Simple, Near-Optimal Quantum Protocols for Die-Rolling

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―――― **Abstract** ――――

Die-rolling is the cryptographic task where two mistrustful, remote parties wish to generate a random D-sided die-roll over a communication channel. Optimal quantum protocols for this task have been given by Aharon and Silman (New Journal of Physics, 2010) but are based on optimal weak coin-flipping protocols which are currently very complicated and not very well understood. In this paper, we first present very simple classical protocols for die-rolling which have decent (and sometimes optimal) security which is in stark contrast to coin-flipping, bit-commitment, oblivious transfer, and many other two-party cryptographic primitives. We also present quantum protocols based on the idea of integer-commitment, a generalization of bit-commitment, where one wishes to commit to an integer. We analyze these protocols using semidefinite programming and finally give protocols which are very close to Kitaev's lower bound for any $D \geq 3$.

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1 Introduction

Die-rolling is the two-party cryptographic primitive in which two spatially separated parties, Alice and Bob, wish to agree upon an integer $d \in [D] := \{1, \ldots, D\}$, generated uniformly at random, over a communication channel. When designing die-rolling protocols, the security goals are:

1. *Completeness:* If both parties are honest, then their outcomes are the same, uniformly random, and neither party aborts.
2. *Soundness against cheating Bob:* If Alice is honest, then a dishonest (i.e., cheating) Bob cannot influence her protocol outcome away from uniform.
3. *Soundness against cheating Alice:* If Bob is honest, then a dishonest (i.e., cheating) Alice cannot influence his protocol outcome away from uniform.

We note here that Alice and Bob start uncorrelated and unentangled. Otherwise, Alice and Bob could each start with half of the following maximally entangled state

$$\frac{1}{\sqrt{D}} \sum_{d \in [D]} |d\rangle_{\mathcal{A}} |d\rangle_{\mathcal{B}}$$

and measure in the computational basis to obtain a perfectly correlated, uniformly random die-roll. Thus, such a primitive would be trivial if they were allowed to start entangled.

Die-rolling is a generalization of a well-studied primitive known as *coin-flipping* [5] which is the special case of die-rolling when $D = 2$. In this paper, we analyze die-rolling protocols in a similar fashion that is widely adopted for coin-flipping protocols [3, 17, 13, 16, 8, 18, 19]. That is, we assume perfect completeness and calculate the soundness in terms of the *cheating probabilities*, as defined by the symbols:

$P_{B,d}^*$: The maximum probability with which a dishonest Bob can force an honest Alice to accept the outcome $d \in [D]$ by digressing from protocol.

$P_{A,d}^*$: The maximum probability with which dishonest Alice can force an honest Bob to accept the outcome $d \in [D]$ by digressing from protocol.

We are concerned with designing protocols which minimize the maximum of these $2D$ quantities since a protocol is only as good as its worst cheating probability. Coincidentally, all the protocols we consider in this paper have the property that all of Alice's cheating probabilities are equal and similarly for a cheating Bob. Therefore, for brevity, we introduce the following shorthand notation:

$$P_A^* := \max\{P_{A,1}^*, \ldots, P_{A,D}^*\} \quad \text{and} \quad P_B^* := \max\{P_{B,1}^*, \ldots, P_{B,D}^*\}.$$

When $D = 2$, the security definition for die-rolling above aligns with that of *strong* coin-flipping. For strong coin-flipping, it was shown by Kitaev [14] that any quantum protocol satisfies $P_{A,1}^* P_{B,1}^* \geq 1/2$ and $P_{A,2}^* P_{B,2}^* \geq 1/2$, implying that at least one party can cheat with probability at least $1/\sqrt{2}$. It was later shown by Chailloux and Kerenidis [8] that all four cheating probabilities can be made arbitrarily close to $1/\sqrt{2}$ by using optimal quantum protocols for *weak* coin-flipping as discovered by Mochon [16].

As pointed out in [1], Kitaev's proof for the lower bound on coin-flipping extends naturally to die-rolling; it can be shown that for any quantum die-rolling protocol, we have

$$P_{A,d}^* P_{B,d}^* \geq \frac{1}{D}$$

for any $d \in [D]$. This implies the lower bound $\max\{P_A^*, P_B^*\} \geq 1/\sqrt{D}$. In fact, extending the optimal coin-flipping protocol construction in [8], it was shown by Aharon and Silman [1] that for $D > 2$, it is possible to find quantum protocols where the maximum of the $2D$ probabilities is at most $1/\sqrt{D} + \delta$, for any $\delta > 0$.

The optimal protocols in [8] and [1] are not explicit as they rely on using Mochon's optimal weak coin-flipping protocols as subroutines. Moreover, Mochon's protocols are very complicated and not given explicitly, although they have been simplified [2].

The best known *explicit* quantum protocol for die-rolling[1] of which we are aware is given in [1]. It uses three messages and has cheating probabilities

$$P_A^* := \frac{D+1}{2D} \quad \text{and} \quad P_B^* := \frac{2D-1}{D^2}.$$

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[1] The protocols considered in this paper have a much different form than these protocols.

These probabilities have the attractive property of approximating Kitaev's lower bound in the limit, but since $P_A^* \to 1/2$ as $D \to \infty$, the maximum cheating probability is quite large.

This motivates the work in this paper which is to find simple and explicit protocols for die-rolling that approximate Kitaev's lower bound on the maximum cheating probability

$$\max\{P_A^*, P_B^*\} \geq 1/\sqrt{D}.$$

1.1 Simple classical protocols

We first show that simple classical protocols exist with decent security.

