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Online Energy Storage Management: an Algorithmic Approach

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Abstract

Motivated by the importance of energy storage networks in smart grids, we provide an algorithmic study of the online energy storage management problem in a network setting, the first to the best of our knowledge. Given online power supplies, either entirely renewable supplies or those in combination with traditional supplies, we want to route power from the supplies to demands using storage units subject to a decay factor. Our goal is to maximize the total utility of satisfied demands less the total production cost of routed power. We model renewable supplies with the zero production cost function and traditional supplies with convex production cost functions. For two natural storage unit settings, private and public, we design poly-logarithmic competitive algorithms in the network flow model using the dual fitting and online primal dual methods for convex problems. Furthermore, we show strong hardness results for more general settings of the problem. Our techniques may be of independent interest in other routing and storage management problems.

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1 Introduction

With recent advancements in renewable energy generation technologies and smart grids, the problem of energy storage management has become a central problem. While renewable energy sources such as solar and wind are expected to supply a significant portion of electricity demand (by some measure, 50% by 2050 [37, 19, 23]), they have rather intermittent and variable output compared to the traditional fossil-fuel power generators. These uncertainties can lead to supply-demand imbalance and higher reserve requirements and pose a significant challenge to the renewable power supplies' integration to the existing power grids and energy distribution to consumers.

Energy storage provides a solution for maintaining supply-demand balance by providing the flexibility of transporting energy across time, just as a power network provides transportation flexibility over a geographical area. Many storage technologies have been researched and developed: batteries, flywheels, pumped-hydro, and compressed air energy storages [27, 17].

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These technologies can help integrate renewable energy sources, such as solar arrays and wind farms, to the power grids as they become more robust, reliable and economically viable.

In recent years, the energy storage management problem has become a focus of active research by the power systems community. The problem of single storage control has been investigated extensively in various settings and several analytical solutions using stochastic dynamic programming exist (e.g., [34, 39, 35, 21]). For the general case when multiple storage units are available on a power network, analytical solutions with efficient algorithmic implementation are more challenging to obtain. We mostly have heuristics such as Model Predictive Control [41] and look-ahead policies [32] without any performance guarantees with the exception of few cases (e.g., [36] for a long-term average cost minimization problem).

Motivated by the importance of energy storage networks within smart grids, we provide an algorithmic study of an online problem of energy storage management with storage units subject to decay. We consider both private and public storage unit settings and model the unpredictable output of renewable sources as online power supplies and the predictable, say hourly, variation of demand as either online or offline demands. We assume a network flow model which is a good approximation of power flows in the power grids in certain high-voltage operation regimes. To the best of our knowledge, there is no prior study of energy storage networks in an online setting which provides provable guarantees.

Our work is closely related to the classical max-flow and multicommodity flow problems and their generalized flow variants with decay (e.g., [40]) and online variants (e.g., [2, 3, 12]) in terms of the model and techniques. However, these problems have not been studied in combination with storage units subject to a decay factor in an online setting. Our online primal dual approach in the case of public storage units is similar to Buchbinder and Naor [12] and Devanur and Jain [18] and more recent work on online covering and packing problems with convex objectives in [14, 11, 4, 22]. Furthermore, our work is related to inventory problems such as the multi-item lot sizing and joint replenishment problems in the planning aspect. In these problems, one wants to balance the cost of surplus inventory maintenance for future demand against the cost of frequent inventory replenishment; both offline and online settings have been well-studied [26, 31, 7, 10].

Other Related Work

Competitive analysis has been applied in other power management problems. Lu et al. [30] studied the problem of generation scheduling for micro-grids with renewable power sources. Chau et al. [15] designed an online algorithm for single storage operation with uncertain prices and renewable generation. Decay factors have been studied previously in different contexts. Babaioff et al. [5] investigated a variant of the classical secretary problem with discounts. More broadly speaking, deterioration, perishability and lifetime constraints of goods have been studied in numerous mathematical models and optimization problems [33]. In addition to the storage challenges of renewable energy, the economic issues around the energy market and pricing have been investigated by various communities (e.g., [24, 9]).

Our Contributions

In this paper, we develop algorithmic techniques for the *Online Energy Storage Management Problem (OESMP)*. In addition to bridging between the smart grid and TCS communities, we believe our techniques can be of its own interest in analyzing other routing and storage management problems.

Given a network represented by a directed graph $G = (V, E)$ with n nodes and m capacitated edges, we want to route power from online supplies to demands, online or offline, over T time steps using storage units subject to decay factor γ. We consider two main storage unit settings: 1) private storage units which are dedicated storage units co-located with power supplies, and 2) public storage units which are networked units that can store energy from any power supplies with a routing path. Demands have utility functions W and supplies have either zero production cost in the case of renewable power supplies or convex production cost, of the form $Q(z) = az^\rho$, in the case of traditional power supplies.

We assume a network flow model which is equivalent to the widely adopted "DC approximation" of AC power flow for power transmission network [38] when power flow routing is possible. Power flow routing is possible if the flows on all edges can be directly controlled as long as conservation and capacity constraints are satisfied. This holds naturally when the network topology is a tree, which is the case for some bulk transmission networks [16] and is approximately true for transmission networks in which *congestion*[1] does not involve cycles [13]. For transmission networks with general network topologies, this may be enabled by Flexible AC Transmission system (FACTS) devices such as phase shifters or smart wires [25]. The flow model can also be used to approximate power flow on distribution networks when line capacity constraints are considered (cf. simplified DistFlow model [6]). More details are provided in Appendix A.

In OESMP, our goal is to route power from supplies to demands and maximize the total utility of satisfied demands less the total production cost of routed power over a finite time horizon of T. We design online deterministic algorithms with poly-logarithmic[2] competitive ratios against the optimal offline algorithm. For simplicity, we assume a single decay factor γ for storage units and a single power exponent ρ for convex cost functions in the following theorem statements. Furthermore, we assume the marginal utility W' is bounded between $w_{\max}$ and $w_{\min} > 0$ and let $\Delta_w = w_{\max}/w_{\min}$. Note n (m) is the number of nodes (edges).

In the *private storage unit setting* with online demands (Section 3), we consider only renewable sources and maximize the utility of satisfied demands. First, we show that an intuitive greedy algorithm achieves a constant competitive ratio if the utility functions are linear and identical.

▶ **Theorem 1.** *For OESMP with private storage units and online demands, there exists a deterministic online algorithm with the constant competitive ratio of* 3 *in the case of uniform utilities.*

We analyze the greedy algorithm using a dual-fitting approach. We show that the congestion on links and storage units leads to a natural dual construction corresponding to the flow in each time step. However, the decay factor introduces strong dependency between different time steps which requires a global evaluation of the dual vectors across the time. We next extend our result to obtain an algorithm with a logarithmic competitive ratio for the private storage unit setting with concave utility functions.

▶ **Theorem 2.** *For OESMP with private storage units and online demands, there exists a deterministic online algorithm with the competitive ratio of* $O(\log \Delta_w)$ *in the case of concave utilities.*

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[1] In power systems, thermal constraints limit the amount of power that can be routed through a transmission line. If the maximum is reached for a line, we say the line is congested. Similarly, we say a path of multiple consecutive lines is congested if at least one of the lines on the path is congested.

[2] In this paper, we say a factor is *poly-logarithmic* if it is poly-logarithmic with respect to system parameters n, Δ_w and γ; and not necessarily with respect to the input size.

In the *public storage unit setting* (Section 4), we consider both traditional and renewable sources and the storage units that can be used by any sources. We show that online demands are hard to cope with:

▶ **Theorem 3.** *For OESMP with public storage units and online demands, the competitive ratio of any online algorithm is $\Omega(n)$ even in the case of uniform utilities.*

Therefore, for the public storage unit setting, we focus on the offline demand variant. We apply the online primal dual method to a convex program formulation of the problem using Fenchel duality. Our analysis requires connections between the convex production cost functions and their convex conjugates and utilizes similar critical ideas to those recently developed in [18, 14, 11, 4, 22].

We show poly-logarithmic competitive online algorithms when demands are offline. Inspired by the daily power markets, we assume that T is a small constant[3].

▶ **Theorem 4.** *For OESMP with public storage units and offline demands, there exists a deterministic online algorithm with the competitive ratio of $O(\log n + \log \gamma^{-T} + \log \Delta_w)$ in the case of concave utilities.*

▶ **Theorem 5.** *For OESMP with public storage units and offline demands, there exists a deterministic online algorithm with the competitive ratio of $O\left(\rho^{\rho/(\rho-1)}(\log n + \log \gamma^{-T} + \log \Delta_w)\right)$ for $\rho > \rho_0$ in the case of concave utilities and convex costs of the form $Q(z) = az^\rho$, where ρ_0 is some constant arbitrarily close to 1.*[4]

Finally for the general network with cycles where the network flow model does not apply directly, we show a strong lower bound (Appendix B):

▶ **Theorem 6.** *For OESMP with public storage units in the general power network model and offline demands, the competitive ratio of any online algorithm is $\Omega(n^{1/5})$ even in the case of uniform utilities. This holds even if all the links are physically identical.*

2 Notations and Preliminaries

We define the *Online Energy Storage Management Problem (OESMP)* as follows.

