Proving Weak Approximability Without Algorithms*

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Abstract

A predicate $f : \{-1,1\}^k \mapsto \{0,1\}$ with $\rho(f) = \mathbb{E}_{x\in\{-1,1\}^k}[f(x)]$ is said to be *strongly approximation resistant* if, for every $\varepsilon > 0$, given a near-satisfiable instance of $\mathsf{MAX\ k\text{-}CSP}(f)$, it is hard to find an assignment such that the fraction of constraints satisfied is outside the range $[\rho(f)-\varepsilon, \rho(f)+\varepsilon]$. A predicate which is not strongly approximation resistant is known as *weakly approximable*.

We give a new method for proving the weak approximability of predicates, using a simple SDP relaxation, without designing and analyzing new rounding algorithms for each predicate. Instead, we use the recent characterization of strong approximation resistance by Khot et. al [13], and show how to prove that for a given predicate f, certain necessary conditions for strong resistance derived from their characterization, are violated. By their result, this implies the existence of a good rounding algorithm, proving weak approximability.

We show how this method can be used to obtain simple proofs of (weak approximability analogues of) various known results on approximability, as well as new results on weak approximability of symmetric predicates.

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1 Introduction

Constraint Satisfaction Problems (CSPs) are some of the most basic problems in the study of approximation algorithms and inapproximability. The problem $\mathsf{MAX\ k\text{-}CSP}(f)$ is characterized by a Boolean predicate $f : \{-1,1\}^k \to \{0,1\}$. An instance of the problem is described by (say) n variables $x_1,\ldots,x_n$ taking values in $\{-1,1\}$, and a set of (say) m constraints where each constraint C_i is of the form $C_i \equiv f(x_{i_1}\cdot b_{i_1},\ldots,x_{i_k}\cdot b_{i_k})$ for some $b_{i_1},\ldots,b_{i_k} \in \{-1,1\}$. An assignment $\sigma : \{x_1,\ldots,x_n\} \to \{-1,1\}$ is said to satisfy the constraint C_i if $f(\sigma(x_{i_1})\cdot b_{i_1},\ldots,\sigma(x_{i_k})\cdot b_{i_k}) = 1$. Given an instance of the problem, the goal is to find an assignment satisfying the maximum possible number of constraints. For a given instance Φ, we denote the *fraction* of constraints satisfied by the optimal assignment as $\mathsf{OPT}(\Phi)$. An algorithm is said to achieve an approximation factor α if it always produces an assignment satisfying at least $\alpha\cdot\mathsf{OPT}(\Phi)$ fraction of constraints.

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Given an instance of MAX k-CSP, a trivial algorithm is to assign independently to each variable x_i a random value in $\{-1, 1\}$. This satisfies a fraction of constraints concentrated around the quantity $\rho(f) = \mathbb{E}_{x \in \{-1,1\}^k}[f(x)]$. A predicate for which this approximation is best possible i.e. for every $\varepsilon > 0$, given an instance with $\mathsf{OPT} \geq 1 - \varepsilon$ it is (NP/UG-) hard find an assignment satisfying $\rho(f) + \varepsilon$ fraction of constraints, is known as approximation resistant. An even stronger notion of hardness, which was implicit in the literature on hardness of approximation, and was explicitly defined by Khot et. al. [13], is known as strong approximation resistance. A predicate is said to be strongly approximation resistant, if it is hard to find an assignment which *significantly deviates* from a random assignment i.e. for every $\varepsilon > 0$, given an instance with $\mathsf{OPT}(\Phi) \geq 1 - \varepsilon$, it is (NP/UG-) hard to find an assignment satisfying a fraction of constraints outside the interval $[\rho(f) - \varepsilon, \rho(f) + \varepsilon]$. A predicate which is not approximation resistant is known as approximable, and one which is not strongly resistant is known as weakly approximable. Note that for an odd predicate i.e.a predicate satisfying $f(x) = 1 - f(-x)\ \forall x$, the notions of approximability and weak approximability are equivalent.

The notion of approximation resistance has been extensively studied, starting from the celebrated result of Håstad [9] showing that MAX 3-SAT and MAX 3-XOR are approximation resistant. Since then, many predicates have been shown to be approximation resistant (see e.g. [7, 16, 11, 4], all proving NP-hardness). Recently, a remarkable result by Chan [2] proved the approximation resistance of the Hypergraph Linearity Predicate (he shows NP-hardness whereas UG-hardness was shown earlier in [17]).

Assuming the Unique Games Conjecture (UGC) of Khot [12], Austrin and Mossel [1] show that any predicate f for which $f^{-1}(1)$ supports a balanced and pairwise independent distribution on $\{-1, 1\}^k$, is approximation resistant. In addition to the above results, a very general result by Raghavendra [15] also shows that assuming the UGC, the best possible approximation for any problem of the form MAX k-CSP(f) (for any f) can be obtained by a certain Semidefinite Programming (SDP) relaxation (see Fig. 1) known as the basic SDP. Thus, assuming the UGC, a predicate is approximation resistant if and only if one cannot do better than the trivial algorithm, using the basic SDP. All of the above results on approximation resistance in fact prove that the predicates in question are *strongly resistant*[1].

For the case of approximability, it follows from the algorithm of Goemans and Williamson [5] (and was shown by Håstad for every alphabet size [10]) that every predicate on 2 inputs is approximable. A classification for all predicates of arity 3 follows from the work of Zwick (see [19, 18]), and a large number of predicates of arity 4 were classified by Hast [8]. Hast also gave a general sufficient condition (discussed later) for a predicate of arity k to be approximable. He provided a rounding algorithm for an SDP relaxation, which achieves a good approximation assuming the above condition.

For predicates with large arity, approximability results are known for various special cases. The case when f is the sign of a linear polynomial in the variables, was studied by Cherghachi et. al.[3]. They defined a special subclass, which they called “Chow-robust” predicates, for which approximability follows from the sufficient condition of Hast. They studied the approximability curve for these predicates, adapting Hast’s algorithm to obtain the best possible approximation. Austrin et. al.[14] proved approximability for the case when f is the sign of a quadratic polynomial which is symmetric in all the variables (with constant term 0), again by using the algorithm of Hast (for which they gave a simpler analysis). They also studied a new predicate, known as the “Monarchy” predicate, for which approximability

[1] To the best of our knowledge, this is true for all known results proving approximation resistance.

does not follow from Hast's algorithm. They gave a new algorithm to obtain a non-trivial approximation for the Monarchy predicate.