▶ **Protocol 1** (Classical protocol).
- *Bob chooses a subset $S \subseteq [D]$ with $|S| = m$, uniformly at random, and sends S to Alice. If $|S| \neq m$, Alice aborts.*
- *Alice selects $d \in S$ uniformly at random and tells Bob her selection. If $d \notin S$, Bob aborts.*
- *Both parties output d.*

We see that this is a valid die-rolling protocol as each party outputs the same value $d \in [D]$ and each value occurs with equal probability. As for the cheating probabilities, it is straightforward to see that

$$P_A^* = \frac{m}{D} \quad \text{and} \quad P_B^* = \frac{1}{m}.$$

Besides being extremely simple, this protocol has the following interesting properties:
- The product $P_{A,d}^* P_{B,d}^* = 1/D$, for any $d \in [D]$, saturates Kitaev's lower bound for every $d \in [D]$.
- For D square and $m = \sqrt{D}$, we have $P_A^* = P_B^* = 1/\sqrt{D}$, yielding an optimal protocol!
- If D is not square, then one party has a cheating advantage, i.e., $P_A^* \neq P_B^*$.

Note that to minimize $\max\{P_A^*, P_B^*\}$, it does not make sense to choose large m (greater than $\lceil \sqrt{D} \rceil$) or small m (less than $\lfloor \sqrt{D} \rfloor$). We can see that for $D = 3$, $D = 7$, or $D = 8$, for example, that choosing the ceiling is better while for $D = 5$ or $D = 10$ choosing the floor is better. Thus, we keep both the cases and summarize the overall security of the above protocol in the following lemma.

▶ **Lemma 2.** *For $D \geq 2$, there exists a classical die-rolling protocol satisfying*

$$\frac{1}{\sqrt{D}} \leq \max\{P_A^*, P_B^*\} = \min\left\{ \frac{\lceil \sqrt{D} \rceil}{D}, \frac{1}{\lfloor \sqrt{D} \rfloor} \right\} \tag{1}$$

which is optimal when D is square.

Note that the special case of $D = 2$ has either Alice or Bob able to cheat perfectly, which is the case for all classical coin-flipping protocols. However, Kitaev's bound on the product of cheating probabilities is still (trivially) satisfied. For $D = 3$, we can choose $m = 2$ to obtain $\max\{P_A^*, P_B^*\} = 2/3$ proving that even classical protocols can have nontrivial security, which is vastly different than the $D = 2$ case. The values from (1) for $D \in \{2, \ldots, 10\}$ are later presented in Table 1.

We are not aware of other lower bounds for classical die-rolling protocols apart from those implied by Kitaev's bounds above. We see that sometimes classical protocols can be optimal, for example when D is square. We now consider how to design (simple) quantum protocols and see what levels of security they can offer.

1.2 Simple quantum protocols

Many of the best known explicit protocols for strong coin-flipping are based on the idea of *bit-commitment* [4, 20, 13, 19]. Optimal protocols are known for bit-commitment as well [9], but are again based on weak coin-flipping and are thus very complicated.

In this paper, we generalize the above simple, explicit protocols such that Alice commits to an *integer* instead of a bit. More precisely, our quantum protocols have the following form.

▶ **Protocol 3** (Quantum protocol). *A quantum die-rolling protocol based on the idea of integer-commitment, denoted here as* DRIC*, is defined as follows:*

▬ *Alice chooses a random $a \in [D]$ and creates the state*

$$|\psi_a\rangle \in \mathcal{A} \otimes \mathcal{B}$$

and sends the subsystem $\mathcal{B}$ to Bob.

▬ *Bob sends a uniformly random $b \in [D]$ to Alice.*

▬ *Alice reveals a to Bob and sends him the subsystem $\mathcal{A}$.*

▬ *Bob checks if $\mathcal{A} \otimes \mathcal{B}$ is in state $|\psi_a\rangle$ using the measurement*

$$\{\Pi_a := |\psi_a\rangle \langle \psi_a|, \Pi_{abort} := I - \Pi_a\}.$$

Bob accepts/rejects a based on his measurement outcome.

▬ *If Bob does not abort, Alice and Bob output*

$$d := (a + b) \bmod D + 1 \in [D].$$

The special case of $D = 2$ yields the structure of the simple, explicit coin-flipping protocols mentioned above. Indeed, these protocols are very easy to describe, one needs only the knowledge of the D states $|\psi_a\rangle$ and, implicitly, the systems they act on, $\mathcal{A}$ and $\mathcal{B}$.

We start by formulating the cheating probabilities of a DRIC-protocol using semidefinite programming. Once we have established the semidefinite programming cheating strategy formulations, we are able to analyze the security of DRIC-protocols. Furthermore, we are able to analyze *modifications* to such protocols and the corresponding changes in security.

In this paper, we present a DRIC-protocol with near-optimal security. We develop this protocol in several steps described below.

The first step is to start with a protocol with decent security. To do this, we show how to create a DRIC-protocol with the same cheating probabilities as Protocol 1.

▶ **Proposition 4.** *There exists a* DRIC*-protocol with the same cheating probabilities as in Protocol 1.*

The second step is to give a process which (approximately) balances the maximum cheating probabilities of Alice and Bob. We accomplish this by modifying the protocol in order to decrease the overall maximum cheating probability (while possibly increasing lesser cheating probabilities).