Storage Management Problem

We consider a power transmission network with nodes and edges over a finite time horizon T.[5] The network is represented by a directed graph $G = (V, E)$ with n nodes and m edges.[6] Following the DC-approximation framework for high-voltage regimes, we have edge capacities $C : E \to \mathbb{R}^+$ modeling the thermal constraints on transmission lines. A node can be a supply or demand node at different times and some nodes have storage units.

We denote the set of vertices with nonzero power supply (demand) at time step i by S^i (D^i). Supply sets S^i are given online. Each supply node s has a convex production cost

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[3] For many markets, $T = 24$ and $\gamma \in [0.9, 1]$. Hence, note $\log \gamma^{-T} < 1.1$ in practice.

[4] We can remove the condition $\rho > \rho_0$ completely and obtain a poly-logarithmic competitive ratio in terms of m, the number of edges, instead of n. The condition arises from our technical analysis and the constant ρ_0 is accompanied by a correspondingly large constant in the big O notation.

[5] For our applications, T is a small constant, e.g., $T = 24$ for 24 hours.

[6] For a directed edge (u, v), we assume power can only move from u to v on this edge. Forcing direction on the edges makes the model only stronger since a bidirectional link between nodes u and v can be simulated in this model by putting two directed edges (u, v) and (v, u).

function $Q_s^i(z)$ for generating z units of power. If s is a renewable power supply, then it has zero production cost up to the generation capacity at that time. If s is a traditional power supply, then $Q_s(z) = a_s z^{\rho_s}$ for some constants a_s and ρ_s, omitting the superscript i.[7] We would need traditional power generators when the renewable ones alone are not enough.

Demand sets D^i may be given online or offline. We assume that S^i and D^i are disjoint after locally satisfying the demand when there also exists some supply at the same node. Each demand node d has a concave utility function W_d (positive and non-decreasing) such that it would receive $W_d(y)$ units of utility for y units of power received. Since W_d is concave, we have (weakly) diminishing returns on each additional unit of power routed. We assume the slope of the utility function is bounded, i.e., $\frac{dW_d(y)}{dy} \in [w_{\min}, w_{\max}]$ where $w_{\min} > 0$ for $i \in [T]$ and $d \in D^i$. Let $\Delta_w := w_{\max}/w_{\min}$.[8]

We denote the set of nodes with storage units by R. Each storage node has a maximum capacity, a decay factor, charging/discharging inefficiency factors, and ramping constraints. A storage node r can store at most L_r units of power across a time step and maintains γ_r fraction of power stored in the process. When charging or discharging, we are subject to inefficiency factors and ramping constraints in that we lose some fraction of the power in these operations and are limited to some maximum amount of charge/discharge rate per time step. Without loss of generality, we focus on the maximum capacity L_r and decay factor γ_r as we can treat the inefficiency factors and ramping constraints similarly.[9]

Our goal is to route power from supplies to demands and maximize the total utility of satisfied demands less the total production cost of routed power. We model power flows using the standard network flow model, which is equivalent to the widely adopted DC approximation to AC power flow when power routing is possible (see Appendix A).

For analysis, we may represent S^i and D^i with additional nodes and edges. For example, for a renewable supply s with $Q_s(z) = 0$ for $z \leq \tau$ and ∞ otherwise, we create a new node s' with zero production cost and connect it to s with an edge of capacity τ. Similarly, for a demand d with, say, a linear utility function up until a threshold τ, we create d' and connect it to d with an edge of capacity τ. We treat supplies and demands on the same node across different time steps separately as independent supplies and demands.

Private and Public Storage Units

We consider two main storage unit settings. In the private storage setting, we have dedicated storage units co-located with power supplies such that each supply node can use only its own storage unit, if it has one[10]. In this setting, only renewable sources are considered. In the public storage setting, we have networked storage units that can store energy from any

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7 For many real-life applications, we model $Q_s(z) = a_s z^2 + b_s z$ for $a_s, b_s > 0$ when considering just the traditional power generators.

8 In real-life applications, Δ_w describes the difference between the marginal cost of generation for traditional sources over time, and $\log \Delta_w$ is often a small constant in the scope of power management problems.

9 Suppose a storage node r is subject to charging/discharging inefficiency factors (γ_+/γ_-) and ramping constraints (τ_+/τ_-). In our model, we would add a new node r' and the following edges: an edge (r, r') with capacity τ_+ and decay factor γ_+, and an edge (r', r) with capacity τ_- and decay factor γ_-. Node r becomes an ordinary connection node and node r' becomes a storage node with the same operation characteristics as the original r.

10 This corresponds to the important practical case where the storage is co-located with some renewable generation source and is used to smooth the output of random renewable generation [8]. Given the physical size of the inverter and other equipments in such settings, it is natural to assume that power always flows from the generation sites (with both renewable generation and storage) to the grid but not in the other direction.

power supplies (including traditional sources) with a routing path. Consequently, a unit of power can be stored on multiple storage units over time.

Competitive Analysis

In the online paradigm, supply set S^i arrives at each time step i. Demand set D^i may be given online or offline depending on the storage unit setting. Upon the arrival of S^i, we need to dispatch power from supplies in S^i and storage units to demands in D^i, and possibly to other storage units. To evaluate the performance of online algorithms, we use the standard competitive analysis framework. An *online* algorithm is *c-competitive* if its achieved objective value ALG is at least $\frac{1}{c}$ fraction of the optimal *offline* algorithm's value OPT (which knows the entire input in advance) up to an additive constant in all problem instances, i.e., there exist c and a constant c_0 such that $\mathsf{ALG} \geq \frac{1}{c}\mathsf{OPT} - c_0$.

3 Private Storage Units

In this section, we consider the private storage setting in which the storage units used by a supply node are located at that same node. Our goal is to maximize the total utility of satisfied demands using only the renewable power supplies. As discussed in Section 2, we assume, without loss of generality, that the renewable supplies S^i have unbounded zero-cost power generation.

We show strong competitive guarantees for the private storage setting given that both the supply nodes S^i and the demand nodes D^i are arriving online. In Section 3.1, we analyze an intuitive greedy algorithm in the simplified case with uniform utility functions where all demands have the same linear utility function $W(y) = y$ up to the limit $y = 1$, i.e., each demand requires at most 1 unit of power. We show that the greedy algorithm has a constant competitive ratio against the optimal offline algorithm which knows all the supplies and demands in advance.

In contrast to the uniform utility case, we can show that the greedy algorithm is not competitive when the utility functions are arbitrary concave functions. In Section 3.2, we show that we can still design a competitive algorithm for the concave utility case using the constant competitive greedy algorithm as a black-box. More precisely, we show that an algorithm with the competitive ratio c for the uniform utility case can be modified to obtain an algorithm with the competitive ratio of $O(c \cdot \log \Delta_w)$ for the general concave utility case.

3.1 Uniform Utilities

We consider the following primal-dual linear program formulation of the problem with private storage units. Assume that every supply node $s \in S^i$ is connected to a private storage unit r^s. For time step i and demand node $d \in D^i$, let $P^+(i,d)$ denote the set of all paths ending at d and starting from a supply node or a storage node. For time step i and storage node r, let $p^+(i,r)$ denote the path ending at r and starting from the supply s which owns the storage node. Let $P^-(i,r)$ denote the set of paths from r to a demand node in D^i. Note that since the storage units are co-located with supplies, the paths $p^+(i,r)$ are edge-disjoint. We define $P_D^+ = \bigcup_{i,d \in D^i} P^+(i,d)$, $P^+ = P_D^+ \cup \bigcup_{i,r} \{p^+(i,r)\}$, and $P^- = \bigcup_{i,r} P^-(i,r)$.

For a path p, the variable x_p denotes the amount of flow passing through the path. Furthermore, for a path $p \in P^+$ the parameters $i(p)$ and $v(p)$ denote the time step in which p lies and the node at the beginning of p. For a storage node r and a time step i, y_r^i denotes the amount of power stored in the node r at the *end* of time step i.