A related result is the study of approximation resistance for *symmetric predicates* in k variables, by Guruswami and Lee [6]. They study whether the sufficient condition of Austrin and Mossel [1] of $f^{-1}(1)$ supporting a balanced pairwise independent distribution, is also necessary for the case of symmetric predicates[2]. They show this to be the case when f is a *even* symmetric predicate, or corresponds to an *interval* of values for $\sum_{i=1}^{k} x_i$. Since the Austrin-Mossel condition is sufficient for approximation resistance, showing that it is necessary corresponds to proving an approximability result. As before, this is also proved using the condition (and algorithm) by Hast [8].

A characterization of strong approximation resistance

The starting point for our work is a characterization of strong approximation resistance, recently given by Khot et. al.[13]. Their characterization is in terms of a polytope $\mathbb{C}(f)$ associated with the predicate f. For a distribution μ supported on a subset of $f^{-1}(1)$, let $\zeta = \zeta(\mu)$ denote the $(k+1) \times (k+1)$ moment matrix with $\zeta(i,j) = \mathbb{E}_{x\sim\mu}[x_i x_j]$ and $\zeta(0,i) = \zeta(i,0) = \mathbb{E}_{x\sim\mu}[x_i]$. Then $\mathbb{C}(f)$ is defined as the convex polytope

$$\mathcal{C}(f) = \{\zeta(\mu) \;:\; supp(\mu) \subseteq f^{-1}(1)\}\,.$$

The condition of Khot et. al.says that a predicate f is strongly approximation resistant *if and only if* there exists a probability measure Λ on $\mathbb{C}(f)$, satisfying certain symmetry properties. These properties amount to saying for a set of k linear transformations $L_1, \ldots, L_k$ (with L_t depending on Fourier coefficients for sets of size t), we get $L_t(\Lambda) \equiv 0$ for all $t \in [k]$. We refer to such a measure Λ as a vanishing measure.

The results of Khot et. al.in fact characterize approximability with respect to the basic SDP relaxation. They show that if a vanishing measure exists, then for every $\varepsilon > 0$, there exist instances such that the value of the SDP relaxation is at least $1-\varepsilon$, but for every assignment to the variables, the fraction of constraints satisfied is in the interval $[\rho(f)-\varepsilon, \rho(f)+\varepsilon]$. By the results of Raghavendra [15], this shows that the predicate is strongly approximation resistant (assuming the UGC). Conversely, they also show that if such a measure does not exist, then there exists a (randomized) rounding algorithm for the basic SDP, which given an SDP solution with value at least $1-\varepsilon$, produces an assignment whose value deviates from $\rho(f)$ by at least ε (in expectation).

Our results

The goal of this work is to show how various results on (weak) approximability can be proved using the characterization of Khot et. al., without designing a new rounding algorithm in each case. Given their result, it suffices to show the nonexistence of a vanishing measure, to show the existence of a good rounding algorithm. For a variety of cases including the condition of Hast [8] and the Monarchy predicate, we show that the showing the nonexistence of a vanishing measure turns out to be much simpler to prove. We also derive some new results on weak approximability results of symmetric predicates, as described below.

We prove the following results corresponding to the approximability condition of Hast. We note that our proof only gives weak approximability, while the proofs by Hast [8] and

[2] Guruswami and Lee consider CSPs both with and without negation. However, we only discuss the former here.

Austrin et. al.[14] based on a rounding algorithm, give approximability under the same condition on f. The result below also gives weak approximability for any results based on Hast's condition.

▶ **Theorem 1.** *Let $f : \{-1,1\} \to \{0,1\}$ be a predicate. Suppose there exists $\eta \in \mathbb{R}$, such that*

$$\frac{2\eta}{\sqrt{2\pi}} \cdot \sum_i \hat{f}(\{i\}) \cdot x_i + \frac{2}{\pi} \cdot \sum_{i<j} \hat{f}(\{i,j\}) \cdot x_i x_j \ > \ 0$$

for all $x \in f^{-1}(x)$. Then f is weakly approximable.

For the Monarchy predicate, we prove the following result (proved by Austrin et. al.[14] using a different rounding algorithm than the one used for the above result)

▶ **Theorem 2.** *Let f be the Monarchy predicate defined as*

$$f(x) \ := \ \frac{1 + sgn\left((k-2) \cdot x_1 + x_2 + \cdots + x_k\right)}{2}.$$

Then f is approximable using the basic SDP.

Note that since Monarchy is an odd predicate, the notions of approximability and weak approximability are equivalent in this case. Finally, we prove that for a symmetric predicate f with non-zero Fourier mass on sets of size 1 and 2, the condition of Austrin and Mossel is tight for strong approximation resistance i.e.f is strongly approximation resistant if and only if $f^{-1}(1)$ supports a balanced pairwise independent distribution. As discussed before, the condition is known to be sufficient for strong approximation resistance, and thus showing that it is necessary is a result about (weak) approximability.

▶ **Theorem 3.** *Let $f : \{-1,1\}^k \to \{0,1\}$ be a symmetric predicate such that either $\hat{f}(\{1\}) = \hat{f}(\{2\}) = 0$, or $\hat{f}(\{1\}) \neq 0$ and $\hat{f}(\{1,2\}) \neq 0$. Then f is strongly approximation resistant if and only if $f^{-1}(1)$ supports a balanced pairwise independent distribution.*

We remark that in the first case of the above theorem, the uniform distribution on $f^{-1}(1)$ is balanced and pairwise independent. Hence, the interesting part of the result is in the case Fourier coefficients are non-zero at both the levels.

We conclude this section with two brief remarks on our techniques and the issue of approximability vs. weak approximability. First, note that the idea of proving a nonexistence result (for a vanishing measure) instead of an existence result (for an algorithm) seems counterintuitive, since we switch from an existential quantifier to a universal one. However, we in fact show the non-existence of a vanishing measure by showing the *existence* of a function h and a $t \in [k]$ such that $\int h \cdot L_t(\Lambda) \neq 0$ (and hence $L_t(\Lambda) \not\equiv 0$ showing that Λ is not a vanishing measure). The function h turns out to be a simpler "core" object which encodes all the required information for a rounding algorithm, but is easier to argue about We characterize the class of functions h which we search over in Section 3, providing a single framework to capturing various known results.

Secondly, we remark that results in this work only prove weak approximability, and do not necessarily find the best approximation threshold for a problem. However, in many cases, the reason for proving approximability, is in fact to rule out approximation resistance. In such cases, it also seems interesting to rule out strong approximation resistance (i.e.prove weak approximability) since the known techniques for proving approximation resistance, seem to also prove strong resistance.