▶ **Proposition 5.** *If there exists a* DRIC*-protocol with cheating probabilities $P_A^* = \alpha$ and $P_B^* = \beta$, then there exists a* DRIC*-protocol with maximum cheating probability*

$$\max\{P_A^*, P_B^*\} \leq \frac{D \max\{\beta, \alpha\} - \min\{\beta, \alpha\}}{D|\beta - \alpha| + D - 1} \leq \max\{\beta, \alpha\}.$$

Moreover, the last inequality is strict when $\alpha \neq \beta$ yielding a strictly better protocol.

■ **Table 1** Values of our bounds (as truncated percentages) for various protocols and values of D. We see that the quantum protocol performs very well, even for D as small as 3.

D	2	3	4	5	6	7	8	9	10
Explicit Protocol in [1]	75%	66%	62%	60%	58%	57%	56%	55%	55%
Our Classical Protocol	100%	66%	50%	50%	50%	42%	37%	33%	33%
Our Quantum Protocol	75%	60%	50%	46%	44%	40%	36%	33%	32%
Kitaev's lower bound	70%	57%	50%	44%	40%	37%	35%	33%	31%

By combining the above two propositions, we are able to obtain the main result of this paper.

▶ **Theorem 6.** *For any $D \geq 2$, there exists a (quantum)* DRIC*-protocol satisfying*

$$\frac{1}{\sqrt{D}} \leq \max\{P_A^*, P_B^*\} \leq \min\left\{\frac{D + \lfloor\sqrt{D}\rfloor}{D(\lfloor\sqrt{D}\rfloor + 1)}, \frac{1 + \lceil\sqrt{D}\rceil}{D + \lceil\sqrt{D}\rceil}\right\}$$

which is strictly better than Protocol 1 when D is not square.

Since $\min\left\{\dfrac{D + \lfloor\sqrt{D}\rfloor}{D(\lfloor\sqrt{D}\rfloor + 1)}, \dfrac{1 + \lceil\sqrt{D}\rceil}{D + \lceil\sqrt{D}\rceil}\right\} \approx \dfrac{1}{\sqrt{D}}$ for large D, this bound is very close to optimal. To compare numbers, we list the values for $D \in \{2, \dots, 10\}$, below.

1.3 Related literature

Quantum protocols for a closely related cryptographic task known as string-commitment have been considered [12, 21, 22, 7, 11]. Technically, this is the case of integer-commitment when $D = 2^n$ (if the string has n bits). It is worth noting that the quantum protocols considered in this paper are quite similar, but the security definitions are very different. Roughly speaking, the references above are concerned with quantum protocols where Alice is able to "cheat" on a bits and Bob is able to "learn" b bits of information about the n bit string. Multiple protocols and security trade-offs are given in the above references.

The use of semidefinite programming has been very valuable in the study of quantum cryptographic protocols, see for example [14, 15, 16, 10, 18, 19]. Roughly speaking, if one is able to formulate cheating probabilities as semidefinite programs, then the problem of analyzing cryptographic security can be translated into a concrete mathematical problem. Moreover, one then has the entire theory of semidefinite programming at their disposal. This is the approach taken in this work, to shine new light on a cryptographic task using the lens of semidefinite programming.

2 Semidefinite programming cheating strategy formulations

In this section, we use the theory of semidefinite programming to formulate Alice and Bob's maximum cheating probabilities for a DRIC-protocol. The formulations in this section are a generalization of those for bit-commitment, see [19] and the references therein for details about this special case.

2.1 Semidefinite programming

Semidefinite programming is the theory of optimizing a linear function over a positive semidefinite matrix variable subject to finitely many affine constraints. A semidefinite

program (SDP) can be written in the following form without loss of generality:

$$p^* := \sup\{\langle C, X \rangle : \mathcal{A}(X) = B, \ X \succeq 0\} \tag{2}$$

where $\mathcal{A}$ is a linear transformation, C and B are Hermitian, and $X \succeq Y$ means that $X - Y$ is (Hermitian) positive semidefinite.

Associated with every SDP is a dual SDP:

$$d^* := \inf\{\langle B, Y \rangle : \mathcal{A}^*(Y) = C + S, \ S \succeq 0, \ Y \text{ is Hermitian}\} \tag{3}$$

where $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$.

We refer to the optimization problem (2) as the *primal* or *primal SDP* and to the optimization problem (3) as the *dual* or *dual SDP*. We say that the primal is *feasible* if there exists an X satisfying the (primal) constraints

$$\mathcal{A}(X) = B \quad \text{and} \quad X \succeq 0$$

and we say the dual is *feasible* if there exists (Y, S) satisfying the (dual) constraints

$$\mathcal{A}^*(Y) = C + S, \quad S \succeq 0, \quad \text{and} \quad Y \text{ is Hermitian.}$$

If further we have X positive definite, then the primal is said to be *strictly* feasible. If further we have S positive definite, then the dual is said to be *strictly* feasible.

Semidefinite programming has a rich and powerful duality theory. In particular, we use the following:

Weak duality: If the primal and dual are both feasible, then $p^* \leq d^*$.

Strong duality: If the primal and dual are both *strictly* feasible, then $p^* = d^*$ *and both attain an optimal solution.*

For more information about semidefinite programming and its duality theory, the reader is referred to [6].

2.2   Cheating strategy formulations

To study a fixed DRIC-protocol, it is convenient to define the following reduced states

$$\rho_a := \text{Tr}_{\mathcal{A}}(|\psi_a\rangle \langle \psi_a|)$$

for all $a \in [D]$. We show that they appear in both the case of cheating Alice and cheating Bob.

