|\chi_{\vec{l}}\rangle = \frac{1}{\sqrt{|T|-1}}\sum_{m=1}^{|T|-1}|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle,$$

and pick arbitrary orthonormal vectors $|\hat{S}^2\rangle,\ldots,|\hat{S}^{|T|-1}\rangle$ to form a basis for the subspace $\mathrm{span}(\{|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle : m \in [1:|T|-1]\})$. Note that $|\langle\vec{b},\chi_{\vec{l}}|\hat{S}^1,\chi_{\vec{l}}\rangle|^2 = \frac{|T|-1}{|T|}$, which implies that one gets $U_S(|\hat{S}^1\rangle|\chi_{\vec{l}}\rangle)$ with probability $\frac{|T|-1}{|T|}$ after applying U_S and measuring in the standard basis. In that case, applying $U_S^\dagger$ results in the input vector $\vec{b}$. Therefore, given a unitary mapping this new basis $\{|S^0,\chi_{\vec{l}}\rangle,|\hat{S}^1,\chi_{\vec{l}}\rangle,\ldots,|\hat{S}^{|T|-1},\chi_{\vec{l}}\rangle\}$ to standard basis, the algorithm returns an answer $\vec{b}' \neq \vec{b}$ happens with probability $1/|T|$. The number of solutions $|T|$ can be very large for larger densities.

Next we show that if we limit the size of T, then random collision can be solved.

▶ **Corollary 5.3.** *Suppose Algorithm 1 is run with $\tilde{U}_S$. If $\tilde{U}_S$ is an efficient unitary operator which maps every state in $\tilde{\mathcal{B}}$ to an arbitrary state in the standard basis, then on input $\vec{l},\vec{b}$, the algorithm solves the collision problem with probability at least $\frac{1}{|T_{\vec{l},p}|}(1 - \frac{1}{|T_{\vec{l},p}|})$, where $p = \vec{l}\cdot\vec{b}$. In particular, when $k \leq \log N + c\log\log N$, the random collision problem can be solved in quantum polynomial time.*

Proof. Similar to the proof for Theorem 5.2, first represent $|\vec{b},\chi_{\vec{l}}\rangle$ as a linear combination of states in $\tilde{\mathcal{B}}$ as follows:

$$|\vec{b},\chi_{\vec{l}}\rangle = \frac{1}{\sqrt{|T|}}|S^0\rangle|\chi_{\vec{l}}\rangle + \sqrt{\frac{|T|-1}{|T|}}\left(\sum_{m=1}^{|T|-1}c_m|\tilde{S}^m\rangle\right)|\chi_{\vec{l}}\rangle,$$

where $T = T_{\vec{l},p}$ and $\tilde{S}^m = \tilde{S}_{\vec{l},p}^m$.

After applying the unitary $\tilde{U}_S$, the state is

$$\tilde{U}_S|\vec{b},\chi_{\vec{l}}\rangle = \frac{1}{\sqrt{|T|}}|D^0\rangle + \sqrt{\frac{|T|-1}{|T|}}\left(\sum_{m=1}^{|T|-1}c_m|D^m\rangle\right), \tag{8}$$

where D^m for $m \in [0:|T|-1]$ are arbitrary distinct states in the standard basis. By measuring the state in the Equation (8) in the standard basis, $|D^0\rangle$ is measured with probability $1/|T|$. Then applying $\tilde{U}_S^\dagger$ and measuring the output state in the standard basis gives $\vec{b}' \neq \vec{b}$ with

probability $\frac{|T|-1}{|T|}$. Based on the Claim 1.1, $|T| = poly(k)$ with high probability. Thus, given a random input $\vec{b}$ and $\vec{l}$, the probability to get $\vec{b}' \neq \vec{b}$ using $\tilde{U}_S$ in Algorithm 1 is at least $\frac{|T|-1}{|T|^2} = 1/poly(k)$.   ◀

5.2   Deciding membership in C

The unitary U_S illustrated how our algorithm works and used the subset sum basis, but U_S may not be useful for distinguishing C from $C^\perp$ in general. Next we consider a unitary U_C that can distinguish C from $C^\perp$. Suppose U_C works on a larger Hilbert space to have work space and exactly distinguishes $\mathcal{B}^0$ from $\mathcal{B}^\perp$ in the first qubit as follows:

▶ **Definition 5.4.** Let U_C be a unitary operator such that

$$U_C(|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle|0\rangle) = \begin{cases} |0\rangle|\psi_{\vec{l},p,0}\rangle & \text{if } m = 0 \\ |1\rangle|\psi_{\vec{l},p,m}\rangle & \text{otherwise} \end{cases}$$

where $\{|\psi_{\vec{l},p,m}\rangle : \vec{l} \in \mathbb{Z}_N^k, p \in \mathbb{Z}_N, m \in [0 : |T_{\vec{l},p}| - 1]\}$ are states resulting from applying U_C and the third register is a workspace initialized to $|0\rangle$.

We modify Algorithm 1 so that only the first bit is measured in step four, and U_C is used instead of U_S.

▶ **Algorithm 2.** *On input $\vec{l} \in \mathbb{Z}_N^k$ and $\vec{b} \in \mathbb{Z}_2^k$:*
1. *Prepare the quantum state $|\vec{b}, \vec{l}\rangle$.*
2. *Apply QFT_N^k on $\vec{l}$, then the state becomes $|\vec{b}\rangle|\chi_{\vec{l}}\rangle$.*
3. *Apply U_C to $|\vec{b}\rangle|\chi_{\vec{l}}\rangle$.*
4. *Measure the first qubit of $U_C(|\vec{b}\rangle|\chi_{\vec{l}}\rangle)$ in the standard basis.*
5. *Apply $U_C^\dagger$.*
6. *Measure the first register in the standard basis.*

▶ **Theorem 5.5.** *If U_C can be implemented efficiently, then Algorithm 2 solves the collision problem on input $\vec{l}, \vec{b}$ with probability $\frac{2}{|T_{\vec{l},p}|}(1 - \frac{1}{|T_{\vec{l},p}|})$, where $p = \vec{l} \cdot \vec{b}$. In particular, when $k \leq \log N + c \log\log N$ the random collision problem can be solved in quantum polynomial time.*

Proof. Given $\vec{l}$ and $\vec{b}$ as input, let $p = \vec{l} \cdot \vec{b}$ and $T = T_{\vec{l},p}$. For $\vec{b} = \vec{b}^{(j_0)} \in T$, after computing the Fourier transform of the second register, we can write the resulting state $|\vec{b}^{(j_0)}, \chi_{\vec{l}}\rangle$ in the subset sum basis as follows:

$$|\vec{b}^{(j_0)}, \chi_{\vec{l}}\rangle = \frac{1}{\sqrt{|T|}} \sum_{m=0}^{|T|-1} \omega^{-j_0 m} |S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle.$$

Applying U_C to this state plus a work register results in

$$\frac{1}{\sqrt{|T|}}(|0\rangle|\psi_{\vec{l},p,0}\rangle + \sum_{m=1}^{|T|-1} \omega^{-j_0 m}|1\rangle|\psi_{\vec{l},p,m}\rangle). \tag{9}$$

Measuring the first qubit of the state in Equation (9) gives $|0\rangle|\psi_{\vec{l},p,m}\rangle$ with probability $1/|T|$ and $\frac{1}{\sqrt{|T|-1}} \sum_{m=1}^{|T|-1} \omega^{-j_0 m}|1\rangle|\psi_{\vec{l},p,m}\rangle$ with probability $1 - 1/|T|$.

Applying $U_C^\dagger$ to the result gives $|S_{\vec{l},p}^0\rangle|\chi_{\vec{l}}\rangle$ in the first case and

$$\frac{1}{\sqrt{|T|-1}}\sum_{m=1}^{|T|-1}\omega^{-j_0 m}|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle$$

in the second case.