2 Preliminaries

2.1 Constraint Satisfaction Problems

▶ **Definition 4.** For a predicate $f : \{-1,1\}^k \to \{0,1\}$, an instance Φ of MAX k-CSP (f) consists of a set of variables $\{x_1,\ldots,x_n\}$ and a set of constraints $C_1,\ldots,C_m$ where each constraint C_i is over a k-tuple of variables $\{x_{i_1},\ldots,x_{i_k}\}$ and is of the form

$$C_i \;\equiv\; f(x_{i_1}\cdot b_{i_1},\ldots,x_{i_k}\cdot b_{i_k})$$

for some $b_{i_1},\ldots,b_{i_k} \in \{-1,1\}$. For an assignment $\sigma : \{x_1,\ldots,x_n\} \mapsto \{-1,1\}$, let $\mathsf{sat}(\sigma)$ denote the fraction of constraints satisfied by σ. The maximum fraction of constraints that can be simultaneously satisfied is denoted by $\mathsf{OPT}(\Phi)$, i.e.

$$\mathsf{OPT}(\Phi) = \max_{\sigma:\{x_1,\ldots,x_n\}\mapsto\{-1,1\}} \mathsf{sat}(\sigma).$$

The density of the predicate is $\rho(f) = \frac{|f^{-1}(1)|}{2^k}$.

▶ **Definition 5.** A predicate $f : \{-1,1\}^k \to \{0,1\}$ is called approximable if there exists a constant $\varepsilon > 0$ and a polynomial time algorithm, possibly randomized, that given an $(1-\varepsilon)$-satisfiable instance of MAX k-CSP (f), outputs an assignment A such that $\mathbb{E}_A[\,\mathsf{sat}(A)\,] \geq \rho(f)+\varepsilon$. Here the expectation is over the randomness used by the algorithm. The predicate is called weakly approximable if the output of the algorithm deviates from $\rho(f)$ in expectation, i.e. $\mathbb{E}_A[\,|\mathsf{sat}(A)-\rho(f)|\,] \geq \varepsilon$.

Note that the two notions are equivalent for an odd predicate satisfying $f(x) = 1 - f(-x)$ for all $x \in \{-1,1\}^k$

A predicate that is not approximable is said to be approximation resistant and a predicate that is not weakly approximable is said to be strongly approximation resistant. However, since these conditions require the *non-existence* of algorithms, one can only define them under certain conjectures such as the Unique Games conjecture of Khot [12] (and $P \neq NP$), or for a specific family of algorithms.

It follows from a result of Raghavendra [15] that approximation resistance with respect to a specific algorithm given by a basic SDP relaxation, discussed in the next section, is equivalent to approximation resistance assuming the UGC. It was observed in [13] that this is also true for strong resistance. Thus, we will limit ourselves to discussion of resistance with respect to the basic SDP relaxation. Since the goal here is to prove approximability, we in fact prove that the problems in question are approximable using the basic SDP.

2.2 Fourier Analysis

Recall that a function $f : \{-1,1\}^k \to \mathbb{R}$ can be written as

$$f \;=\; \sum_{S\subseteq[k]} \hat{f}(S)\cdot\chi_S\,,$$

where the functions $\chi_S(x) = \prod_{i\in S} x_i$ form an orthonormal basis for the space of functions $f : \{-1,1\}^k \to \mathbb{R}$ under the inner product $\langle f,g\rangle = \mathbb{E}_{x\in\{-1,1\}^k}[f(x)g(x)]$. The coefficients $\hat{f}(S)$ are known as Fourier coefficients and can be computed as

$$\hat{f}(S) \;=\; \langle f,\chi_S\rangle \;=\; \mathbb{E}\left[\prod_{i\in S} x_i\cdot f(x)\right].$$

$$
\begin{array}{rcll}
\text{maximize} & & \underset{C\in\Phi}{\mathbb{E}}\left[\displaystyle\sum_{\alpha\in\{-1,1\}^k} f(\alpha\cdot b_C)\cdot x_{(S_C,\alpha)}\right] & \\
\text{subject to} & & & \\
\left\langle \mathbf{v}_{(i,1)}, \mathbf{v}_{(i,-1)} \right\rangle & = & 0 & \forall i\in[n] \\
\mathbf{v}_{(i,1)} + \mathbf{v}_{(i,-1)} & = & \mathbf{v}_{(\emptyset,\emptyset)} & \forall i\in[n] \\
\left\| \mathbf{v}_{(\emptyset,\emptyset)} \right\|^2 & = & 1 & \\
\displaystyle\sum_{\substack{\alpha\in\{-1,1\}^{S_C} \\ \alpha(i_1)=b_1,\alpha(i_2)=b_2}} x_{(S_C,\alpha)} & = & \left\langle \mathbf{v}_{(i_1,b_1)}, \mathbf{v}_{(i_2,b_2)} \right\rangle & \forall C\in\Phi, i_1\neq i_2\in S_C, b_1,b_2\in\{-1,1\} \\
x_{(S_C,\alpha)} & \geq & 0 & \forall C\in\Phi,\ \forall\alpha\in\{-1,1\}^{S_C}
\end{array}
$$

Figure 1 Basic Relaxation for MAX k-CSP (f).

2.3 The Basic SDP Relaxation for CSPs

We present below the basic SDP relaxation considered by Raghavendra [15]. The relaxation is includes non-negative variables $x_{(S_C,\alpha)}$ are included for sets S_C corresponding to the set of CSP variables for some constraint C, and an assignment $\alpha\in\{-1,1\}^{S_C}$. The variables $\{x_{(S_C,\alpha)}\}_{\alpha\in\{-1,1\}^{S_C}}$ add up to 1, thus defining a distribution on the assignments to the CSP variables in the set S_C.

The relaxation also has vectors $\mathbf{v}_{(i,b)}$ for each $i\in[n]$ and $b\in\{-1,1\}$, such that the inner products $\langle \mathbf{v}_{(i_1,b_1)}, \mathbf{v}_{(i_2,b_2)}\rangle$ correspond to the probability that $x_{i_1}=b_1$ and $x_{i_2}=b_2$. The relaxation (after a minor rewriting) is shown in Fig. 1.

For an SDP relaxation of MAX k-CSP, and for a given instance Φ of the problem, we denote by $\mathsf{FRAC}(\Phi)$ the SDP (fractional) optimum. For the particular instance Φ, the integrality gap is defined as $\mathsf{FRAC}(\Phi)/\mathsf{OPT}(\Phi)$. The integrality gap of the relaxation is the supremum of integrality gaps over all instances. The integrality gap thus defined is in terms of a ratio whereas we are concerned with the specific *g*ap location $1-o(1)$ versus $\rho(f)+o(1)$ and also with the *s*trong integrality gap as defined below.