2.2.1   Cheating Bob

To see how Bob can cheat, notice that he only has one message he sends to Alice. Thus, he must send $b \in [D]$ to force the outcome he wishes. For example, if he wishes to force the outcome d, he would send b such that $d = (a + b) \bmod D + 1$. Therefore, he must extract the value of a from $\mathcal{B}$ to accomplish this. Suppose he measures $\mathcal{B}$ with the measurement

$$\{M_1, \ldots, M_D\}$$

where the outcome of the measurement corresponds to Bob's guess for a. If Alice chose $a \in [D]$, he succeeds in cheating if his guess is correct, which happens with probability

$$\langle M_a, \rho_a \rangle.$$

Since the choice of Alice's integer a is uniformly random, we can calculate Bob's optimal cheating probability as

$$P_B^* = \max \left\{ \frac{1}{D} \sum_{a \in [D]} \langle M_a, \rho_a \rangle : \sum_{a \in [D]} M_a = I_\mathcal{B}, \, M_a \succeq 0, \forall a \in [D] \right\} \qquad (4)$$

noting that the variables being optimized over correspond to a POVM measurement. Note that the maximum is attained since the set of feasible $(M_1, \ldots, M_D)$ forms a compact set.

Now that Bob's optimal cheating probability is stated in terms of an SDP, we can examine its dual as shown in the lemma below. Note that the lemma below follows from strong duality (details in the full version).

▶ **Lemma 7.** *For any* DRIC-*protocol, we have*

$$P_B^* = \min \left\{ \mathrm{Tr}(X) : X \succeq \frac{1}{D} \rho_a, \forall a \in [D] \right\}. \qquad (5)$$

We refer to the optimization problem (4) as *Bob's primal SDP* and to the optimization problem (5) as *Bob's dual SDP*. The utility of having dual SDP formulations is that any feasible solution yields an *upper bound* on the maximum cheating probability. Proving upper bounds on cheating probabilities would otherwise be a very hard task.

2.2.2 Cheating Alice

If Alice wishes to force Bob to accept outcome $d \in [D]$, she must convince him that the state in $\mathcal{A} \otimes \mathcal{B}$ is indeed $|\psi_a\rangle$ where a is such that $d = (a + b) \bmod D + 1$. Note that this choice of a is determined after learning b from Bob, which occurs with uniform probability.

To quantify the extent to which Alice can cheat, we examine the states Bob has during the protocol. We know that Bob measures and accepts a with the measurement operator $\Pi_a := |\psi_a\rangle \langle \psi_a|$. Let $(a, \mathcal{A})$ be Alice's last message. Then Bob's state at the end of the protocol is given by a density operator σ_a acting on $\mathcal{A} \otimes \mathcal{B}$ which is accepted with probability $\langle \sigma_a, |\psi_a\rangle \langle \psi_a| \rangle$. Note that Alice's first message $\mathcal{B}$ is in state $\sigma := \mathrm{Tr}_\mathcal{A}(\sigma_a)$ which is independent of a (since Alice's first message does not depend on a when she cheats). Thus, the states under Bob's control are subject to the constraints

$$\mathrm{Tr}_\mathcal{A}(\sigma_a) = \sigma, \, \forall a \in [D], \quad \mathrm{Tr}(\sigma) = 1, \quad \sigma, \sigma_1 \ldots, \sigma_D \succeq 0. \qquad (6)$$

(Note that $\mathrm{Tr}(\sigma_a) = 1$, for all $a \in [D]$, is implied by the constraints above, and is thus omitted.) On the other hand, if Alice maintains a purification of the states above, then using Uhlmann's Theorem [23] she can prepare any set of states satisfying conditions (6).

Thus, we have

$$P_A^* = \max \left\{ \frac{1}{D} \sum_{a \in [D]} \langle \sigma_a, |\psi_a\rangle \langle \psi_a| \rangle : \mathrm{Tr}_\mathcal{A}(\sigma_a) = \sigma, \, \forall a \in [D], \, \mathrm{Tr}(\sigma) = 1, \, \sigma, \sigma_1 \ldots, \sigma_D \succeq 0 \right\}. \qquad (7)$$

Again, since the set of feasible $(\sigma, \sigma_1, \ldots, \sigma_D)$ is compact, the above SDP attains an optimal solution.

Similar to the case of cheating Bob, we can view the dual of Alice's cheating SDP above as shown in the lemma below. Again, the lemma below follows by strong duality (details in the full version).

▶ **Lemma 8.** *For any* DRIC-*protocol, we have*

$$P_A^* = \min \left\{ s : sI_{\mathcal{B}} \succeq \sum_{a \in [D]} Z_a, \ I_{\mathcal{A}} \otimes Z_a \succeq \frac{1}{D} |\psi_a\rangle \langle \psi_a|, \ \forall a \in [D], \ Z_a \text{ is Hermitian} \right\}. \quad (8)$$

We refer to the optimization problem (7) as *Alice's primal SDP* and the optimization problem (8) as *Alice's dual SDP*.

Note that every solution feasible in Alice's dual SDP has Z_a being positive semidefinite, for all $a \in [D]$. We can further assume that each Z_a is positive definite if we sacrifice the attainment of an optimal solution. This is because we can take an optimal solution $(s, Z_1, \ldots, Z_D)$ and consider $(s + \varepsilon D, Z_1 + \varepsilon I_{\mathcal{B}}, \ldots, Z_D + \varepsilon I_{\mathcal{B}})$ which is also feasible for any $\varepsilon > 0$, and $s + \varepsilon D$ approaches $s = P_A^*$ as ε decreases to 0.