Finally, the state is measured in the standard basis. In the first case, when a zero is measured in the first bit, which happens with probability $1/|T|$, a vector $\vec{b}' \neq \vec{b}$ is measured with probability $1 - 1/|T|$ in the last step. In the second case when a one is measured the amplitude of $|\vec{b}^{(j_0)}, \chi_{\vec{l}}\rangle$ in $\frac{1}{\sqrt{|T|-1}}\sum_{m=1}^{|T|-1}\omega^{-j_0 m}|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle$ is

$$\frac{1}{\sqrt{|T|-1}}\sum_{m=1}^{|T|-1}\omega^{-j_0 m}\langle\vec{b}^{(j_0)}, \chi_{\vec{l}}|S_{\vec{l},p}^m\rangle|\chi_{\vec{l}}\rangle = \frac{1}{\sqrt{|T|-1}}\sum_{m=1}^{|T|-1}\omega^{-j_0 m}\langle\vec{b}^{(j_0)}|S_{\vec{l},p}^m\rangle$$

$$= \frac{1}{\sqrt{(|T|-1)|T|}}\sum_{m=1}^{|T|-1}\omega^{-j_0 m}\omega^{j_0 m} = \frac{|T|-1}{\sqrt{(|T|-1)|T|}}.$$

Thus, the probability that the measurement gives $\vec{b}' \neq \vec{b}$ is $1 - \frac{(|T|-1)^2}{(|T|-1)|T|} = 1/|T|$. Therefore, the probability the algorithm returns $\vec{b}' \neq \vec{b}$ is $\frac{2}{|T|}(1 - \frac{1}{|T|})$.

By Claim 1.1 the probability that a randomly chosen $\vec{l}$ satisfies $|T_{\vec{l},p}| \leq poly(k)$ is at least $1/poly(k)$ when $k = \log N + c\log\log N$. Thus, the random collision problem can be solved by repeating the algorithm $poly(k)$ times. ◀

Now we consider the case where an arbitrary basis can be used within each subspace spanned by solutions of a given subset sum instance $\vec{l}, p$ as in Definition 5.1. Let $\tilde{U}_C$ be a unitary that maps every state in $\tilde{\mathcal{B}}$ to quantum state whose first qubit indicates if the state is in $\tilde{\mathcal{B}}^0$ or $\tilde{\mathcal{B}}^\perp$

▶ **Corollary 5.6.** *If Algorithm 2 is run with $\tilde{U}_C$ on input $\vec{l}, \vec{b}$, then it solves the collision problem with probability at least $\frac{1}{|T_{\vec{l},p}|}(1 - \frac{1}{|T_{\vec{l},p}|})$, where $p = \vec{l}\cdot\vec{b}$. In particular, if $k \leq \log N + c\log\log N$ then it solves the random collision problem in quantum polynomial time.*

Proof. Suppose $\tilde{U}_C$ maps $|S_{\vec{l},p}^0\rangle$ to a state $|0\rangle|\psi_{\vec{l},p,0}\rangle$ and maps $|\tilde{S}_{\vec{l},p}^m\rangle$ to $|1\rangle|\psi_{\vec{l},p,m}\rangle$ for $m \in [1 : |T|-1]$, where the set of vectors $\{|\psi_{\vec{l},p,m}\rangle : m \in [1 : |T|-1]\}$ are an arbitrary orthonormal set of quantum states. The analysis is similar to the proof above, but we only consider the case when the state collapses to $m = 0$. Specifically, after applying $\tilde{U}_C$ to $|\vec{b}, \chi_{\vec{l}}\rangle$, the state is

$$\tilde{U}_C|\vec{b}, \chi_{\vec{l}}\rangle = \frac{1}{\sqrt{|T|}}|0\rangle|\psi_{\vec{l},p,0}\rangle + \sqrt{\frac{|T|-1}{|T|}}(\sum_{m=1}^{|T|-1}c_m|1\rangle|\psi_{\vec{l},p,m}\rangle). \tag{10}$$

The probability the state collapses to $|0\rangle|\psi_{\vec{l},p,0}\rangle$ after measuring the first qubit is $1/|T|$. After applying $\tilde{U}_C^\dagger$ and measuring the state a vector $\vec{b}' \neq \vec{b}$ is measured with probability $1 - 1/|T|$. In total the probability of success is at least $\frac{|T|-1}{|T|^2}$. For the choice of k given, this is at least $1/poly(k)$ with probability $1/poly(k)$ by Claim 1.1. ◀