The first set of primal constraints ensure that the total flow routing through an edge does not violate the edge capacity. The second set of constraints bounds the amount of energy in storage units. The last set of primal constraints ensures the feasibility of flow over time: we may assume that a storage node sends the stored energy to the next time step and receives energy from the previous time step. Intuitively, the last constraint implies that the total outgoing flow cannot be more than the total incoming flow.

$$\begin{aligned}
&\max \sum_{p \in P_D^+} x_p && (\mathbb{P}) \\
&\forall e \quad \sum_{p \ni e} x_p \le C_e && (\alpha_e) \\
&\forall i, r \quad y_r^i \le L_r && (\beta_r^i) \\
&\forall i, r \quad y_r^i + \sum_{p \in P^-(i,r)} x_p \le \gamma_r y_r^{i-1} + x_{p^+(i,r)} && (\tau_r^i) \\
&y_r^0 = 0; \quad x_p, y_r^i \ge 0
\end{aligned}$$

$$\begin{aligned}
&\min \sum_e C_e \alpha_e + \sum_{i,r} L_r \beta_r^i && (\mathbb{D}) \\
&\forall p \in P_D^+ \quad \sum_{e \in p} \alpha_e + \tau_{v(p)}^{i(p)} \ge 1 && (\text{D1}) \\
&\forall i, r \quad \sum_{e \in p^+(i,r)} \alpha_e \ge \tau_r^i && (\text{D2}) \\
&\forall i, r \quad \beta_r^i + \tau_r^i \ge \gamma_r \tau_r^{i+1} && (\text{D3}) \\
&\alpha, \beta, \tau \ge 0; \forall v \in S^i : \tau_{i,v} = 0
\end{aligned}$$

Greedy Algorithm

Upon the arrival of S^i and D^i, we route the maximum power flow from S^i and the storage units to the demand nodes in D^i. We store the residual power generated at a supply in its private storage unit, up to the storage capacity and edge capacities on the path from the supply to the storage. Recall that based on the simplifications in our model, there can be two edges on the path from a supply to its dedicated storage unit.

Dual Construction

We now use a dual-fitting argument to show that the greedy algorithm is indeed constant competitive. For a time step i, let $P^+(i) := \bigcup_{d \in D^i} P^+(i, d)$. After the run of the greedy algorithm, consider the flow paths $\hat{x}$ and the storage amounts $\hat{y}$ chosen by the greedy algorithm, corresponding to the primal solution. Let $\hat{x}^i$ denote the selected flow at time step i. Furthermore, let $\hat{x}(D^i)$ denote the selected flow that satisfies the demands in D^i. We note that by definition, $\hat{x}(D^i)$ is a sub-flow of $\hat{x}^i$. In what follows, we say a flow f on a graph separates a vertex u from v, if f cannot be augmented by a u–v flow in the residual network. We now describe the construction of our dual solution. We then show the feasibility of the dual solution and the bound on its objective.

DC Part I. Let R_ϕ^i denote the set of storage nodes that are *not* separated from D^i by $\hat{x}(D^i)$. We choose a minimum cut separating $S^i \cup (R \setminus R_\phi^i)$ from D^i. For every edge e in this cut, we set $\alpha_e = 1$.

DC Part II. We repeat the following process for every j by starting from $j = T$ and ending at $j = 1$: For every $r \in R_\phi^j$, set $\tau_r^j = 1$. Let i denote the last step before j for which $\hat{y}_r^i = L_r$ in the greedy solution. If no such i exists, set $i = 0$. For $k = \{j-1, j-2, \dots, i+2, i+1\}$, set the dual variable $\tau_r^k = \gamma_r \tau_r^{k+1}$. Furthermore, if $i > 0$ set $\beta_r^i = \gamma_r \tau_r^{i+1}$. We note that in this process, we may reassign values multiple times to τ_r^i for some i and r[11]. Hence, it is important to iterate j from T to 1.

[11] In fact, the assigned values can only be non-decreasing.

DC Part III. For every $i \in [T]$ and storage node r, if the path $p^+(i, r)$ is congested in $\hat{x}^i$, we choose a congested edge $e \in p^+(i, r)$. If α_e is not set in Part I, we set $\alpha_e = \tau_r^i$.

We now need to prove the feasibility of the constructed dual vector. See Appendix C for proofs.

▶ **Lemma 7.** *The dual vector $\langle \alpha, \beta, \tau \rangle$ constructed by DC-Parts I-III is feasible.*

It now remains to prove a bound on the dual objective. Let GA denote the total primal value of the greedy solution.

Proof of Theorem 1. We analyze the cost we incur at every part of the dual construction separately. We show that the dual cost in each part can be upper bounded by GA.

For a flow F, let $|F|$ denote the amount of flow routed by F. Recall that for a flow F on a graph, if there is no augmenting path from a set U to a set V (i.e., F separates U from V), then the size of the minimum cut separating U and V is upper bounded by $|F|$. Now consider $\hat{x}(D^i)$ for some i. By definition, $R \setminus R_\phi^i$ is separated by $\hat{x}(D^i)$ from D^i. Furthermore, since the greedy algorithm chooses a maximum flow, we know that $\hat{x}^i(D^i)$ cannot be augmented by a path from a supply node in S^i to a demand node. Therefore $\hat{x}(D^i)$ is separating $S^i \cup (R \setminus R_\phi^i)$ from D^i. The total dual cost of the minimum cut we choose in Part I for some i is upper bounded by $|\hat{x}^i(D^i)|$. Hence, the total cost we incur in Part I is bounded by $\sum_i |\hat{x}^i(D^i)| \leq \text{GA}$.

The cost we incur in Part II is $\sum_{i,r} L_r \beta_r^i$. The variable β_r^i is positive only if r is full at time i, i.e., $\hat{y}_r^i = L_r$. By Lemma 14, we know that the stored power will be used in at most $\log_{\gamma_r}(\beta_r^i)$ steps. Therefore the primal gain from the power stored at time step i in r is at least $\gamma_r^{\log_{\gamma_r}(\beta_r^i)} \hat{y}_r^i = L_r \beta_r^i$. Furthermore, suppose for two time steps $i_2 > i_1$, we have that $\beta_r^{i_1} > 0$ and $\beta_r^{i_2} > 0$. By Part II of the construction, we know that r should become empty for some time step $j \in \{i_1 + 1, \ldots, i_2 - 1\}$. Thus the power stored in r in time steps i for which $\beta_r^i > 0$, are disjoint. Therefore now we can charge the cost $\sum_{i,r} L_r \beta_r^i$ to the utility we gain from dispatching power stored in the storage units at time steps in which $\beta_r^i > 0$.

The cost we incur in Part III can be upper bounded as follows. Suppose $\tau_r^i > 0$ for some i and r. The cost we incur is $\tau_r^i C_e$ for some congested edge $e \in p^+(i, r)$. We note that the flow passing through e is either satisfying a demand in D^i, or it is being stored in r. In the former, the utility we gain is 1 per unit of power. In the latter, by Lemma 14, we gain utility at rate at least $\gamma_r^{\log_{\gamma_r}(\tau_r^i)} = \tau_r^i$. Therefore our primal gain from the flow routed on e is at least $C_e \tau_r^i$. Now since the storage nodes are co-located, all the paths $p^+(i, r)$ are disjoint. Therefore we can bound the total cost incurred in Part III by GA.

Finally summing over the three parts, we have that the dual objective is at most $3\,\text{GA}$. The theorem follows from Lemma 7 and weak duality. ◀

3.2 Concave Utilities

In this section, we demonstrate a simple reduction from the variant with concave utility functions to the variant with uniform linear functions losing only a logarithmic factor. We then use the algorithm of Theorem 1 as a blackbox to solve the case of concave utilities and obtain the competitive ratio of $O(\log \Delta_w)$.

Proof Sketch of Theorem 2. The main idea is to reduce an instance of the problem with concave utility functions to $O(\log \Delta_w)$ instances of the problem with uniform utility functions. Since ${W_d}'(x) \in [w_{\min}, w_{\max}]$ for each demand node d, we can construct $O(\log \Delta_w)$ instances where each ${W_d}'$ is approximately constant within each instance. Then, we solve each instance independently using a constant competitive algorithm. Due to the space constraints, we defer the complete proof to Appendix C. ◀

4 Public Storage Units

We consider the general setting with public storage units on the network. A supply node can access any storage unit as long as there is a path, and, consequently, a unit of power can be stored on multiple storage units over time. As Theorem 3 shows online demands are hard to satisfy (see Appendix D for proof), we focus on the offline demands in this section. Note offline demands naturally model scenarios where consumer demands', say, hourly variation is predictable.

We investigate two specific cases: the case of renewable power generation in Section 4.1 and the case of combined traditional and renewable power generation in Section 4.2. In the first case, we want to route power from renewable supplies to demands assuming the production cost is zero. In the second case, we still route power but the supplies are equipped with both traditional and renewable power generators and have time-varying convex production costs. Supplies are arriving online while demands are given offline and have concave utility functions.

We design poly-logarithmic competitive online algorithms using the online primal dual method on convex programming formulations. Our approach is closely related to Buchbinder and Naor [12] and Devanur and Jain [18] and more recent work on online covering and packing problems with convex objectives in [14, 11, 4, 22]. For analysis, we use the following bicriteria notion of competitive algorithms: An algorithm is (c_1, c_2)-*competitive* if it routes the total flow of amount at least $\frac{1}{c_1}$ of the optimal and the load on each edge is at most c_2, where the *load* of an edge is the ratio of the total flow going through it divided by its capacity. Ideally, the total bandwidth allocated for flows should not exceed the capacity.