▶ **Definition 6.** Let $\varepsilon>0$ be a constant. A relaxation is said to have a $(1-\varepsilon,\rho(f)+\varepsilon)$-integrality gap if there exists a CSP instance Φ such that $\mathsf{FRAC}(\Phi)\geq 1-\varepsilon$ and $\mathsf{OPT}(\Phi)\leq \rho(f)+\varepsilon$.

The relaxation is said to have a strong $(1-\varepsilon,\rho(f)\pm\varepsilon)$-integrality gap if there exists a CSP instance Φ such that $\mathsf{FRAC}(\Phi)\geq 1-\varepsilon$ and for every assignment σ to the instance, $|\mathsf{sat}(\sigma)-\rho(f)|\leq\varepsilon$.

It was shown by Raghavendra [15] that the integrality gap for the basic relaxation as in Fig. 1 implies a UG-hardness result. It was observed by Khot et. al.[13] that this also holds for strong integrality gaps.

▶ **Theorem 7** ([15]). *If the basic SDP in Fig. 1 has a $(1-\varepsilon,\rho(f)+\varepsilon)$-integrality gap for every $\varepsilon>0$, then f is approximation resistant assuming the UGC. Moreover, if the SDP has a strong $(1-\varepsilon,\rho(f)\pm\varepsilon)$-gap for every $\varepsilon>0$, then f is strongly approximation resistant (assuming the UGC).*

2.4 Approximation Resistance Characterization

In this section, we briefly review the characterization of strong approximation resistance by Khot et. al.[13].

Let μ be a probability distribution over $\{-1,1\}^k$. Then the symmetric matrix of first and second moments $\zeta(\mu)$ is defined as follows

$$\zeta(\mu) = \begin{bmatrix} 1 & \mathbb{E}[x_1] & \mathbb{E}[x_2] & \cdots & \mathbb{E}[x_k] \\ \mathbb{E}[x_1] & 1 & \mathbb{E}[x_1 x_2] & \cdots & \mathbb{E}[x_1 x_k] \\ \mathbb{E}[x_2] & \mathbb{E}[x_1 x_2] & 1 & \cdots & \mathbb{E}[x_2 x_k] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[x_k] & \mathbb{E}[x_1 x_k] & \mathbb{E}[x_2 x_k] & \cdots & 1 \end{bmatrix}$$

with $\mathbb{E}[x_i]$ in the $(0,i)$ entry, and $\mathbb{E}[x_i x_j]$ in the (i,j) entry. All expectations above are with respect to the distribution μ. The characterization of Khot et. al.is in terms of measures on the convex polytope

$$\mathcal{C}(f) = \{\zeta(\mu) \mid supp(\mu) \subseteq f^{-1}(1)\}.$$

To describe the characterization, we first consider three ways of transforming such a matrix ζ. All transformations preserve the symmetry of ζ.

- Projection to a subset S: Fix a nonempty $S \subset [k]$. Then ζ_S is the $|S|+1$ by $|S|+1$ principal submatrix obtained by restricting to rows and columns in $\{0\} \cup S$.
- Permuting rows/columns: Fix a permutation $\pi : S \to S$. Then, $\zeta_{S,\pi}$ is the $|S|+1$ by $|S|+1$ matrix obtained by permuting the rows and columns of ζ_S corresponding to S, according to π^{-1}, i.e.

$$\zeta_{S,\pi}(i,j) = \zeta_S(\pi(i),\pi(j)) \quad \forall i,j \in S \quad \text{and} \quad \zeta_{S,\pi}(i,0) = \zeta_{S,\pi}(0,i) = \zeta_S(0,\pi(i)).$$

- Applying a vector of signs: Fix $b \in \{-1,1\}^S$. Then, $\zeta_{S,\pi,b}$ is the $|S|+1$ by $|S|+1$ matrix obtained by taking the entry-wise product of $\zeta_{S,\pi,b}$ and $(1b)(1b)^T$, i.e.

$$\zeta_{S,\pi,b}(i,j) = b_i b_j \cdot \zeta_{S,\pi}(i,j) \quad \forall i,j \in S \quad \text{and} \quad \zeta_{S,\pi,b}(i,0) = \zeta_{S,\pi,b}(0,i) = b_i \cdot \zeta_{S,\pi}(0,i).$$

Now, let Λ be a probability measure over $\mathcal{C}(f)$. Fix, nonempty $S \subset [k]$, $\pi : S \to S$, and $b \in \{-1,1\}^S$. We define the transformed measure $\Lambda_{S,\pi,b}$ (over $|S|+1$ by $|S|+1$ matrices) defined as

$$\Lambda_{S,\pi,b}(M) \;:=\; \Lambda\left(\{\zeta \in \mathbb{C}(f) \mid \zeta_{S,\pi,b} = M\}\right).$$

We now state the characterization of strong approximation resistance in terms of the basic SDP relaxation in Fig. 1.

▶ **Theorem 8** ([13]). *A given predicate $f : \{-1,1\}^k \to \{0,1\}$ is strongly approximation resistant for the basic SDP relaxation if and only if there exists a probability measure Λ supported on $\mathcal{C}(f)$ such that for all $t \in [k]$ the following function on matrices M is identically 0:*

$$\Lambda^{(t)}(M) \quad = \quad \sum_{|S|=t} \; \sum_{\pi:S\to S} \; \sum_{b\in\{-1,1\}^S} \Lambda_{S,\pi,b}(M) \cdot \hat{f}(S) \cdot \prod_{i\in S} b_i$$

Such a probability measure Λ is called a vanishing measure.

3 Weak Approximability of Predicates

We first derive a necessary condition for strong approximation resistance, using the characterization in Theorem 8. We will then derive various approximability results by showing that the necessary condition is violated.

Suppose $f : \{-1,1\}^k \to \{0,1\}$ is strongly approximation resistant. Theorem 8 implies that there exists a measure Λ supported on the convex body $\mathcal{C}(f)$ such that for all $t \in [k]$:

$$\Lambda^{(t)}(M) = \sum_{|S|=t} \sum_{\pi:S\to S} \sum_{b\in\{-1,1\}^{|S|}} \Lambda_{S,\pi,b}(M) \cdot \hat{f}(S) \cdot \prod_{i\in S} b_i$$

is an identically zero function. Then, for any function h we have

$$\int h(M) \cdot \Lambda^{(t)}(M) \;\; = 0\,.$$

We use this to derive a necessary condition. For $t \in [k]$, let $h : [-1,1]^{(t+1)\times(t+1)} \to \mathbb{R}$ be a function on $(t+1) \times (t+1)$ matrices. We will consider matrices of the form $\zeta_{S,\pi,b}$ where $|S| = t$, $\pi : S \to S$ and $b \in \{-1,1\}^S$. We call h an odd symmetric function if for all $\zeta \in [-1,1]^{(t+1)\times(t+1)}$ with $\zeta(i,i) = 1$, all $\pi : [t] \to [t]$ and $b \in \{-1,1\}^t$, we have

$$h(\zeta_{\pi,b}) \;\; = \;\; \left(\prod_{i\in[t]} b_i\right) \cdot h(\zeta)\,.$$

Note that the permutations π only permute rows and columns $1,\ldots,t$ but do not move the 0^{th} row or column (although the entries in the 0^{th} or column may be permuted). We now state our necessary condition for strong approximation resistance.