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A   Appendix

▶ **Claim 1.1.** *Let $\vec{b} \in \mathbb{Z}_2^k$ be an arbitrary fixed vector with $k = \log N + c \log \log N$ for some constant c. Then over random choices of $\vec{l} \in \mathbb{Z}_N^k$, the probability that $|T_{\vec{l}, \vec{b} \cdot \vec{l}}| \leq poly(k)$ is at least $\frac{1}{poly(k)}$.*

Proof. Fix $\vec{b} \in \mathbb{Z}_2^k$ and let $X_{\vec{b}'}$ be a random variable over $\vec{l}$ such that $X_{\vec{b}'} = 1$ if $\vec{b}' \cdot \vec{l} = \vec{b} \cdot \vec{l}$ and $X_{\vec{b}'} = 0$ otherwise. Then $|T_{\vec{l},\vec{l}\cdot\vec{b}}| = \sum_{\vec{b}' \in \mathbb{Z}_2^k} X_{\vec{b}'} = \sum_{\vec{b}' \in \mathbb{Z}_2^k \setminus \{\vec{b}\}} X_{\vec{b}'} + 1$.

For $\vec{b}' \neq \vec{b}$, the expected value of $X_{\vec{b}'}$ is $\mathrm{E}[X_{\vec{b}'}] = \mathrm{Prob}_{\vec{l}}(X_{\vec{b}'} = 1) = \mathrm{Prob}_{\vec{l}}(\vec{l} \cdot \vec{b}' = \vec{l} \cdot \vec{b}) = \frac{1}{N}$. The last equality can be seen by choosing i such that $b_i' = 1$ and $b_i = 0$ without loss generality ($\vec{b}$ and $\vec{b}'$ can be swapped if needed). Then by fixing l_j for $j \neq i$, and choosing l_i uniformly, $\vec{b} \cdot \vec{l}$ is fixed while $\vec{b}' \cdot \vec{l}$ is uniformly distributed. The variance of $X_{\vec{b}'}$ is $\mathrm{Var}(X_{\vec{b}'}) = \frac{1}{N} - \frac{1}{N^2}$.

Therefore, the expected value of $|T_{\vec{l},\vec{l}\cdot\vec{b}}| - 1$ is $\mathrm{E}[\sum_{\vec{b}' \in \mathbb{Z}_2^k \setminus \{\vec{b}\}} X_{\vec{b}'}] = \sum_{\vec{b}' \in \mathbb{Z}_2^k \setminus \{\vec{b}\}} \mathrm{E}[X_{\vec{b}'}] = \frac{2^k - 1}{N}$, and the variance of $|T_{\vec{l},\vec{l}\cdot\vec{b}}| - 1$ is

$$
\begin{aligned}
\mathrm{Var}(\sum_{\vec{b}' \in \mathbb{Z}_2^k \setminus \{\vec{b}\}} X_{\vec{b}'}) &= \sum_{\vec{b}' \in \mathbb{Z}_2^k \setminus \{\vec{b}\}} \mathrm{Var}(X_{\vec{b}'}) + \sum_{\vec{b}' \neq \vec{b}'', \vec{b}', \vec{b}'' \in \mathbb{Z}_2^k \setminus \{\vec{b}\}} \mathrm{Cov}(X_{\vec{b}'}, X_{\vec{b}''}) \\
&\leq \frac{2^k - 1}{N} + \sum_{\vec{b}'' \neq \vec{b}', \vec{b}', \vec{b}'' \in \mathbb{Z}_2^k \setminus \{\vec{b}\}} \mathrm{Cov}(X_{\vec{b}'}, X_{\vec{b}''}). \quad (11)
\end{aligned}
$$

This results in $\mathrm{Var}(\sum_{\vec{b}' \in \mathbb{Z}_2^k \setminus \{\vec{b}\}} X_{\vec{b}'}) \leq \frac{2^k - 1}{N}$ provided that the covariences are all zero, which we show below. First we finish proving the claim by applying Chebyshev's inequality to get

$$
\mathrm{Prob}(|T_{\vec{l},\vec{b}\cdot\vec{l}}| \geq poly(k)) \leq \frac{2^k - 1}{N} \frac{1}{poly(k)} = \frac{1}{poly(k)},
$$

when $k \leq \log N + c \log \log N$.