Time-Expanded Graph

We use *time-expanded graph* $G^* = (V^*, E^*)$ constructed as follows. For $i = 1, \dots, T$, we create a time-copy of G, $G^i = (V^i, E^i)$. To represent storage, we create *storage edges* between time-copies of G; for each node v and time step i, we create an edge of capacity L_v from v^i to v^{i+1}. Let $S^* = \bigcup_i S^i$ and $D^* = \bigcup_i D^i$. Instead of creating a node for each individual demand as in Section 2, we add a single node d^* as the designated super-demand. For each $i \in [T]$ and demand $d \in D^i$, we add an edge of infinite capacity from d^i to d^*; these are *demand edges* and we use D^* to denote both the demand edges and corresponding demands.

For $s \in S^*$ and $d \in D^*$, let $P(s, d)$ be the set of simple paths in G^* from supply s to demand d. Let $P(s, \cdot) = \bigcup_d P(s, d)$, $P(\cdot, d) = \bigcup_s P(s, d)$, and P be the set of all simple paths from supplies to demands. For a routing path p, we denote the corresponding supply and demand nodes by $s(p)$ and $d(p)$, respectively; we simply use s and d if p is clear from the context. For simplicity, we assume a single decay factor γ for all storage units. We define:

$\gamma(p) :=$ overall decay due to storage edges on p;

$\gamma(p, e) :=$ overall decay due to storage edges on p preceeding edge $e \in p$.

In our case, $\gamma(p) = \gamma^{(\text{number of storage edges on } p)}$ and $\gamma(p, e) = \gamma^{(\text{number of storage edges on } p \text{ before } e)}$. Let $l_{\max} = nT$ be the maximum routing path length, and $\gamma_{\min} = \gamma^T$ be the greatest overall decay.

Algorithm 1 Online Algorithm for Concave Utilities

1: Let $y_d = \sum_{p \in P(\cdot,d)} \gamma(p) x_p, \forall d.$ ▷ y determined in terms of x
2: **for** $i = 1, \dots, T$ **do**
3:     **for** $s \in S^i$ in arbitrary order **do**
4:         **while** $P' = \{p \in P(s,\cdot) : \sum_{e \in p} \gamma(p,e)\alpha_e < \gamma(p) {W_{d(p)}}'(\lambda_{d(p)})\}$ is not empty **do**
5:             Update continuously:
6:                 $p = \arg\max_{p \in P'} \gamma(p) {W_{d(p)}}'(\lambda_{d(p)})$
7:                 $dx_p = 1$ ▷ Increase x_p at a uniform rate
8:                 $\frac{d\lambda_{d(p)}}{dx_p} = \gamma(p)$
9:                 $\frac{d\alpha_e}{dx_p} = c\left(\frac{\gamma(p,e)\alpha_e}{C_e} + \frac{\gamma(p) {W_{d(p)}}'(\lambda_{d(p)})}{l_{\max} C_e}\right), \forall e \in p$ ▷ $c \geq 1$ is some parameter
10:         **end while**
11:     **end for**
12: **end for**

4.1 Concave Utilities

In this section, we consider only renewable power generation with zero production cost and route power from supplies to demands. We model the demand nodes' utility functions with monotonically nondecreasing concave functions W_d such that demand node d gains the utility of $W_d(f)$ for receiving f units of flow. There is diminishing returns on each additional flow routed.

We consider the following primal-dual convex program formulation. For each path p, x_p denotes the amount of flow passing through the path. For demand d, y_d denotes the total amount of flow routed to the demand and is set to equal $y_d = \sum_{p \in P(\cdot,d)} \gamma(p) x_p$. W_d is a monotonically nondecreasing concave function and we define $\widehat{W}_d(\lambda) := W_d(\lambda) - \lambda {W_d}'(\lambda)$.

$$\begin{array}{llcll}
\max & \sum_d W_d(y_d) & \qquad & \min & \sum_e C_e \alpha_e + \sum_d \widehat{W}_d(\lambda_d) \\
\forall e & \sum_{p \ni e} \gamma(p,e) x_p \leq C_e & & \forall p \in P & \sum_{e \in p} \gamma(p,e)\alpha_e \geq \gamma(p) {W_{d(p)}}'(\lambda_{d(p)}) \\
\forall d \in D^* & y_d \leq \sum_{p \in P(\cdot,d)} \gamma(p) x_p & & & \alpha, \beta, \lambda \geq 0 \\
 & x, y \geq 0 & & &
\end{array}$$

We first prove Algorithm 1 is bicriteria competitive.

▶ **Lemma 8.** *For any $c \geq 1$, Algorithm 1 is $\left(2c+1, \frac{O(\log n + \log \gamma^{-T} + \log \Delta_w)}{c}\right)$-competitive.*

Proof. We show that Algorithm 1 returns a primal solution that violates the edge capacity constraints by some factor and a feasible dual solution. From the ratio of the primal and dual objective values, $\mathcal{P}$ and $\mathcal{D}$, we obtain the stated competitive ratio.

For a supply s, P' in Line 4 is nonempty if there is a violated dual constraint. As long as P' is nonempty, we continuously increase x_p and dual variables correspondingly. Since all variables increase monotonically and the first derivatives ${W_d}'$ are monotonically non-increasing, violated dual constraints for $p \in P(s,\cdot)$ eventually become satisfied and no previously satisfied constraints become violated. Hence, the while loop terminates with no violated dual constraints for $p \in P(s,\cdot)$, and the algorithm terminates with a feasible dual solution.

The lemma follows from the following claims (proved in Appendix D) and the weak duality:

▶ **Claim 9.** *The load on each edge is at most* $\frac{O(\log n+\log\gamma^{-T}+\log\Delta_w)}{c}$ *in the primal solution.*

▶ **Claim 10.** *At termination,* $\mathcal{D} \leq (2c+1)\cdot\mathcal{P}$. ◀

We can now prove Theorem 4. For checking the condition and choosing path p in Lines 4–6 in Algorithm 1, we can use a backward variant of Dijkstra's algorithm in polynomial time (details in Appendix D):

Proof of Theorem 4. We choose $c = c' \cdot (\log n + \log\gamma^{-T} + \log\Delta_w)$ for the constant c' that results from the analysis in Lemma 8. Then, we get an $\big(O(\log n + \log\gamma^{-T} + \log\Delta_w), 1\big)$-competitive algorithm. ◀

4.2 Concave Utilities and Convex Costs

We consider the more general case where each supply is equipped with both renewable and traditional power generators. Each supply node s individually manages its own renewable and traditional power generation and only pays cost for using the traditional generators. It generates power according to the production cost function $Q_s(z) = a_s z^{\rho_s}$ which changes from time to time depending on the renewable power generation. The production cost functions are given online.

We consider the following primal-dual convex program formulation. For each path p, x_p denotes the amount of flow passing through the path. For demand d, y_d denotes the total amount of flow routed to the demand and is set to equal $y_d = \sum_{p\in P(\cdot,d)}\gamma(p)x_p$. For supply s, z_s denotes the total power generated using the traditional power generators and is set to equal $z_s = \sum_{p\in P(s,\cdot)} x_p$. We define $\widehat{W}_d(\lambda) := W_d(\lambda) - \lambda {W_d}'(\lambda)$ and Q^* is the convex conjugate of Q defined to be $Q^*(\mu) := \sup_{z\geq 0}\{\mu z - Q(z)\}$.

$$
\begin{array}{llll}
\max & \sum_d W_d(y_d) - \sum_s Q_s(z_s) & \min & \sum_e C_e\alpha_e + \sum_d \widehat{W}_d(\lambda_d) + \sum_s Q_s^*(\mu_s) \\
\forall e\in E & \sum_{p\ni e}\gamma(p,e)x_p \leq C_e & \forall p\in P & \sum_{e\in p}\gamma(p,e)\alpha_e + \mu_{s(p)} \geq \gamma(p){W_{d(p)}}'(\lambda_{d(p)}) \\
\forall d\in D^* & \sum_{p\in P(\cdot,d)}\gamma(p)x_p \geq y_d & & \alpha,\lambda,\mu\geq 0 \\
\forall s\in S^* & \sum_{p\in P(s,\cdot)} x_p \leq z_s & & \\
 & x,y,z\geq 0 & &
\end{array}
$$

We show that Algorithm 2, given in Appendix D, is poly-logarithmic competitive. For concreteness, we prove Theorem 5 with $\rho_0 = 1.44$. The constant ρ_0 can be replaced with a smaller constant arbitrarily close 1 and with a correspondingly large multiplicative constant in the competitive ratio. We first prove the following lemma:

▶ **Lemma 11.** *For any* $c \geq 1$*, Algorithm 2 (in Appendix D) is* $\left((2c+1)\rho^{\rho/(\rho-1)} + 1, \frac{O(\log n+\log\gamma^{-T}+\log\Delta_w)}{c}\right)$*-competitive where* $\rho = \max_s \rho_s > 1.44$.

