The Generalised Colouring Numbers on Classes of Bounded Expansion*

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Abstract

The generalised colouring numbers $\mathrm{adm}_r(G)$, $\mathrm{col}_r(G)$, and $\mathrm{wcol}_r(G)$ were introduced by Kierstead and Yang as generalisations of the usual colouring number, also known as the degeneracy of a graph, and have since then found important applications in the theory of bounded expansion and nowhere dense classes of graphs, introduced by Nešetřil and Ossona de Mendez. In this paper, we study the relation of the colouring numbers with two other measures that characterise nowhere dense classes of graphs, namely with uniform quasi-wideness, studied first by Dawar et al. in the context of preservation theorems for first-order logic, and with the splitter game, introduced by Grohe et al. We show that every graph excluding a fixed topological minor admits a universal order, that is, one order witnessing that the colouring numbers are small for every value of r. Finally, we use our construction of such orders to give a new proof of a result of Eickmeyer and Kawarabayashi, showing that the model-checking problem for successor-invariant first-order formulas is fixed parameter tractable on classes of graphs with excluded topological minors.

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1 Introduction

The *colouring number* $\mathrm{col}(G)$ of a graph G is the minimum k for which there is a linear order $<_L$ on the vertices of G such that each vertex v has *back-degree* at most $k-1$, that is, v has at most $k-1$ neighbours u with $u <_L v$. The colouring number is a measure for uniform sparseness in graphs: we have $\mathrm{col}(G) = k$ if and only if every subgraph H of G has a vertex

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of degree at most $k-1$. Hence, provided $\mathrm{col}(G)=k$, not only G is sparse, but also every subgraph of G is sparse. The colouring number minus one is also known as the *degeneracy*.

Recently, Nešetřil and Ossona de Mendez introduced the notions of *bounded expansion* [12] and *nowhere density* [14] as very general formalisations of uniform sparseness in graphs. Since then, several independent and seemingly unrelated characterisations of these notions have been found, showing that these concepts behave robustly. For example, nowhere dense classes of graphs can be defined in terms of excluded shallow minors [14], in terms of uniform quasi-wideness [2], a notion studied in model theory, or in terms of a game [8] with direct algorithmic applications. The *generalised colouring numbers* adm_r, col_r, and wcol_r were introduced by Kierstead and Yang [11] in the context of colouring and marking games on graphs. As proved by Zhu [17], they can be used to characterise both bounded expansion and nowhere dense classes of graphs.

The invariants adm_r, col_r, and wcol_r are defined similarly to the classic colouring number: for example, the *weak r-colouring* number $\mathrm{wcol}_r(G)$ of a graph G is the minimum integer k for which there is a linear order of the vertices such that each vertex v can reach at most $k-1$ vertices w by a path of length at most r in which w is the smallest vertex on the path.

The generalised colouring numbers found important applications in the context of algorithmic theory of sparse graphs. For example, they play a key role in Dvořák's approximation algorithm for minimum dominating sets [4], or in the construction of sparse neighbourhood covers on nowhere dense classes, a fundamental step in the almost linear time model-checking algorithm for first-order formulas of Grohe et al. [8].

In this paper we study the relation between the colouring numbers and the above mentioned characterisations of nowhere dense classes of graphs, namely with uniform quasi-wideness and the splitter game. We use the generalised colouring numbers to give a new proof that every bounded expansion class is uniformly quasi-wide. This was first proved by Nešetřil and Ossona de Mendez in [13]; however, the constants appearing in the proof of [13] are huge. We present a very simple proof which also improves the appearing constants. Furthermore, for the splitter game introduced in [8], we show that splitter has a very simple strategy to win on any class of bounded expansion, which leads to victory much faster than in general nowhere dense classes of graphs.

Every graph G from a fixed class $\mathcal{C}$ of bounded expansion satisfies $\mathrm{wcol}_r(G) \leq f(r)$ for some function f and all positive integers r. However, the order that witnesses this inequality for G may depend on the value r. We say that a class $\mathcal{C}$ admits *uniform orders* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for each $G \in \mathcal{C}$ there is one linear order that witnesses $\mathrm{wcol}_r(G) \leq f(r)$ for every value of r. We show that every class that excludes a fixed topological minor admits uniform orders that can be computed efficiently.