In the following, we show that $X_{\vec{b}'}$ and $X_{\vec{b}''}$ are independent when $\vec{b}$, $\vec{b}'$, and $\vec{b}''$ are all different values, which implies $\mathrm{Cov}(X_{\vec{b}'}, X_{\vec{b}''}) = 0$. To see this let 1 be a coordinate such that $b_1' = 1$ and $b_1'' = 0$ without loss of generality (b_j' and b_j'' can be swapped). If $b_1 = 0$, then

$$
\begin{aligned}
&\mathrm{Prob}_{\vec{l}}(X_{\vec{b}'} = 1, X_{\vec{b}''} = 1) \\
&= \sum_{l_2,\ldots,l_k=0:}^{N-1} \mathrm{Prob}_{l_1}(X_{\vec{b}'} = 1, X_{\vec{b}''} = 1 | l_2, \ldots, l_k) \cdot \mathrm{Prob}(l_2, \ldots, l_k) \\
&= \frac{1}{N^{k-1}} \sum_{l_2,\ldots,l_k=0}^{N-1} \mathrm{Prob}_{l_1}(X_{\vec{b}'} = 1 | l_2, \ldots, l_k) \cdot \mathrm{Prob}_{l_1}(X_{\vec{b}''} = 1 | l_2, \ldots, l_k) \quad (12) \\
&= \frac{1}{N} \cdot \frac{1}{N^{k-1}} \sum_{l_2,\ldots,l_k=0}^{N-1} \mathrm{Prob}_{l_1}(X_{\vec{b}''} = 1 | l_2, \ldots, l_k) \\
&= \frac{1}{N} \cdot \frac{1}{N^{k-1}} \cdot N^{k-2} = \frac{1}{N^2}. \quad (13)
\end{aligned}
$$

Equation 12 is true because $X_{b''}$ is fixed after fixing $l_2, \ldots, l_k$. For Equation 13 note that $\vec{b}$ and $\vec{b}''$ differ in at least one bit besides position $i = 1$. Therefore a $1/N$ fraction of the N^{k-1} choices for $l_2, \ldots, l_k$ satisfy $\vec{l} \cdot \vec{b} = \vec{l} \cdot \vec{b}''$.

In the case where $b_1 = 1$ the properties of $X_{\vec{b}'}$ and $X_{\vec{b}''}$ are reversed:

$$
\begin{aligned}
&\mathrm{Prob}_{\vec{l}}(X_{\vec{b}'} = 1, X_{\vec{b}''} = 1) \\
&= \sum_{l_2,\ldots,l_k=0}^{N-1} \mathrm{Prob}_{l_1}(X_{\vec{b}'} = 1, X_{\vec{b}''} = 1 | l_2, \ldots, l_k) \cdot \mathrm{Prob}(l_2, \ldots, l_k) \\
&= \frac{1}{N^{k-1}} \sum_{l_2,\ldots,l_k=0}^{N-1} \mathrm{Prob}_{l_1}(X_{\vec{b}'} = 1 | l_2, \ldots, l_k) \cdot \mathrm{Prob}_{l_1}(X_{\vec{b}''} = 1 | l_2, \ldots, l_k) \qquad (14) \\
&= \frac{1}{N} \frac{1}{N^{k-1}} \sum_{l_2,\ldots,l_k=0}^{N-1} \mathrm{Prob}_{l_1}(X_{\vec{b}'} = 1 | l_2, \ldots, l_k) \\
&= \frac{1}{N} \cdot \frac{1}{N^{k-1}} \cdot N^{k-2} = \frac{1}{N^2}. \qquad (15)
\end{aligned}
$$

Equation 14 is true because $X_{\vec{b}'}$ is fixed to 0 or 1 for all l_1. Equation 15 is true because $\vec{b}$ and $\vec{b}'$ differ in at least one bit besides $i = 1$.

Therefore, the covariance of $X_{\vec{b}'}$ and $X_{\vec{b}''}$ is 0.   ◀