Proof. The proof is similar to that of Lemma 8. We need to prove claims about the load and competitive ratio. Analyzing the competitive ratio is more difficult because of the convex cost functions and requires a different approach.

The lemma follows from the following claims (with complete proofs in Appendix D) and the weak duality.

▶ **Claim 12.** *The load on each edge is at most* $\frac{O(\log n+\log\gamma^{-T}+\log\Delta_w)}{c}$ *in the primal solution.*

▶ **Claim 13.** *At termination,* $\mathcal{D} \leq \left((2c+1)\rho^{\rho/(\rho-1)}+1\right)\cdot\mathcal{P}$.

Proof Sketch of Claim 13. Let $\mathcal{D}_0 = \sum_e C_e\alpha_e + \sum_s Q_s^*(\mu_s)$; so, $\mathcal{D} = \mathcal{D}_0 + \sum_d \widehat{W}_d(\lambda_d)$. Assume $\mathcal{D}_0 \leq (2c+1)\rho^{\rho/(\rho-1)}\mathcal{P}$. Since $\widehat{W}_d(z) \leq W_d(z), \forall z \geq 0$, it would follow that $\sum_d \widehat{W}_d(\lambda_d) \leq \sum_d W_d(y_d) \leq \mathcal{P}$. Then, $\mathcal{D} = \sum_d \widehat{W}_d(\lambda_d) + \mathcal{D}_0 \leq \left((2c+1)\rho^{\rho/(\rho-1)}+1\right)\mathcal{P}$, and the claim would follow.

To show $\mathcal{D}_0 \leq (2c+1)\rho^{\rho/(\rho-1)}\mathcal{P}$, we prove $d\mathcal{P} \geq \frac{1}{\sigma}d\mathcal{D}_0$ for $\sigma = (2c+1)\rho^{\rho/(\rho-1)}$ and $\rho > 1.44$ when we route power. This reduces to showing

$$\left(1-\frac{2c}{\sigma}\right)\mu_s - {Q_s}'(z_s) \geq \frac{1}{\sigma}(Q_s^*)'(\mu_s)\frac{d\mu_s}{dz_s}\ ,$$

which is satisfied by our choice of updates to primal and dual variables. For a ρ_0 constant smaller than 1.44, we would need to have a multiplicative factor greater than $2c+1$. Due to the space constraints, we defer the complete proof to Appendix D. ◀

◀

We can now prove Theorem 5. For the path construction routine in Algorithm 2, we find the routing paths in the same manner as in Section 4.1:

Proof of Theorem 5. Let $\rho = \max_s \rho_s$. We choose $c = c'\cdot(\log n + \log\gamma^{-T} + \log\Delta_w)$ for the constant c' that results from the analysis in Lemma 11. Then, we get an $\left(O\left(\rho^{\rho/(\rho-1)}(\log n+\log\gamma^{-T}+\log\Delta_w)\right), 1\right)$-competitive algorithm. ◀

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A From Electric Power Flow to Network Flow

Bulk electric power grids are operated with Alternating current (AC) power flow, where the physical quantities of interest, such as voltage, current and power, are sinusoidal signals of time. Economic and market operations of the grid are usually solved and scheduled for a slow timescale, such that the controls or set points for the system are modified at a frequency of every 5 minutes or lower. Consequently the sinusoidal signals have stabilized into a steady state for almost all time points in each time slot, and are characterized by constant-frequency sinusoidal waveforms and permit a *phasor representation* [20, Section 2.1]. The AC voltage is then represented as a phasor, denoted by $\mathcal{V}\exp(\mathbf{i}\theta) \in \mathbb{C}$, where $\mathcal{V} \in \mathbb{R}_+$ is the (root-mean-square) voltage magnitude, $\theta \in \mathbb{R}$ is the voltage phase angle, and $\mathbf{i} = \sqrt{-1}$. Provided that current has a similar phasor representation, the resulting power is a complex number, whose real part is referred to as *real power* $\mathcal{P}$ and imaginary part is referred to as *reactive power* $\mathcal{Q}$. Intuitively speaking, real power can be thought of as the actual power that serves the load, while reactive power is a consequence of the phase difference between the current and voltage phasors. Thus the majority of the economic aspect of the grid operation has centered around real power, with reactive power considered often for other purposes such as power quality.

Given an electric grid represented by a graph $G = (V, E)$ consisting of a set V of buses and a set E of lines connecting the buses, the *AC power flow equation* is a set of $2|V|$ nonlinear equations relating the voltage phasors $(\mathcal{V}_v, \theta_v)$ at each bus $v \in V$ and the corresponding complex powers $(\mathcal{P}_v, \mathcal{Q}_v)$, $v \in \mathcal{V}$. Together with additional operational constraints such as line capacity and suitable objectives, one can formulate corresponding optimization problems for the set points of the controllable devices on the grid. In general, such optimization problems are often referred to as *AC optimal power flow* (AC-OPF) problems. Given the nonlinear nature of the AC power flow equations, AC OPF is in general nonconvex and NP-hard [28, 29], so that practical solutions have relied on approximation of the AC power flow equations.

The most widely adopted approximation for bulk electric power networks (transmission networks) is a particular linearization of the AC power flow equations referred to as DC approximation to AC power flow [38]. Assuming that i) the voltage magnitudes over all buses are held constant, ii) all lines are purely inductive (i.e., there is no real power losses due to resistance), and iii) voltage phase differences between buses are small, we can obtain a linear relationship between the nodal real power injections $\mathcal{P}_v$, $v \in V$, and the voltage phase angles θ_v, $v \in V$. In particular, the real power flow through any line $e = (v, u) \in E$ can be written as

$$e \in E, \quad f_e = \mathcal{B}_e(\theta_v - \theta_u), \tag{1}$$

where $\mathcal{B}_e$ is the reciprocal of the reactance of line e. Consequently, by flow conservation on node $v \in V$, we have

$$v \in V, \quad \mathcal{P}_v = \sum_{e=(v,u)} \mathcal{B}_e(\theta_v - \theta_u).$$

Therefore, the differences of bus voltage phase angles θ_v, $v \in V$ determine all the line flows and nodal power injections, and hence fully characterize the operation condition of the system under the DC approximation. As phase angle differences are invariant to a constant shift, we can without loss of generality set the phase angle at a bus v_0, called slack bus, to be 0 and work with the remaining phase angles θ_v, $v \in V \backslash \{v_0\}$. When the flow capacities of lines are considered, together with the fact that the nodal power injection at certain buses may not be controllable, the feasible set of phase angles is then characterized by the constraints

$$v \in V^{\text{Fix}}, \quad g_v - d_v = \sum_{e=(v,u)} \mathcal{B}_e(\theta_v - \theta_u), \tag{2}$$

$$e \in E, \quad \mathcal{B}_e(\theta_v - \theta_u) \leq C_e, \tag{3}$$

where $V^{\text{Fix}} \subset V$ is the set of buses only connected to uncontrollable devices, g_v is the uncontrollable generation at bus v, d_v is the uncontrollable demand at bus v, and C_e is the real power capacity of line e.

To convert the formulation in (2) and (3) in terms of phase angles θ_v, $v \in V$, into a network flow formulation which uses the flows f_e, $e \in E$, as the variables, we observe that, using (1), (2) and (3) can be written as

$$v \in V^{\text{Fix}}, \quad g_v - d_v = \sum_{e=(v,u)} f_e, \tag{4}$$

$$e \in E, \quad f_e \leq C_e. \tag{5}$$

This set of constraints, however, in general is not an equivalent formulation to (2) and (3) as not every collection of flows f_e, $e \in E$ that satisfies these equations will induce a feasible collection of phase angles θ_v, $v \in V$. In particular, when $V^{\text{Fix}} = \emptyset$, we know that the set of feasible θ_v, $v \in V$ lives in an affine subspace of dimension $|V| - 1$. However for general graph with $|E| > |V| - 1$, the set of feasible flows is of dimension $|E|$. Consequently the mapping between f_e, $e \in E$ and θ_v, $v \in V$, defined by (1) is not one-to-one. This mapping would indeed be one-to-one if the graph G is a tree. In this case, it is easy to check that (4) and (5), which is the standard *network flow* constraints, equivalently characterize the set of feasible operation conditions of the system under the DC approximation.

B Hardness of Networks with Cycles

In this section, we show a hardness result for general instances of OESMP when congested transmission lines can form a cycle in the network. We assume the general network flow model described in Appendix A and the network flow formulation given by (4)-(5) where C_e is the capacity and $\mathcal{B}_e$ is the susceptance, the reciprocal of the reactance, of edge e.