Next, we use an analysis similar to one found in [15] and [24] to simplify the constraint

$$I_{\mathcal{A}} \otimes Z_a \succeq |\psi_a\rangle \langle \psi_a|$$

when Z_a is positive definite. Since $X \to ZXZ^{-1}$ is an automorphism of the set of positive semidefinite matrices for any fixed positive definite Z, we have

$$I_{\mathcal{A}} \otimes Z_a \succeq \frac{1}{D} |\psi_a\rangle \langle \psi_a| \iff I_{\mathcal{A} \otimes \mathcal{B}} \succeq (I_{\mathcal{A}} \otimes Z_a^{-1/2}) \left(\frac{1}{D} |\psi_a\rangle \langle \psi_a| \right) (I_{\mathcal{A}} \otimes Z_a^{-1/2}). \quad (9)$$

Note that since the quantity on the right is positive semidefinite with rank at most 1, its largest eigenvalue is equal to its trace which is equal to

$$\frac{1}{D} \langle I_{\mathcal{A}} \otimes Z_a^{-1}, |\psi_a\rangle \langle \psi_a| \rangle = \frac{1}{D} \langle Z_a^{-1}, \text{Tr}_{\mathcal{A}}(|\psi_a\rangle \langle \psi_a|) \rangle = \frac{1}{D} \langle Z_a^{-1}, \rho_a \rangle.$$

Thus, we can rewrite (9) as

$$I_{\mathcal{A}} \otimes Z_a \succeq \frac{1}{D} |\psi_a\rangle \langle \psi_a| \iff \frac{1}{D} \langle Z_a^{-1}, \rho_a \rangle \leq 1 \iff \langle Z_a^{-1}, \rho_a \rangle \leq D.$$

Therefore, we have the following lemma.

▶ **Lemma 9.** *For any* DRIC-*protocol, we have*

$$P_A^* = \inf \left\{ s : sI_{\mathcal{B}} \succeq \sum_{a \in [D]} Z_a, \ \langle Z_a^{-1}, \rho_a \rangle \leq D, \forall a \in [D], \ Z_a \text{ is positive definite}, \ \forall a \in [D] \right\}.$$
$$\quad (10)$$

We also refer to the optimization problem (10) as Alice's dual SDP and we distinguish them by equation number.

3 Finding a decent DRIC-protocol

In this section, we exhibit a DRIC-protocol which has the same cheating probabilities as Protocol 1:

$$P_B^* = \frac{1}{m} \quad \text{and} \quad P_A^* = \frac{m}{D}.$$

To do this, define T_m to be the subsets of $[D]$ of cardinality m and note that $|T_m| = \binom{D}{m}$. Consider the following states

$$|\psi_a\rangle := \frac{1}{\sqrt{\binom{D-1}{m-1}}} \sum_{S \in T_m \,:\, a \in S} |S\rangle\,|S\rangle \in \mathcal{A} \otimes \mathcal{B},$$

for $a \in [D]$, where $\mathcal{A} = \mathcal{B} = \mathbb{C}^{|T_m|}$. Notice that

$$\rho_a := \mathrm{Tr}_\mathcal{A}\left(|\psi_a\rangle\,\langle\psi_a|\right) = \frac{1}{\binom{D-1}{m-1}} \sum_{S \in T_m \,:\, a \in S} |S\rangle\,\langle S|.$$

We now use the cheating SDPs developed in the previous section to analyze the cheating probabilities of this protocol.

3.1 Cheating Bob

To prove that Bob can cheat with probability at least $1/m$, suppose he measures his message from Alice in the computational basis. He then obtains a random subset $S \in T_m$ such that $a \in S$. He then guesses which integer is a and responds with the appropriate choice for b to get his desired outcome. He succeeds if and only if his guess for a (from the m choices in S) is correct. This strategy succeeds with probability $1/m$. Thus, $P_B^* \geq 1/m$.

To prove Bob cannot cheat with probability greater than $1/m$, notice that

$$X = \frac{1}{D\binom{D-1}{m-1}} I_\mathcal{B}$$

satisfies

$$X \succeq \frac{1}{D}\rho_a, \ \forall a \in [D],$$

and thus is feasible in Bob's dual (5). Therefore, $P_B^* \leq \mathrm{Tr}(X) = 1/m$, as desired.

3.2 Cheating Alice

Alice can cheat by creating the maximally entangled state

$$|T_m\rangle := \frac{1}{\sqrt{|T_m|}} \sum_{S \in T_m} |S\rangle\,|S\rangle \in \mathcal{A} \otimes \mathcal{B}$$

and sending $\mathcal{B}$ to Bob. After learning b, she sends a such that $(a + b) \bmod D + 1$ is her desired outcome. She also sends $\mathcal{A}$ to Bob (without altering it in any way). Thus, her cheating probability is precisely the probability of her passing Bob's cheat detection which is

$$\langle \Pi_a, |T_m\rangle\,\langle T_m| \rangle = \langle |\psi_a\rangle\,\langle\psi_a|, |T_m\rangle\,\langle T_m| \rangle = |\langle T_m|\psi_a\rangle|^2 = \frac{m}{D}.$$

Therefore, this cheating strategy succeeds with probability m/D, proving $P_A^* \geq m/D$.