Finally, based on our construction of uniform orders for graphs that exclude a fixed topological minor, we provide an alternative proof of a very recent result of Eickmeyer and Kawarabayashi [6], that the model-checking problem for successor-invariant first-order (FO) formulas is fixed-parameter tractable on such classes (we obtained this result independently of, but later than, [6]). Successor-invariant logics have been studied in database theory and finite model theory, and successor-invariant FO is known to be more expressive than plain FO [15]. The model-checking problem for successor-invariant FO is known to be fixed-parameter tractable parameterized by the size of the formula on any graph class that excludes a fixed minor [7]. Very recently, this result was lifted to classes that exclude a fixed topological minor by Eickmeyer and Kawarabayashi [6]. The key point of their proof is to use the decomposition theorem for graphs excluding a fixed topological minor, due to Grohe and Marx [9]. Our approach is similar to that of [6]. However, we employ new constructions based on the generalised colouring numbers and use the decomposition theorem of [9] only implicitly. In particular, we do not construct a graph decomposition in order to solve the model-checking problem. Therefore, we believe that our approach may be easier to extend further to classes of bounded expansion, or even to nowhere dense classes of graphs.

2 Preliminaries

Notation. We use standard graph-theoretical notation; see e.g. [3] for reference. All graphs considered in this paper are finite, simple, and undirected. For a graph G, by $V(G)$ and $E(G)$ we denote the vertex and edge sets of G, respectively. A graph H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any $M \subseteq V(G)$, by $G[M]$ we denote the subgraph induced by M. We write $G - M$ for the graph $G[V(G) \setminus M]$ and if $M = \{v\}$, we write $G - v$ for $G - M$. For a non-negative integer ℓ, a *path of length* ℓ in G is a sequence $P = (v_1, \ldots, v_{\ell+1})$ of pairwise different vertices such that $v_i v_{i+1} \in E(G)$ for all $1 \le i \le \ell$. We write $V(P)$ for the vertex set $\{v_1, \ldots, v_{\ell+1}\}$ of P and $E(P)$ for the edge set $\{v_i v_{i+1} : 1 \le i \le \ell\}$ of P and identify P with the subgraph of G with vertex set $V(P)$ and edge set $E(P)$. We say that the path P *connects* its *endpoints* $v_1, v_{\ell+1}$, whereas $v_2, \ldots, v_\ell$ are the *internal vertices* of P. The *length* of a path is the number of its edges. Two vertices $u, v \in V(G)$ are *connected* if there is a path in G with endpoints u, v. The *distance* $\mathrm{dist}(u, v)$ between two connected vertices u, v is the minimum length of a path connecting u and v; if u, v are not connected, we put $\mathrm{dist}(u, v) = \infty$. The *radius* of G is $\min_{u \in V(G)} \max_{v \in V(G)} \mathrm{dist}(u, v)$. The set of all neighbours of a vertex v in G is denoted by $N^G(v)$, and the set of all vertices at distance at most r from v is denoted by $N_r^G(v)$. A graph G is c-*degenerate* if every subgraph $H \subseteq G$ has a vertex of degree at most c. A c-degenerate graph of order n contains an independent set of order at least $n/(c + 1)$.

A graph H with $V(H) = \{v_1, \ldots, v_n\}$ is a *minor* of G, written $H \preccurlyeq G$, if there are pairwise disjoint connected subgraphs $H_1, \ldots, H_n$ of G, called *branch sets*, such that whenever $v_i v_j \in E(H)$, then there are $u_i \in H_i$ and $u_j \in H_j$ with $u_i u_j \in E(G)$. We call $(H_1, \ldots, H_n)$ a *minor model* of H in G. The graph H is a *topological minor* of G, written $H \preccurlyeq^t G$, if there are pairwise different vertices $u_1, \ldots, u_n \in V(G)$ and a family of paths $\{P_{ij} : v_i v_j \in E(H)\}$, such that each P_{ij} connects u_i and u_j, and paths P_{ij} are pairwise internally vertex-disjoint.

Generalised colouring numbers. Let us fix a graph G. By $\Pi(G)$ we denote the set of all linear orders of $V(G)$. For $L \in \Pi(G)$, we write $u <_L v$ if u is smaller than v in L, and $u \le_L v$ if $u <_L v$ or $u = v$. Let $u, v \in V(G)$. For a non-negative integer r, we say that u is *weakly* r-*reachable* from v with respect to L, if there is a path P of length ℓ, $0 \le \ell \le r$, connecting u and v such that u is minimum among the vertices of P (with respect to L). By $\mathrm{WReach}_r[G, L, v]$ we denote the set of vertices that are weakly r-reachable from v w.r.t. L.