▶ **Lemma 9.** *Let $f : \{-1,1\}^k \to \{0,1\}$ be a predicate and let Λ be a vanishing measure on $\mathbb{C}(f)$ satisfying the condition in Theorem 8 for all $t \in [k]$. Let $h : [-1,1]^{(t+1)\times(t+1)} \to \mathbb{R}$ be an odd symmetric function. Then,*

$$\mathop{\mathbb{E}}_{\zeta\sim\Lambda}\left[\sum_{|S|=t} \hat{f}(S) \cdot h\left(\zeta_S\right)\right] \;\; = \;\; 0\,.$$

Proof. The proof is a simple consequence of Theorem 8. Let M be a $(t+1) \times (t+1)$ matrix. Since Λ is a vanishing measure, we know that the signed measure $\Lambda^{(t)}$ should be identically zero. Thus, we have

$$\int \Lambda^{(t)}(M) \cdot h(M) \;\; = \;\; \int \left(\sum_{|S|=t} \;\; \sum_{\pi:S\to S} \;\; \sum_{b\in\{-1,1\}^S} \hat{f}(S) \cdot \prod_{i\in S} b_i \cdot \Lambda_{S,\pi,b}(M)\right) \cdot h(M) \;\; = \;\; 0\,.$$

From the definition of $\Lambda_{S,\pi,b}$, we know that

$$\int \Lambda_{S,\pi,b}(M) \cdot h(M) \;\; = \;\; \mathop{\mathbb{E}}_{\zeta\sim\Lambda}\left[h\left(\zeta_{S,\pi,b}\right)\right]\,.$$

Thus, we have

$$
\begin{aligned}
& \underset{\zeta\sim\Lambda}{\mathbb{E}}\left[\sum_{|S|=t}\ \sum_{\pi:S\to S}\ \sum_{b\in\{-1,1\}^S} \hat{f}(S)\cdot\prod_{i\in S} b_i\cdot h\left(\zeta_{S,\pi,b}\right)\right] \ = \ 0 \\
\Rightarrow\ & \underset{\zeta\sim\Lambda}{\mathbb{E}}\left[\sum_{|S|=t}\ \sum_{\pi:S\to S}\ \sum_{b\in\{-1,1\}^S} \hat{f}(S)\cdot h\left(\zeta_S\right)\right] \ = \ 0 \\
\Rightarrow\ & \underset{\zeta\sim\Lambda}{\mathbb{E}}\left[\sum_{|S|=t} \hat{f}(S)\cdot h\left(\zeta_S\right)\right] \ = \ 0\,,
\end{aligned}
$$

where the first implication uses the fact that h is an odd symmetric function. ◀

▶ Remark. The restriction to odd symmetric functions in the above lemma is actually without loss of generality. Starting from an arbitrary function g, we would get

$$
\underset{\zeta\sim\Lambda}{\mathbb{E}}\left[\sum_{|S|=t}\ \sum_{\pi:S\to S}\ \sum_{b\in\{-1,1\}^S} \hat{f}(S)\cdot\prod_{i\in S} b_i\cdot g\left(\zeta_{S,\pi,b}\right)\right] \ = \ 0\,,
$$

where

$$
h\left(\zeta_S\right) \ = \ \sum_{\pi:S\to S}\ \sum_{b\in\{-1,1\}^S}\ \prod_{i\in S} b_i\cdot g\left(\zeta_{S,\pi,b}\right)
$$

is an odd symmetric function of ζ_S.

We shall use Lemma 9 with different functions h to derive the required approximability results.

3.1 Low Degree Advantage

A widely used general condition for proving approximability is due to Hast [8]. A simplified proof was also given by Austrin et. al.[14] using an SDP rounding algorithm. This condition was also used in the study of approximability of symmetric predicates by Guruswami and Lee [6].

▶ **Theorem 10** ([8, 14])**.** *Let $f:\{-1,1\}\to\{0,1\}$ be a predicate. Suppose there exists $\eta\in\mathbb{R}$, such that*

$$
\frac{2\eta}{\sqrt{2\pi}}\cdot\sum_i \hat{f}(\{i\})\cdot x_i + \frac{2}{\pi}\cdot\sum_{i<j}\hat{f}(\{i,j\})\cdot x_i x_j \ > \ 0
$$

for all $x\in f^{-1}(1)$. Then f is approximable.

We show that the weak approximability analogue of the above theorem follows directly from Lemma 9.

▶ **Theorem 11.** *Let $f:\{-1,1\}\to\{0,1\}$ be a predicate. Suppose there exists $\eta\in\mathbb{R}$, such that*

$$
\frac{2\eta}{\sqrt{2\pi}}\cdot\sum_i \hat{f}(\{i\})\cdot x_i + \frac{2}{\pi}\cdot\sum_{i<j}\hat{f}(\{i,j\})\cdot x_i x_j \ > \ 0
$$

for all $x\in f^{-1}(1)$. Then f is weakly approximable.

Proof. Suppose f is strongly approximation resistant. Then, by Theorem 8, there exists a vanishing measure Λ on $\mathbb{C}(f)$.

We first apply Lemma 9 with $t = 1$. Note that this case corresponds to $|S| = 1$. The matrices ζ_S are 2×2 matrices with diagonal entries 1 and off-diagonal entries equal to $\zeta(0, i)$ when $S = \{i\}$. We take the function $h(M) = M(0, 1)$ (equal to the off diagonal entry). Since there are no nontrivial permutations, and multiplying row 1 and column 1 by $b \in \{-1, 1\}$ multiplies $M(0, 1)$ by b, h is an odd symmetric function. Thus, we get

$$\mathop{\mathbb{E}}_{\zeta \sim \Lambda} \left[\sum_{i \in [k]} \hat{f}(\{i\}) \cdot h(\zeta_{\{i\}}) \right] \;=\; \mathop{\mathbb{E}}_{\zeta \sim \Lambda} \left[\sum_{i \in [k]} \hat{f}(\{i\}) \cdot \zeta(0, i) \right] \;=\; 0\,.$$