Proof of Theorem 6. We prove the lower bound by constructing a hard example. We first consider the network in Figure 1 and then construct a related network with identical edges with the same susceptance and capacity. In the network in Figure 1, the edges are bidirectional and have the same susceptance of (sufficiently large) $\mathcal{B}$ but different capacities, 1 or M, as indicated.

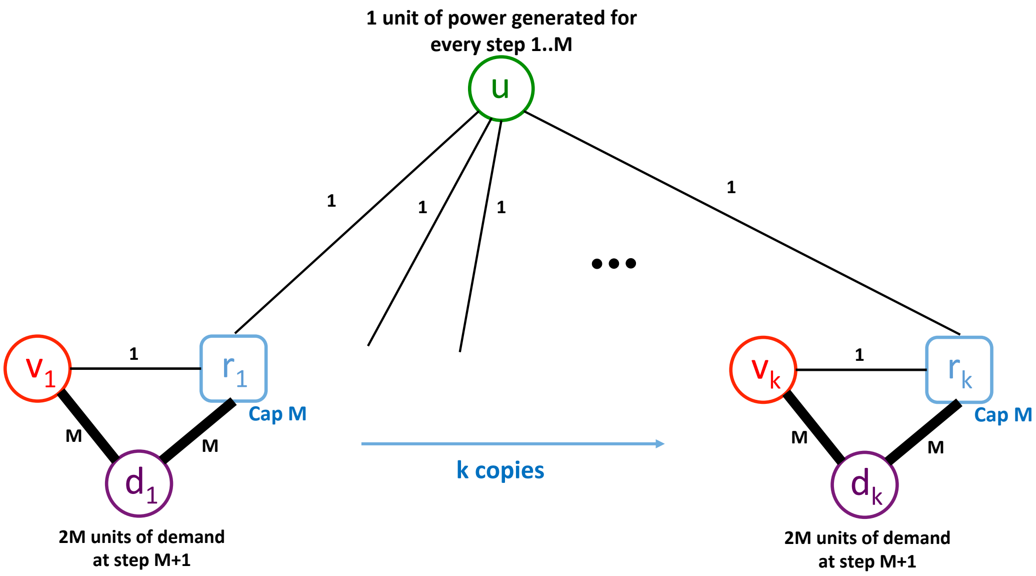

Figure 1 A hard example for general electrical networks with online supplies.

Let M be a large integer and let $k = \lfloor\sqrt{M}\rfloor$. In each time step $i = 1, \ldots, M$, one unit of power is generated at supply node u. The power can be stored in any of the storage units at $r_1, \ldots, r_k$ with capacity M. At time $i = M + 1$, there are $2M$ units of demand at all the demand nodes $d_1, \ldots, d_k$. The demand nodes' utility functions are linear and uniform.

Consider an arbitrary randomized online algorithm. The algorithm distributes the first M units of power onto the storage units. Let r_j be the storage unit with the least expected amount of power stored at the beginning of time step $i = M + 1$. Note the expected amount of power stored at r_j is at most $\frac{M}{k} = O(\sqrt{M})$. At time step $i = M + 1$, we assume M units of power at supply v_j and zero unit at all other supplies. Then, the algorithm can route at most $O(\sqrt{M})$ units of power in total to the demand nodes. The algorithm routes $O(\sqrt{M})$ units of power to demand d_j. Since the voltage phase angle difference between v_j and r_j can be at most 1, due to the edge of capacity 1 between them, the algorithm routes $O(\sqrt{M})$ units of newly generated power from v_j to d_j and $O(\sqrt{M})$ units of stored power from r_j to d_j. Similarly, the algorithm further routes at most 2 units of power to other demand nodes.

On the other hand, the optimal offline algorithm routes $2M$ units of power by first storing the M units of generated power from u on r_j and then routing to d_j. The resulting competitive ratio is $\Omega(\sqrt{M})$.

We construct a network with identical edges with the same susceptance and capacity. Note that we can model an edge of capacity $2c$ and susceptance $\mathcal{B}$ using four edges of capacity c and susceptance $\mathcal{B}$ arranged in the diamond shape.[12] We recursively use the diamond construction $\log M$ times to reduce the edges of capacity M to edges of capacity 1. Per an edge of capacity M, we get $4^{\log M}$ new edges and $\Theta(4^{\log M}) = \Theta(M^2)$ new nodes. Therefore, this network has $\Theta(M^{5/2})$ nodes and the lower bound becomes $\Omega(n^{1/5})$ where n is the number of nodes. ◀

[12] In the diamond shape, the four edges are arranged as (a, b), (b, d), (a, c), and (c, d).

C Missing Materials from Section 3

We provide the missing proofs from Section 3.

C.1 Uniform Utilities

We prove Lemma 7 below. We first prove the following structural lemma:

▶ **Lemma 14.** *Consider an arbitrary storage node r for which $\tau_r^i > 0$ or $\beta_r^i > 0$ at a time step i. Then, all the power stored in r at time i will be dispatched to the demands in no more than $\log_{\gamma_r}(\max(\beta_r^i, \tau_r^i))$ time steps. Furthermore, if $\tau_r^i > 0$, then the path $p^+(i,r)$ is congested in $\hat{x}^i$.*

Proof. Let $j \geq i$ denote the first step after i for which $r \in R_\phi^j$. By the recursive construction in Part II, we know that $\max(\beta_r^i, \tau_r^i) = \gamma_r^{j-i}$ and one of variables is zero. On the other hand, since $r \in R_\phi^j$, congestion does not block all the routes from r to D^j. Since we have not routed more flow from r to D^j at that time step, we know that $\hat{y}_r^j = 0$. This indeed proves the first part of the lemma. Furthermore, if $\tau_r^i > 0$, we know that r never gets full between time steps i to j, otherwise τ_r^i would have been zero. Thus the reason that the supply $s \in S^i$, which owns r, is not supplying r with more power is that $p^+(i,r)$ is congested; which completes the proof. ◀

Proof of Lemma 7. We check the feasibility of each set of dual constraints in the program $\mathbb{D}$ separately. Consider an arbitrary path $p \in P_D^+$ at time interval i. Constraint D1 enforces a lower bound of one on $\sum_{e \in p} \alpha_e + \tau_{v(p)}^{i(p)}$. We distinguish between two cases. If $v(p) \in R_\phi^i$, the constraint is satisfied in DC-Part II by setting $\tau_{v(p)}^i = 1$. Otherwise, $v(p) \in S^i \cup (R \setminus R_\phi^i)$, for which DC-Part I satisfies the constraint.

The feasibility of D2 constraints follows from Lemma 14; which shows that if $\tau_r^i > 0$ for some i and r, then $p^+(i,r)$ is congested. Therefore DC Part III satisfies D2.

Finally, the feasibility of D3 constraints follows directly from the iterative construction in Part II. ◀

C.2 Concave Utilities

Proof of Theorem 2. Under the concave utility model, scaling down all the storage and edge capacities by a factor ρ may only change the optimal solution by at worst a factor $1/\rho$. Recall that we assume $\frac{dW_d(x)}{dx} \in [w_{\min}, w_{\max}]$ for every demand. Let $\rho = \lfloor \log \Delta_w \rfloor + 1$ where $\Delta_w = \frac{w_{\max}}{w_{\min}}$. Given an instance of the general flow problem with concave utility functions and optimal solution OPT^*, we construct ρ instances of the problem with uniform utility as follows.

For every i and $d \in D^i$, let $\ell_d(q) = \inf\{x : \frac{dW_d(x)}{dx} \leq 2^q w_{\min}\}$ for every integer $q \in \{0, \ldots, \rho\}$. We note that ℓ_d is monotone non-increasing. For every $q \in \{1, \ldots, \rho\}$, we construct an uniform-utility instance in which a new node d' is connected to d with an edge (d, d') with capacity $\ell_d(q) - \ell_d(q-1)$. The node d is not a demand node anymore and instead d' is a demand node. Furthermore, we scale down all the storage and edge capacities by a factor of ρ. Let OPT_q denote the optimal solution to the q^{th} instance.

▶ **Claim 15.** $\frac{\mathsf{OPT}^*}{2\rho} \leq \sum_{q=1}^{\rho} 2^{q-1} w_{\min} \mathsf{OPT}_q \leq \mathsf{OPT}^*$.