To prove this strategy is optimal, we use Alice's dual (10). Define

$$Z_a := \frac{1}{D} \sum_{S \in T_m \,:\, a \in S} |S\rangle\,\langle S| + \varepsilon \sum_{S \in T_m \,:\, a \notin S} |S\rangle\,\langle S|$$

where ε is a small positive constant. Z_a is invertible and we can write

$$Z_a^{-1} := D \sum_{S \in T_m \ : \ a \in S} |S\rangle \langle S| + \frac{1}{\varepsilon} \sum_{S \in T_m \ : \ a \notin S} |S\rangle \langle S|.$$

We see that each Z_a satisfies $\langle Z_a^{-1}, \rho_a \rangle = D$, for all $a \in [D]$. Also,

$$Z_a \preceq \frac{1}{D} \sum_{S \in T_m \ : \ a \in S} |S\rangle \langle S| + \varepsilon I_{\mathcal{B}}$$

thus

$$\sum_{a \in [D]} Z_a \preceq \frac{1}{D} \sum_{a \in [D]} \sum_{S \in T_m \ : \ a \in S} |S\rangle \langle S| + \varepsilon D \, I_{\mathcal{B}} = \left(\frac{m}{D} + \varepsilon D \right) I_{\mathcal{B}}.$$

Thus, $s = \dfrac{m}{D} + \varepsilon D$ satisfies

$$s \, I_{\mathcal{B}} \succeq \sum_{a \in [D]} Z_a$$

proving $P_A^* \leq s = \dfrac{m}{D} + \varepsilon D$, for all $\varepsilon > 0$. Therefore, $P_A^* = m/D$, as desired.

4   Balancing Alice and Bob's cheating probabilities

This section is comprised of two parts. We first focus on reducing Bob's cheating probabilities, then Alice's.

4.1   Building new protocols that reduce Bob's cheating

We start with a lemma.

▶ **Lemma 10.** *If there exists a* DRIC-*protocol with cheating probabilities* $P_A^* = \alpha$ *and* $P_B^* = \beta$, *then there exists another* DRIC-*protocol with cheating probabilities* $P_A^* = \alpha'$ *and* $P_B^* = \beta'$ *where*

$$\beta' \leq (1 - t)\beta + \frac{t}{D} \quad \text{and} \quad \alpha' \leq (1 - t)\alpha + t$$

for any $t \in (0, 1)$.

We sketch the proof here. Fix a DRIC-protocol with cheating probabilities $P_A^* = \alpha$ and $P_B^* = \beta$ defined by the states $|\psi_a\rangle \in \mathcal{A} \otimes \mathcal{B}$, for $a \in [D]$. Extend each of the Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$ by another basis vector $|\bot\rangle$ and denote these Hilbert spaces by $\mathcal{A}'$ and $\mathcal{B}'$, respectively. In short, $\mathcal{A}' := \mathcal{A} \oplus \text{span}\{|\bot\rangle\}$ and $\mathcal{B}' := \mathcal{B} \oplus \text{span}\{|\bot\rangle\}$. Note that

$$\langle \bot, \bot | \psi_a \rangle = 0, \quad \text{for all } a \in [D].$$

We now analyze the cheating probabilities of Alice and Bob in the new DRIC-protocol defined by the states

$$|\psi_a'\rangle := \sqrt{1 - t} \, |\psi_a\rangle + \sqrt{t} \, |\bot, \bot\rangle \in \mathcal{A}' \otimes \mathcal{B}', \quad \text{for all } a \in [D]$$

as a function of $t \in (0, 1)$. For this, note that

$$\rho_a' := \text{Tr}_{\mathcal{A}} \left(|\psi_a'\rangle \langle \psi_a'| \right) = (1 - t) \, \rho_a + t \, |\bot\rangle \langle \bot|,$$

where $\rho_a := \text{Tr}_{\mathcal{A}} \left(|\psi_a\rangle \langle \psi_a| \right)$.

To show how Bob's cheating probability changes, consider an optimal solution X to Bob's dual SDP (5) corresponding to the original protocol. Then one can show that

$$X' := (1-t)X + \frac{t}{D}\left|\bot\right\rangle\left\langle\bot\right|$$

is feasible in Bob's dual SDP after the protocol has been modified. This proves that

$$P_B^* \leq (1-t)\beta + t/D$$

for the new protocol.

Concerning cheating Alice, let $(s, Z_1, \ldots, Z_D)$ be a feasible solution for Alice's dual (10) for the original protocol. Then one can show that

$$
\begin{aligned}
s' &:= s(1-t) + t \\
Z_1' &:= ((1-t) + t/s)\, Z_1 + \left(\frac{s(1-t) + t}{D}\right)\left|\bot\right\rangle\left\langle\bot\right| \\
&\ \ \vdots \\
Z_D' &:= ((1-t) + t/s)\, Z_D + \left(\frac{s(1-t) + t}{D}\right)\left|\bot\right\rangle\left\langle\bot\right|
\end{aligned}
$$

is feasible for Alice's dual for the new protocol. Thus,

$$P_A^* \leq s' = s(1-t) + t$$

and since s can be taken arbitrarily close to α, the result follows.

Intuitively, Alice can cheat more if the states ρ_a are "close" to each other and Bob can cheat more if they are "far apart". What this protocol modification does is make all the states closer together to increase Alice's cheating probability but to decrease Bob's.