Similarly, for the case of $t = 2$, we consider the function $h(M) = M(1, 2)$. The only nontrivial permutation of $1, 2$ swaps the two indices. Thus, for symmetric matrices ζ_S with $|S| = 2$, this is an odd symmetric function. Hence, we get

$$\mathop{\mathbb{E}}_{\zeta \sim \Lambda} \left[\sum_{i<j} \hat{f}(\{i, j\}) \cdot h(\zeta_{\{i,j\}}) \right] \;=\; \mathop{\mathbb{E}}_{\zeta \sim \Lambda} \left[\sum_{i<j} \hat{f}(\{i, j\}) \cdot \zeta(i, j) \right] \;=\; 0\,.$$

Combining the two conditions, we get

$$\mathop{\mathbb{E}}_{\zeta \sim \Lambda} \left[\frac{2\eta}{\sqrt{2\pi}} \cdot \sum_{i} \hat{f}(\{i\}) \cdot \zeta(0, i) + \frac{2}{\pi} \cdot \sum_{i<j} \hat{f}(\{i, j\}) \cdot \zeta(i, j) \right] \;=\; 0\,.$$

Let ζ_0 denote the matrix $\mathbb{E}_{\zeta \sim \Lambda}[\zeta]$. Then, by linearity of expectation

$$\frac{2\eta}{\sqrt{2\pi}} \cdot \sum_{i} \hat{f}(\{i\}) \cdot \zeta_0(0, i) + \frac{2}{\pi} \cdot \sum_{i<j} \hat{f}(\{i, j\}) \cdot \zeta_0(i, j) \;=\; 0\,.$$

Since $\mathbb{C}(f)$ is a convex polytope, $\zeta_0 \in \mathbb{C}(f)$. Thus, there exists a distribution μ_0 with $supp(\mu_0) \subseteq f^{-1}(1)$ satisfying $\zeta_0(0, i) = \mathbb{E}_{x \sim \mu_0}[x_i]$ and $\zeta_0(i, j) = \mathbb{E}_{x \sim \mu_0}[x_i \cdot x_j]$ for all $i, j \in [k]$. Thus, the above condition can we written as

$$\mathop{\mathbb{E}}_{x \sim \mu_0} \left[\frac{2\eta}{\sqrt{2\pi}} \cdot \sum_{i} \hat{f}(\{i\}) \cdot x_i + \frac{2}{\pi} \cdot \sum_{i<j} \hat{f}(\{i, j\}) \cdot x_i x_j \right] \;=\; 0\,,$$

which is a contradiction since the inner quantity is positive for all $x \in f^{-1}(1)$ by assumption. ◀

3.2 Symmetric Predicates

Recall that $f : \{-1, 1\}^k \to \{0, 1\}$ is a symmetric predicate if permuting the input bits of x does not change the value of $f(x)$. Alternatively, $f(x)$ only depends on $\sum_i x_i$. We will also use the fact that for a symmetric function f, $\hat{f}(S)$ only depends on $|S|$.

The approximability of symmetric predicates was studied by Guruswami and Lee [6]. They consider both the cases with and without negation. For the case with negation, as considered in this paper, they show that when f is *even* or corresponds to an *interval* (i.e.there is an interval $I \subseteq [-k, k]$ such that $f(x) = 1 \;\Leftrightarrow\; \sum_i x_i \in I$), f is approximation resistant if and only if there exists a balanced pairwise independent distribution distribution μ supported in $f^{-1}(1)$. Note that this condition was shown to be sufficient by Austrin and

Mossel [1]. They show that for the cases of intervals and even predicates, this condition is also necessary.

We study a different class of symmetric predicates, which either have non-zero mass on both of the first two Fourier levels i.e.$\hat{f}(\{1\}) \neq 0$ and $\hat{f}(\{1,2\}) \neq 0$, or have $\hat{f}(\{1\}) = \hat{f}(\{1,2\}) = 0$. We show that any such predicate f is approximation resistant if and only if $f^{-1}(1)$ supports a balanced pairwise independent distribution. We first consider the case when $\hat{f}(\{1\}) = \hat{f}(\{1,2\}) = 0$. In this case, it is easy to see that $f^{-1}(1)$ supports a balanced pairwise independent distribution, and hence f is approximation resistant.

▶ **Theorem 12.** *Let $f : \{-1,1\}^k \to \{0,1\}$ be a symmetric predicate such that $\hat{f}(\{1\}) = \hat{f}(\{1,2\}) = 0$. Then, the uniform distribution on $f^{-1}(1)$ is balanced and pairwise independent.*

Proof. Let μ denote the uniform distribution on $f^{-1}(1)$. Then, for any $i \in [k]$

$$\mathop{\mathbb{E}}_{x\sim\mu}[x_i] \;=\; \frac{2^k}{|f^{-1}(1)|} \cdot \mathop{\mathbb{E}}_{x\in\{-1,1\}^k}[f(x)\cdot x_i] \;=\; \frac{2^k}{|f^{-1}(1)|} \cdot \hat{f}(\{i\}) \;=\; 0\,.$$

Similarly, we also have that $\mathbb{E}_{x\sim\mu}[x_i x_j] = 0$ for all $i \neq j$. ◀

Next, we consider the case when both $\hat{f}(\{1\})$ and $\hat{f}(\{1,2\})$ are nonzero.

▶ **Theorem 13.** *Let $f : \{-1,1\}^k \to \{0,1\}$ be a symmetric predicate such that $\hat{f}(\{1\}) \neq 0$ and $\hat{f}(\{1,2\}) \neq 0$. Then f is strongly approximation resistant if and only if $f^{-1}(1)$ supports a balanced pairwise independent distribution.*

Proof. We only need to prove that strong approximation resistance implies the existence of a balanced pairwise distribution supported in $f^{-1}(1)$, since the other direction follows from the result of Austrin and Mossel [1].