Proof. We partition the optimal solution into ρ separate flows of power. Suppose that at every step of the algorithm, the flows of power are routed continuously. For $q \in [\rho]$, let F_q

denote the flows corresponding to units of power that are ending up satisfying a demand d during the interval that the total power received by d is between $[\ell_d(q), \ell_d(q-1))$. We note that

$$\frac{\mathsf{OPT}^*}{2} \leq \sum_q 2^{q-1} w_{\min} |F_q| \leq \mathsf{OPT}^* \ .$$

Let F'_q denote the flow obtained from F_q by reducing the flows on every edge and storage unit by a factor ρ. We note that F'_q is indeed a feasible solution for the q^{th} instance of the problem constructed above: we simply need to re-route the flow coming to d to d'. This completes the proof since $\mathsf{OPT}_q \geq |F'_q| \geq \frac{|F'_q|}{\rho}$ ◀

Now suppose we have an online algorithm with competitive ratio c for the uniform-utility model. Given a general instance, for every $q \in \{1, \ldots, \rho\}$, we separately execute the uniform weight algorithm on the q-th instance of the problem constructed as above. Let ALG_q denote the solution corresponding to the q-th instance. Since all capacities are scaled down in each instance, we can route the flows in all instances simultaneously. At every time step we simply output the union of flows output by the instances of the uniform weight algorithm. The final utility is therefore at least

$$\sum_{q=1}^{\rho} 2^{q-1} w_{\min} \mathsf{ALG}_q \geq \sum_{q=1}^{\rho} 2^{q-1} w_{\min} \frac{\mathsf{OPT}_q}{c} \geq \frac{\mathsf{OPT}^*}{2c\rho} \ ,$$

which completes the proof since Theorem 1 gives an algorithm with constant c. ◀

D Missing Materials from Section 4

We provide the missing materials from Section 4.

D.1 Hardness of Online Demands

We show that online demands are hard to satisfy in the public storage setting by proving the following theorem:

Proof of Theorem 3. Our lower bound instance is a network similar to that considered in Theorem 6. Consider a network with n nodes where there exist one supply node with renewable power generators and $n-1$ demand nodes that are connected to the "root" supply node. Each demand node has a public storage unit with unit capacity and the decay factor of 1; these are the only storage units on the network. We assume the case of uniform utilities where each demand node requires at most 1 unit of power, that is, the utility functions are of the form $W(y) = y$ up to the limit $y = 1$.

At time $i = 1$, the supply node generates a unit of power. At time $i = 2$, exactly one demand node requests power while other nodes do not. Consider an arbitrary randomized online algorithm and let d be the demand node with the least expected amount of power stored at the beginning of time step $i = 2$. Note the expected amount of power stored at d is at most $O(1/n)$. We let d be the only demand node to request power at time $i = 2$, and the online algorithm routes $O(1/n)$ units of power.

However, the optimal offline algorithm knows where the demand is going to be and can always satisfy the power demand and obtain total utility of 1. Hence, a lower bound of $\Omega(n)$ on the competitive ratio follows. ◀

D.2 Concave Utilities

Proof of Claim 9. For each edge e, we show that $\alpha_e \geq \omega_e$ at all times where

$$\omega(e) := \frac{w_{\min}\gamma_{\min}}{l_{\max}}\left(e^{\frac{c}{C_e}\sum_{p\ni e}\gamma(p,e)x_p} - 1\right).$$

Initially, $\alpha_e = \omega_e = 0$. Note both α_e and ω_e increase only if some path p containing e is used for routing. Assume $\alpha_e \geq \omega_e$ and we show the inequality still holds after the updates due to routing through p, say, from supply s to demand d. Note that

$$\begin{aligned}\frac{d\omega_e}{dx_p} &= \frac{c\gamma(p,e)}{C_e}\frac{w_{\min}\gamma_{\min}}{l_{\max}}e^{\frac{c}{C_e}\sum_{p\ni e}\gamma(p,e)x_p} \\ &= \frac{c\gamma(p,e)}{C_e}\omega_e + \frac{cw_{\min}\gamma_{\min}\gamma(p,e)}{l_{\max}C_e} \\ &\leq c\left(\frac{\gamma(p,e)\alpha_e}{C_e} + \frac{\gamma(p)W_d{}'(\lambda_d)}{l_{\max}C_e}\right) \\ &= \frac{d\alpha_e}{dx_p}.\end{aligned}$$

Hence, α_e increases at a faster rate than ω_e and it follows that $\alpha_e \geq \omega_e$ throughout Algorithm 1.

If $\alpha_e = w_{\max}$, the dual constraints for any path p containing edge e are satisfied and α_e is not further increased. Then, $w_{\max} \geq \alpha_e \geq \omega_e$. It follows that

$$\begin{aligned} w_{\max} &\geq \frac{w_{\min}\gamma_{\min}}{l_{\max}}\left(e^{\frac{c}{C_e}\sum_{p\ni e}\gamma(p,e)x_p} - 1\right) \\ \log\left(\frac{l_{\max}\Delta_w}{\gamma_{\min}} + 1\right) &\geq \frac{c}{C_e}\sum_{p\ni e}\gamma(p,e)x_p \\ O\left(\log\left(\frac{l_{\max}\Delta_w}{\gamma_{\min}}\right)\right)\frac{C_e}{c} &\geq \sum_{p\ni e}\gamma(p,e)x_p.\end{aligned}$$

Since $l_{\max} = nT$ and $\gamma_{\min} = \gamma^T$, we see that

$$\sum_{p\ni e}\gamma(p,e)x_p \leq \frac{O(\log nT + \log\gamma^{-T} + \log\Delta_w)}{c}\cdot C_e \leq \frac{O(\log n + \log\gamma^{-T} + \log\Delta_w)}{c}\cdot C_e,$$

where the last inequality follows from the fact that $\log\gamma^{-T}$ dominates $\log T$. ◀

Proof of Claim 10. Let $\mathcal{D}_0 = \sum_e C_e\alpha_e$; so, $\mathcal{D} = \mathcal{D}_0 + \sum_d \widehat{W}_d(\lambda_d)$.

We first show that $\mathcal{D}_0 \leq 2c\mathcal{P}$. Initially, $\mathcal{P} = \mathcal{D}_0 = 0$. Assume we route an infinitesimal amount through path p from supply s to demand d and correspondingly update dual variables. We compute corresponding changes in $\mathcal{P}$ and $\mathcal{D}_0$. Note $d\mathcal{P} = W_d{}'(y_d)dy_d = \gamma(p)W_d{}'(y_d)dx_p$ since $dy_d = \gamma(p)dx_p$. Furthermore, $d\mathcal{D}_0 = \sum_{e\in p} C_e d\alpha_e = \sum_{e\in p} c\left(\gamma(p,e)\alpha_e + \frac{w_{\min}\gamma_{\min}}{l_{\max}}\right)dx_p \leq 2c\gamma(p)W_d{}'(\lambda_d)dx_p$. By construction, we increase y_d and λ_d at the same rate and hence, $y_d = \lambda_d$ for all demand $d \in D^*$. It follows that $d\mathcal{D}_0 \leq 2cd\mathcal{P}$ and $\mathcal{D}_0 \leq 2c\mathcal{P}$ at termination.

Since $\widehat{W}_d(z) \leq W_d(z), \forall z \geq 0$, it follows that $\sum_d \widehat{W}_d(\lambda_d) \leq \sum_d W_d(y_d) = \mathcal{P}$. Then, $\mathcal{D} = \sum_d \widehat{W}_d(\lambda_d) + \mathcal{D}_0 \leq (2c+1)\mathcal{P}$. ◀

Path Construction

We show how to check the condition in Line 4 and choose a routing path p in Line 6 in Algorithm 1 in polynomial time. More specifically, we show how to check the condition for paths in $P(s,d)$ for supply s and demand d and find a simple path p, if it exists, such that $\sum_{e\in p}\gamma(p,e)\alpha_e - \gamma(p){W_d}'(\lambda_d) < 0$. This is equivalent to finding a discounted variant of shortest path where α_e are the edge lengths and γ_e are the discount factors, from which $\gamma(p,e)$ can be defined (cf. [1]).

Given s and d, we run a backward variant of Dijkstra's algorithm starting with d with initialization $\sigma(d) = -{W_d}'(\lambda_d)$ and computing iteratively the discounted "distance" $\sigma(u) = \min_{p\in P(u,d)}\sum_{e\in p}\gamma(p,e)\alpha_e - \gamma(p){W_d}'(\lambda_d)$ and corresponding successor $\pi(u)$ for each node u. We compute σ and π in stages such that if s is in time t_1 and d is in time t_2, we process the time-copies $G^{t_2}, G^{t_2-1}, \ldots, G^{t_1}$ in that order. For $t = t_2, \ldots, t_1$, we iteratively initialize σ and π based on nodes in G^{t+1} via storage edges and then compute them for all nodes in G^t.

The correctness follows from the shortest discounted paths' optimality property in the time-expanded graph. Note that the shortest discounted paths have an optimal substructure property similar to that of shortest paths in that any suffix of a shortest discounted path is a shortest discounted path. If p is a shortest discounted path from u to d and $p = (u,v)\cup p'$, p' is a shortest discounted path from v to d. Otherwise, we can find a shorter discounted path from u to d via a shorter path $p'' \in P(v,d)$, since $\sum_{e\in p}\gamma(p,e)\alpha_e - \gamma(p){W_d}'(\lambda_d) = \alpha_{(u,v)} + \gamma_{(u,v)}\left(\sum_{e\in p'}\gamma(p',e)\alpha_e - \gamma(p'){W_d}'(\lambda_d)\right)$. Furthermore, a shortest discounted path's length increases monotonically within each time-copy G^t when it is extended to another shortest discounted path; in other words, there are no "negative-weight" edges in G^t. As storage edges are the only edges with discount factors and do not form cycles in the time-expanded graph, the path lengths within each time-copy are correctly computed with the backward variant of Dijkstra's algorithm which runs in polynomial time.