Note that this lemma is useful when $\beta > \alpha$. In this case, one can choose

$$t = \frac{\beta - \alpha}{(1 - 1/D) + (\beta - \alpha)} \in (0, 1)$$

to equate the upper bounds. If $\alpha > \beta$, then no choice of $t \in (0, 1)$ will make the two upper bounds in Lemma 10 equal. We summarize in the following corollary.

▶ **Corollary 11.** *If there exists a* DRIC-*protocol with cheating probabilities* $P_A^* = \alpha$ *and* $P_B^* = \beta$, *with* $\beta > \alpha$, *then there exists another* DRIC-*protocol with maximum cheating probability*

$$\max\{P_A^*, P_B^*\} \leq \frac{D\beta - \alpha}{D\beta - D\alpha + D - 1} < \beta.$$

4.2 Building new protocols that reduce Alice's cheating

In this subsection, we show how to reduce Alice's cheating probabilities in a DRIC-protocol.

▶ **Lemma 12.** *If there exists a* DRIC-*protocol with cheating probabilities* $P_A^* = \alpha$ *and* $P_B^* = \beta$, *then there exists another* DRIC-*protocol with cheating probabilities* $P_A^* = \alpha'$ *and* $P_B^* = \beta'$ *where*

$$\beta' \leq (1-t)\beta + t \quad \text{and} \quad \alpha' \leq (1-t)\alpha + \frac{t}{D},$$

for $t \in (0, 1)$.

We sketch the proof here. Fix a DRIC-protocol with cheating probabilities $P_A^* = \alpha$ and $P_B^* = \beta$ defined by the states $|\psi_a\rangle \in \mathcal{A} \otimes \mathcal{B}$, for $a \in [D]$. Extend each of the Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$ by the set of orthogonal basis vectors $\{|\bot_a\rangle : a \in [D]\}$, and denote these new Hilbert spaces by $\mathcal{A}'$ and $\mathcal{B}'$, respectively. In other words,

$$\mathcal{A}' := \mathcal{A} \oplus \text{span}\{|\bot_1\rangle, \ldots, |\bot_D\rangle\} \quad \text{and} \quad \mathcal{B}' := \mathcal{B} \oplus \text{span}\{|\bot_1\rangle, \ldots, |\bot_D\rangle\}.$$

Note that

$$\langle \bot_{a''}, \bot_{a'} | \psi_a \rangle = 0, \quad \text{for all } a, a', a'' \in [D].$$

Again, we analyze the cheating probabilities of Alice and Bob in the new DRIC-protocol defined by the states

$$|\psi_a'\rangle := \sqrt{1-t}\,|\psi_a\rangle + \sqrt{t}\,|\bot_a\rangle\,|\bot_a\rangle \in \mathcal{A}' \otimes \mathcal{B}'$$

for $a \in [D]$. The reduced states are

$$\rho_a' := (1-t)\,\rho_a + t\,|\bot_a\rangle\,\langle\bot_a|$$

for $a \in [D]$, recalling that $\rho_a := \text{Tr}_{\mathcal{A}}(|\psi_a\rangle\,\langle\psi_a|)$. We now analyze the cheating probabilities of this new protocol as a function of $t \in (0,1)$.

To show how Bob's cheating probability changes, we can use a similar argument. Consider an optimal solution X to Bob's dual (5) for the original protocol. Then one can show that

$$X' := (1-t)X + \frac{t}{D} \sum_{a \in [D]} |\bot_a\rangle\,\langle\bot_a|$$

is feasible for Bob's dual for the modified protocol. This shows that

$$P_B^* \leq (1-t)\beta + t.$$

Concerning cheating Alice, let $(s, Z_1, \ldots, Z_D)$ be a feasible solution for Alice's dual (10) for the original protocol. Then one can show that

$$
\begin{aligned}
s' &:= (1-t)s + t/D + \zeta(D-1) \\
Z_1' &:= \left((1-t) + \frac{t}{Ds}\right)Z_1 + \left((1-t)s + \frac{t}{D}\right)|\bot_1\rangle\,\langle\bot_1| + \zeta \sum_{c \in [D], c \neq 1} |\bot_c\rangle\,\langle\bot_c| \\
&\ \ \vdots \\
Z_D' &:= \left((1-t) + \frac{t}{Ds}\right)Z_D + \left((1-t)s + \frac{t}{D}\right)|\bot_D\rangle\,\langle\bot_D| + \zeta \sum_{c \in [D], c \neq D} |\bot_c\rangle\,\langle\bot_c|
\end{aligned}
$$

is feasible for Alice's dual for the new protocol for $\zeta > 0$ a small constant. Thus,

$$P_A^* \leq (1-t)s + t/D$$

and since s can be taken arbitrarily close to α, the result follows.

Intuitively, this protocol modification works in the opposite manner of the last. Here, we are making the states farther apart as to decrease Alice's cheating at the expense of increasing Bob's.

As opposed to Lemma 10, the above lemma is useful when $\alpha > \beta$. Similarly, if $\beta > \alpha$, then no choice of $t \in (0,1)$ will make the two upper bounds in Lemma 12 equal.

By symmetry, we have the following corollary.