Let f be approximation resistant and let Λ be the corresponding vanishing measure on $\mathbb{C}(f)$. For a permutation $\pi : [k] \to [k]$, recall that Λ_π denotes the measure

$$\Lambda_\pi(\zeta) \;=\; \Lambda(\zeta_\pi)\,.$$

By the symmetry of the variables in f, if Λ is a vanishing measure, then so is Λ_π. Since the conditions in Theorem 8 are linear in the measure Λ, we get that $\mathbb{E}_{\pi:[k]\to[k]}[\Lambda_\pi]$ is also a vanishing measure. Thus, we can assume without loss of generality that for the given vanishing measure, we have

$$\mathop{\mathbb{E}}_{\zeta\sim\Lambda}[\zeta(0,i)] \;=\; \mathop{\mathbb{E}}_{\zeta\sim\Lambda}[\zeta(0,j)] \;\; \forall i\neq j \quad \text{and} \quad \mathop{\mathbb{E}}_{\zeta\sim\Lambda}[\zeta(i_1,j_1)] \;=\; \mathop{\mathbb{E}}_{\zeta\sim\Lambda}[\zeta(i_2,j_2)] \;\; \forall i_1\neq j_1, i_2\neq j_2\,. \tag{1}$$

As in Theorem 11, we apply Lemma 9 with $t = 1$ using $h(M) = M(0,1)$, and with $t = 2$ using $h(M) = M(1,2)$, to get the conditions

$$\mathop{\mathbb{E}}_{\zeta\sim\Lambda}\left[\sum_{i\in[k]} \hat{f}(\{i\})\cdot\zeta(0,i)\right] \;=\; 0 \quad \text{and} \quad \mathop{\mathbb{E}}_{\zeta\sim\Lambda}\left[\sum_{i<j} \hat{f}(\{i,j\})\cdot\zeta(i,j)\right] \;=\; 0\,.$$

Using the symmetry of the Fourier coefficients, and Eq. (1), this gives

$$\hat{f}(\{1\})\cdot\mathop{\mathbb{E}}_{\zeta\sim\Lambda}[\zeta(0,i)] \;=\; 0 \quad \text{and} \quad \hat{f}(\{1,2\})\cdot\mathop{\mathbb{E}}_{\zeta\sim\Lambda}[\zeta(i,j)] \;=\; 0 \qquad \forall i,j\in[k],\; i\neq j\,.$$

Since $\hat{f}(\{1\}) \neq 0$ and $\hat{f}(\{1,2\}) \neq 0$, we get that $\mathbb{E}_{\zeta\sim\Lambda}[\zeta(0,i)] = 0$ and $\mathbb{E}_{\zeta\sim\Lambda}[\zeta(i,j)] = 0$ for all $i \neq j \in [k]$. Let $\zeta_0 = \mathbb{E}_{\zeta\sim\Lambda}[\zeta]$. As before, we know that $\zeta_0 \in \mathbb{C}(f)$ by convexity and hence there exists μ_0 supported in $f^{-1}(1)$ such that ζ_0 corresponds to the moments of μ_0. Hence,

$$\mathop{\mathbb{E}}_{x\sim\mu_0}[x_i] \;=\; \zeta_0(0,i) \;=\; 0 \quad \text{and} \quad \mathop{\mathbb{E}}_{x\sim\mu_0}[x_i \cdot x_j] \;=\; \zeta_0(i,j) \;=\; 0 \qquad \forall i,j \in [k],\; i \neq j\,.$$

Thus, μ_0 is a balanced pairwise independent distribution supported in $f^{-1}(1)$. ◀

3.3 Monarchy

Next, we consider the Monarchy predicate, which was proved to be approximable by Austrin et. al.[14]. The predicate is a halfspace defined as

$$f(x) \;:=\; \frac{1 + \operatorname{sgn}\left((k-2)\cdots x_1 + x_2 + \cdots + x_k\right)}{2}\,.$$

The predicate is determined by the value of x_1 unless $x_2 = \cdot = x_k = -x_1$. Austrin et. al.considered this predicate as an example of a predicate to which Hast's condition (discussed in the previous section) does not apply. Moreover, it did not seem amenable to the rounding scheme used in the proof of Hast's result and they provide a new rounding algorithm to prove the approximability of this predicate.

We show that the approximability of Monarchy follows from Lemma 9. Moreover, since it is an odd predicate, weak approximability is equivalent to approximability. We shall use the following observation by Austrin et. al.

▶ **Lemma 14** ([14]). *Let f be the monarchy predicate and let μ be a distribution on $\{-1,1\}^k$ with $supp(\mu) \subseteq f^{-1}(1)$. Then for all $i > 1$,*

$$\mathop{\mathbb{E}}_{x\sim\mu}[x_i] \;\geq\; -\mathop{\mathbb{E}}_{x\sim\mu}[x_1]\,.$$

Proof. If x is a satisfying assignment then either $x_1 = 1$, or for all $i > 2$ $x_i = 1$. In both cases, we have $x_i \geq -x_1$ for all $i > 2$. The claim follows by linearity of expectation. ◀

We will also need the following facts about the Fourier coefficients of the Monarchy predicate.

▶ **Lemma 15.** *Let f be the Monarchy predicate as defined above. Then*
1. *$\hat{f}(\{1\}) = 1/2 - 1/2^{k-1}$ and $\hat{f}(\{2\}) = \cdots = \hat{f}(\{k\}) = 1/2^{k-1}$.*
2. *$\hat{f}(S) = 0$ for all S such that $|S| = 2$.*
3. *For S with $|S| = 3$, $\hat{f}(S) = -1/2^{k-1}$ if $1 \in S$ and $\hat{f}(S) = 1/2^{k-1}$ otherwise.*

We can now prove that Monarchy is approximable.

▶ **Theorem 16.** *Let f be the Monarchy predicate as defined above. Then f is approximable using the basic SDP.*

Proof. Suppose that f is not approximable. Then, by Theorem 8, there exists a vanishing measure Λ on the polytope $\mathbb{C}(f)$. For $s \in \{-1,0,1\}$, define the probabilities $p(s) := \Lambda(\{\zeta \mid \operatorname{sgn}(\zeta(0,1)) = s\})$. We first prove the following.

▶ **Lemma 17.** *There exist $\beta_1 \geq 1$ and $\beta_0 \geq 0$ such that $p(-1) = \beta_1 \cdot p(1) + \beta_0 \cdot p(0)$. Moreover, we must have $p(-1) > 0$.*

Proof. We apply Lemma 9 for $t = 1$ and the function $h(M) = \operatorname{sgn}(M_{0,1})$. Then, since h is an odd symmetric function, we get

$$\underset{\zeta\sim\Lambda}{\mathbb{E}}\left[\hat{f}(\{1\})\cdot\operatorname{sgn}(\zeta(0,1)) + \sum_{i>1}\hat{f}(\{i\})\cdot\operatorname{sgn}(\zeta(0,i))\right] \;=\; 0$$

$$\Rightarrow \sum_{s\in\{-1,0,1\}} p(s)\cdot\mathbb{E}\left[\hat{f}(\{1\})\cdot\operatorname{sgn}(\zeta(0,1)) + \sum_{i>1}\hat{f}(\{i\})\cdot\operatorname{sgn}(\zeta(0,i)) \mid \operatorname{sgn}(\zeta(0,1)) = s\right] \;=\; 0\,.$$