D.3 Concave Utilities and Convex Costs

We present the missing algorithm:

Algorithm 2 Online Algorithm for Concave Utilities and Convex Costs

Let $y_d = \sum_{p\in P(\cdot,d)}\gamma(p)x_p, \forall d$; $z_s = \sum_{p\in P(s,\cdot)} x_p, \forall s$ ▷ y, z determined in terms of x
Let $\mu_s = {Q_s}'(\tau_s z_s)$ for $\tau_s = \rho_s^{1/(\rho_s-1)}, \forall s$ ▷ μ determined in terms of z
for $i = 1, \ldots, T$ **do**
  for $s \in S^i$ in arbitrary order **do**
    while $P' = \{p \in P(s,\cdot) : \sum_{e\in p}\gamma(p,e)\alpha_e + \mu_s < \gamma(p){W_{d(p)}}'(\lambda_{d(p)})\} \neq \emptyset$ **do**
      Update continuously:
        $p = \arg\max_{p\in P'}\gamma(p){W_{d(p)}}'(\lambda_{d(p)})$
        $dx_p = 1$ ▷ Increase x_p at a uniform rate
        $\frac{d\lambda_{d(p)}}{dx_p} = \gamma(p)$
        $\frac{d\alpha_e}{dx_p} = c\left(\frac{\gamma(p,e)\alpha_e}{C_e} + \frac{\gamma(p){W_{d(p)}}'(\lambda_{d(p)})}{l_{\max}C_e}\right), \forall e \in p$ ▷ $c \geq 1$ is some parameter
    end while
  end for
end for

Proof of Claim 12. For each edge e, we show that $\alpha_e \geq \omega_e$ at all times where

$$\omega(e) := \frac{w_{\min}\gamma_{\min}}{l_{\max}} \left(e^{\frac{c}{C_e}\sum_{p\ni e}\gamma(p,e)x_p} - 1 \right).$$

Initially, $\alpha_e = \omega_e = 0$. Note both α_e and ω_e increase only if some path p containing e is used for routing. Assume $\alpha_e \geq \omega_e$ and we show the inequality still holds after the updates due to routing through path p, say, from supply s to demand d. Note that

$$\begin{aligned}\frac{d\omega_e}{dx_p} &= \frac{c\gamma(p,e)}{C_e}\frac{w_{\min}\gamma_{\min}}{l_{\max}} e^{\frac{c}{C_e}\sum_{p\ni e}\gamma(p,e)x_p} \\ &= \frac{c\gamma(p,e)}{C_e}\omega_e + \frac{cw_{\min}\gamma_{\min}\gamma(p,e)}{l_{\max}C_e} \\ &\leq c\left(\frac{\gamma(p,e)\alpha_e}{C_e} + \frac{{W_d}'(\lambda_d)\gamma(p)}{l_{\max}C_e}\right) \\ &= \frac{d\alpha_e}{dx_p}.\end{aligned}$$

Hence, α_e increases at a faster rate than ω_e and it follows that $\alpha_e \geq \omega_e$ throughout Algorithm 2.

If $\alpha_e = w_{\max}$, the dual constraints for any path p containing edge e are satisfied and α_e is not further increased. Then, $w_{\max} \geq \alpha_e \geq \omega_e$. On the same line of reasoning as in Lemma 8, we see that

$$\frac{O(\log n + \log\gamma^{-T} + \log\Delta_w)}{c} \cdot C_e \geq \sum_{p\ni e}\gamma(p,e)x_p. \qquad ◀$$

Proof of Claim 13. Let $\mathcal{D}_0 = \sum_e C_e\alpha_e + \sum_s Q_s^*(\mu_s)$; so, $\mathcal{D} = \mathcal{D}_0 + \sum_d \widehat{W}_d(\lambda_d)$. Assume $\mathcal{D}_0 \leq (2c+1)\rho^{\rho/(\rho-1)}\mathcal{P}$. Since $\widehat{W}_d(z) \leq W_d(z), \forall z \geq 0$, it would follow that $\sum_d \widehat{W}_d(\lambda_d) \leq \sum_d W_d(y_d) \leq \mathcal{P}$. Then, $\mathcal{D} = \sum_d \widehat{W}_d(\lambda_d) + \mathcal{D}_0 \leq \left((2c+1)\rho^{\rho/(\rho-1)} + 1\right)\mathcal{P}$, and the claim would follow.

We now show $\mathcal{D}_0 \leq (2c+1)\rho^{\rho/(\rho-1)}\mathcal{P}$. For $\sigma = (2c+1)\rho^{\rho/(\rho-1)}$ and $\rho > 1.44$, we show $d\mathcal{P} \geq \frac{1}{\sigma}d\mathcal{D}_0$. Initially, $\mathcal{P} = \mathcal{D}_0 = 0$. Assume we route an infinitesimal amount through path p, say, from supply s to demand d and correspondingly update primal and dual variables. We compute the resulting changes in the primal objective $\mathcal{P}$ and (partial) dual objective $\mathcal{D}_0$: $d\mathcal{P} = {W_d}'(y_d)dy_d - {Q_s}'(z_s)dz_s$; $d\mathcal{D}_0 = \sum_{e\in p} C_e d\alpha_e + (Q_s^*)'(\mu_s)d\mu_s$. Note $dz_s = dx_p$ and $dy_d = \gamma(p)dx_p$.

Note $d\mathcal{P} \geq \frac{1}{\sigma}d\mathcal{D}_0$ is equivalent to

$$ {W_d}'(y_d)dy_d - {Q_s}'(z_s)dz_s \geq \frac{1}{\sigma}\left(\sum_{e\in p} C_e d\alpha_e + (Q_s^*)'(\mu_s)d\mu_s\right). \tag{6}$$

By the dual variables' updates, (6) is equivalent to

$$ {W_d}'(y_d)dy_d - {Q_s}'(z_s)dz_s \geq \frac{1}{\sigma}\left(c\cdot dx_p \cdot \sum_{e\in p}\left(\gamma(p,e)\alpha_e + \frac{{W_d}'(\lambda_d)\gamma(p)}{l_{\max}}\right) + (Q_s^*)'(\mu_s)d\mu_s\right).$$

Since $\sum_{e\in p}\gamma(p,e)\alpha_e + \mu_s < \gamma(p){W_d}'(\lambda_d)$, the right hand side is upper bounded by $\frac{1}{\sigma}\left(2c\gamma(p){W_d}'(\lambda_d)dx_p + (Q_s^*)'(\mu_s)d\mu_s\right)$. It is sufficient to show

$$\left(1 - \frac{2c}{\sigma}\right)\gamma(p){W_d}'(\lambda_d)dx_p - {Q_s}'(z_s)dz_s \geq \frac{1}{\sigma}(Q_s^*)'(\mu_s)d\mu_s \ .$$

Since $\sum_{e \in p} \gamma(p,e)\alpha_e + \mu_s < \gamma(p) {W_d}'(\lambda_d)$, it suffices to show

$$\left(1 - \frac{2c}{\sigma}\right)\mu_s - {Q_s}'(z_s) \geq \frac{1}{\sigma}(Q_s^*)'(\mu_s)\frac{d\mu_s}{dz_s} \quad . \tag{7}$$

If $Q_s(z) = a_s z^{\rho_s}$, then $Q_s^*(\mu) = \frac{\rho_s - 1}{\rho_s}\frac{1}{(a_s\rho_s)^{1/(\rho_s-1)}}\mu^{\rho_s/(\rho_s-1)}$. Also, $\mu_s = {Q_s}'(\tau_s z_s)$. Then, (7) reduces to $\sigma \geq \frac{{\tau_s}^{\rho_s}(\rho_s-1)}{{\tau_s}^{\rho_s-1}-1} + \frac{2c{\tau_s}^{\rho_s-1}}{{\tau_s}^{\rho_s-1}-1}$. For $\tau_s = {\rho_s}^{1/(\rho_s-1)}$, the right hand side is equal to ${\rho_s}^{\rho_s/(\rho_s-1)} + \frac{2c\rho_s}{\rho_s-1}$. For $\rho_s \geq 1.44$, it is upper bounded by $(2c+1){\rho_s}^{\rho_s/(\rho_s-1)}$ which is exactly the value of σ chosen. Therefore, (7) holds and $d\mathcal{P} \geq \frac{1}{\sigma} d\mathcal{D}_0$. For a ρ_0 constant smaller than 1.44, we would need to have a multiplicative factor greater than $(2c+1)$ in the penultimate step. ◀