▶ **Corollary 13.** *If there exists a* DRIC-*protocol with cheating probabilities* $P_A^* = \alpha$ *and* $P_B^* = \beta$, *with* $\alpha > \beta$, *then there exists another* DRIC-*protocol with maximum cheating probability*

$$\max\{P_A^*, P_B^*\} \leq \frac{D\alpha - \beta}{D\alpha - D\beta + D - 1} < \alpha.$$

Note that if $\alpha = \beta$, the quantity $\frac{D\alpha - \beta}{D\alpha - D\beta + D - 1}$ is equal to $\alpha(= \beta)$. Thus, we still have

$$\max\{P_A^*, P_B^*\} \leq \frac{D\alpha - \beta}{D\alpha - D\beta + D - 1}$$

holding, although no protocol modification is necessary. Therefore, Proposition 5 now follows from combining Corollaries 11 and 13 and the comment above.

──── **References** ────

1 N. Aharon and J. Silman. Quantum dice rolling: a multi-outcome generalization of quantum coin flipping. *New Journal of Physics*, 12(3):033027, 2010.

2 Dorit Aharonov, André Chailloux, Maor Ganz, Iordanis Kerenidis, and Loïck Magnin. A simpler proof of existence of quantum weak coin flipping with arbitrarily small bias. *SIAM Journal of Computing*, to appear, 2016.

3 Dorit Aharonov, Amnon Ta-Shma, Umesh Vazirani, and Andrew Chi-Chih Yao. Quantum bit escrow. In *Proceedings of 32nd Annual ACM Symposium on the Theory of Computing*, pages 705–714. ACM, 2000. `doi:10.1145/335305.335404`.

4 Andris Ambainis. A new protocol and lower bounds for quantum coin flipping. In *Proceedings of 33rd Annual ACM Symposium on the Theory of Computing*, pages 134–142. ACM, 2001. `doi:10.1109/FOCS.2004.13`.

5 Manuel Blum. Coin flipping by telephone. In Allen Gersho, editor, *Advances in Cryptology: A Report on CRYPTO 81, CRYPTO 81, IEEE Workshop on Communications Security, Santa Barbara, California, USA, August 24-26, 1981*, pages 11–15. U. C. Santa Barbara, Dept. of Elec. and Computer Eng., ECE Report No. 82-04, 1982, 1981.

6 Stephen Boyd and Lieven Vandenberghe. *Convex Optimization.* Cambridge University Press, 2004.

7 Harry Buhrman, Matthias Christandl, Patrick Hayden, Hoi-Kwong Lo, and Stephanie Wehner. Possibility, impossibility, and cheat-sensitivity of quantum bit string commitment. *Phys. Rev. A*, 78:022316, 2008.

8 André Chailloux and Iordanis Kerenidis. Optimal quantum strong coin flipping. In *Proceedings of 50th IEEE Symposium on Foundations of Computer Science*, pages 527–533. IEEE Computer Society, 2009.

9 André Chailloux and Iordanis Kerenidis. Optimal bounds for quantum bit commitment. In *Proceedings of the 52nd Annual IEEE Symposium on Foundations of Computer Science*, pages 354–362. IEEE Computer Society, 2011. `doi:10.1109/FOCS.2011.42`.

10 André Chailloux, Iordanis Kerenidis, and Jamie Sikora. Lower bounds for quantum oblivious transfer. *Quantum Information & Computation*, 13(1 & 2):158–177, 2013.

11 Rahul Jain. New binding-concealing trade-offs for quantum string commitment. *Journal of Cryptology*, 21:579–592, 2008.

12 Adrian Kent. Quantum bit string commitment. *Phys. Rev. Lett.*, 90:237901, 2003.

13 Iordanis Kerenidis and Ashwin Nayak. Weak coin flipping with small bias. *Information Processing Letters*, 89(3):131–135, 2004. `doi:10.1016/j.ipl.2003.07.007`.

14 Alexei Kitaev. Quantum coin-flipping. Unpublished result. Talk at the 6th Annual workshop on Quantum Information Processing (QIP 2003), 2002.

15 Carlos Mochon. A large family of quantum weak coin-flipping protocols. *Physical Review A*, 72:022341, 2005. URL: `http://arxiv.org/abs/quant-ph/0502068`, `doi:10.1103/PhysRevA.72.022341`.

16 Carlos Mochon. Quantum weak coin flipping with arbitrarily small bias. Available as arXiv.org e-Print quant-ph/0711.4114, 2007.

17 Ashwin Nayak and Peter W. Shor. Bit-commitment based quantum coin flipping. *Physical Review A*, 67:012304, 2003. `doi:10.1103/PhysRevA.67.012304`.

18 Ashwin Nayak, Jamie Sikora, and Levent Tunçel. Quantum and classical coin-flipping protocols based on bit-commitment and their point games. Available as arXiv.org e-Print quant-ph/1504.04217, 2015.

19 Ashwin Nayak, Jamie Sikora, and Levent Tunçel. A search for quantum coin-flipping protocols using optimization techniques. *Mathematical Programming*, 156(1):581–613, 2016.

20 Robert W. Spekkens and Terence Rudolph. Degrees of concealment and bindingness in quantum bit commitment protocols. *Physical Review A*, 65:012310, 2001. `doi:10.1103/PhysRevA.65.012310`.

21 Toyohiro Tsurumaru. Implementable quantum bit-string commitment protocol. *Phys. Rev. A*, 71:012313, 2005.

22 Toyohiro Tsurumaru. Group covariant protocols for quantum string commitment. *Phys. Rev. A*, 74:042307, 2006.

23 A. Uhlmann. The "transition probability" in the state space of a *-algebra. *Reports on Mathematical Physics*, 9(2):273–279, 1976.

24 John Watrous. Semidefinite programs for completely bounded norms. *Theory of Computing*, 5:217–238, 2009.