Using the facts that $\hat{f}(\{2\}) = \cdots = \hat{f}(\{k\})$ and $\zeta_i \geq -\zeta_1$ $\forall i > 2$ by Lemma 14, we get

$$p(-1)\cdot\Big(-\hat{f}(\{1\}) + (k-1)\cdot\hat{f}(\{2\})\Big) + p(0)\cdot\Big(a\cdot\hat{f}(\{2\})\Big) + p(1)\cdot\Big(\hat{f}(\{1\}) + b\cdot\hat{f}(\{2\})\Big) \;=\; 0\,,$$

for some $a \in [0, k-1]$ and $b \in [-(k-1), k-1]$. Thus, we get $p(-1) = \beta_1\cdot p(1) + \beta_0\cdot p(0)$, where

$$\beta_1 \;=\; \frac{\hat{f}(\{1\}) + b\cdot\hat{f}(\{2\})}{\hat{f}(\{1\}) - (k-1)\cdot\hat{f}(\{2\})} \;\geq\; 1 \quad \textit{and} \quad \beta_0 \;=\; \frac{a\cdot\hat{f}(\{2\})}{\hat{f}(\{1\}) - (k-1)\cdot\hat{f}(\{2\})} \;\geq\; 0$$

To prove the second part of the claim, we again apply Lemma 9 with $t = 1$ and $h(M) = M(0,1)$. This gives,

$$\underset{\zeta\sim\Lambda}{\mathbb{E}}\left[\hat{f}(\{1\})\cdot\zeta(0,1) + \sum_{i>1}\hat{f}(\{i\})\cdot\zeta(0,i)\right] \;=\; 0.$$

By the definition of the Monarchy predicate, we also know that for any $\zeta \in \mathbb{C}(f)$,

$$(k-2)\cdot\zeta(0,1) + \sum_{i>1}\zeta(0,i) \;>\; 0\,.$$

Using the fact that $\hat{f}(\{i\}) = \hat{f}(\{2\})$ for all $i > 1$, we get

$$\left((k-2) - \frac{\hat{f}(\{1\})}{\hat{f}(\{2\})}\right)\cdot\underset{\zeta\sim\Lambda}{\mathbb{E}}[\zeta(0,1)] \;>\; 0 \quad\Rightarrow\quad \underset{\zeta\sim\Lambda}{\mathbb{E}}[\zeta(0,1)] \;<\; 0\,.$$

Hence, we must have $p(-1) \;=\; \mathbb{P}[\zeta(0,1) < 0] \;>\; 0.$ ◀

Next, we apply Lemma 9 with $t = 3$ and $h(M) = \prod_{j=1}^{3} M(0,j)$. This gives

$$\underset{\zeta\sim\Lambda}{\mathbb{E}}\left[\sum_{|S|=3}\hat{f}(S)\cdot\prod_{i\in S}\operatorname{sgn}(\zeta(0,i))\right] \;=\; 0$$

$$\Rightarrow \sum_{s\in\{-1,0,1\}} p(s)\cdot\mathbb{E}\left[\sum_{|S|=3}\hat{f}(S)\cdot\prod_{i\in S}\operatorname{sgn}(\zeta(0,i)) \;\Big|\; \operatorname{sgn}(\zeta(0,1)) = s\right] \;=\; 0\,. \tag{2}$$

We analyze the terms for each $s \in \{-1,0,1\}$ separately. For $s = -1$, we have $\zeta(0,1) < 0$ and hence, $\zeta(0,i) > 0$ for all $i > 1$, by Lemma 14. Since the Fourier coefficients are negative when $1 \in S$ and positive otherwise (Lemma 15), we get that

$$E(-1) \;=\; \mathbb{E}\left[\sum_{|S|=3}\hat{f}(S)\cdot\prod_{i\in S}\operatorname{sgn}(\zeta(0,i)) \;\Big|\; \operatorname{sgn}(\zeta(0,1)) = -1\right] \;=\; \sum_{|S|=3}\left|\hat{f}(S)\right|\,.$$

For $s = 0$, we have $\zeta(0,1) = 0$ and hence $\zeta(0,i) \geq 0$ for all $i > 1$. This gives

$$\begin{aligned} E(0) &= \mathbb{E}\left[\sum_{|S|=3} \hat{f}(S) \cdot \prod_{i \in S} \operatorname{sgn}(\zeta(0,i)) \;\middle|\; \operatorname{sgn}(\zeta(0,1)) = 0\right] \\ &= \mathbb{E}\left[\sum_{\substack{|S|=3 \\ 1 \notin S}} \hat{f}(S) \cdot \prod_{i \in S} \operatorname{sgn}(\zeta(0,i)) \;\middle|\; \operatorname{sgn}(\zeta(0,1)) = 0\right] \;\geq\; 0\,, \end{aligned}$$

since $\hat{f}(S) \geq 0$ for all S with $|S| = 3$ and $1 \notin S$. Finally, for $s = 1$, we note that since $\hat{f}(S) < 0$ for $1 \in S$, we must have

$$\left|\sum_{\substack{|S|=3 \\ 1 \in S}} \hat{f}(S) \prod_{i \in S} \operatorname{sgn}(\zeta(0,i))\right| \;<\; \sum_{\substack{|S|=3 \\ 1 \in S}} \left|\hat{f}(S)\right|\,,$$

since $\operatorname{sgn}(\zeta(0,i)) \cdot \operatorname{sgn}(\zeta(0,j))$ cannot be simultaneously negative for all $i, j > 1$. This gives,

$$|E(1)| \;=\; \left|\mathbb{E}\left[\sum_{|S|=3} \hat{f}(S) \cdot \prod_{i \in S} \operatorname{sgn}(\zeta(0,i)) \;\middle|\; \operatorname{sgn}(\zeta(0,1)) = 1\right]\right| \;<\; \sum_{|S|=3} \left|\hat{f}(S)\right| \;=\; E(-1)\,.$$

We will show that this implies a contradiction to Eq. (2). By Lemma 17, we have that $p(-1) = \beta_1 \cdot p(1) + \beta_0 \cdot p(0)$ for $\beta_1 \geq 1$ and $\beta_0 \geq 0$. Thus, we have

$$\begin{aligned} & p(-1) \cdot E(1) + p(0) \cdot E(0) + p(1) \cdot E(1) \\ =\; & (\beta_1 p(1) + \beta_0 p(0)) \cdot E(-1) + p(0) \cdot E(0) + p(1) \cdot E(1) \\ \geq\; & p(1) \cdot (\beta_1 E(-1) - |E(1)|) + p(0) \cdot (\beta_0 E(-1) + E(0))\,, \end{aligned}$$

which is strictly greater than 0 (thus contradicting Eq. (2)) unless $p(1) = 0$ and $\beta_0 = 0$. However, this would imply that $p(-1) = \beta_1 \cdot p(1) + \beta_0 \cdot p(0) = 0$, which is impossible by Lemma 17. ◀

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