Vertex u is *strongly* r-*reachable* from v with respect to L, if there is a path P of length ℓ, $0 \le \ell \le r$, connecting u and v such that $u \le_L v$ and such that all internal vertices w of P satisfy $v <_L w$. Let $\mathrm{SReach}_r[G, L, v]$ be the set of vertices that are strongly r-reachable from v w.r.t. L. Note that we have $v \in \mathrm{SReach}_r[G, L, v] \subseteq \mathrm{WReach}_r[G, L, v]$.

For a non-negative integer r, we define the *weak* r-*colouring number* $\mathrm{wcol}_r(G)$ of G and the r-*colouring number* $\mathrm{col}_r(G)$ of G respectively as follows:

$$\begin{aligned} \mathrm{wcol}_r(G) &:= \min_{L \in \Pi(G)} \max_{v \in V(G)} \left|\mathrm{WReach}_r[G, L, v]\right|, \\ \mathrm{col}_r(G) &:= \min_{L \in \Pi(G)} \max_{v \in V(G)} \left|\mathrm{SReach}_r[G, L, v]\right|. \end{aligned}$$

For a non-negative integer r, the *r-admissibility* $\mathrm{adm}_r[G, L, v]$ of v w.r.t. L is the maximum size k of a family $\{P_1, \ldots, P_k\}$ of paths of length at most r that start in v, end at a vertex w with $w \le_L v$, and satisfy $V(P_i) \cap V(P_j) = \{v\}$ for all $1 \le i < j \le k$. As for $r > 0$ we can always let the paths end in the first vertex smaller than v, we can assume that the internal vertices of the paths are larger than v. Note that $\mathrm{adm}_r[G, L, v]$ is an integer, whereas $\mathrm{WReach}_r[G, L, v]$ and $\mathrm{SReach}_r[G, L, v]$ are vertex sets. The *r-admissibility* $\mathrm{adm}_r(G)$ of G is

$$\mathrm{adm}_r(G) \quad = \quad \min_{L \in \Pi(G)} \max_{v \in V(G)} \mathrm{adm}_r[G, L, v].$$

The generalised colouring numbers were introduced by Kierstead and Yang [11] in the context of colouring and marking games on graphs. The authors also proved that the generalised colouring numbers are related by the following inequalities:

$$\mathrm{adm}_r(G) \le \mathrm{col}_r(G) \le \mathrm{wcol}_r(G) \le (\mathrm{adm}_r(G))^r. \tag{1}$$

Shallow minors, bounded expansion, and nowhere denseness. A graph H with $V(H) = \{v_1, \ldots, v_n\}$ is a *depth-r minor* of G, denoted $H \preccurlyeq_r G$, if there is a minor model $(H_1, \ldots, H_n)$ of H in G such that each H_i has radius at most r. We write $d(H)$ for the *average degree* of H, that is, for the number $2|E(H)|/|V(H)|$. A class $\mathcal{C}$ of graphs has *bounded expansion* if there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for all non-negative integers r we have $d(H) \le f(r)$ for every $H \preccurlyeq_r G$ with $G \in \mathcal{C}$. A class $\mathcal{C}$ of graphs is *nowhere dense* if for every real $\epsilon > 0$ and every non-negative integer r, there is an integer n_0 such that if H is an n-vertex graph with $n \ge n_0$ and $H \preccurlyeq_r G$ for some $G \in \mathcal{C}$, then $d(H) \le n^\epsilon$.

Bounded expansion and nowhere dense classes of graphs were introduced by Nešetřil and Ossona de Mendez as models for uniform sparseness of graphs [12, 14]. As proved by Zhu [17], the generalised colouring numbers are tightly related to densities of low-depth minors, and hence they can be used to characterise bounded expansion and nowhere dense classes.

▶ **Theorem 1** (Zhu [17]). *A class $\mathcal{C}$ of graphs has bounded expansion if and only if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{wcol}_r(G) \le f(r)$ for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$.*

Due to (1), we may equivalently demand that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{adm}_r(G) \le f(r)$ or $\mathrm{col}_r(G) \le f(r)$ for all non-negative integers r and all $G \in \mathcal{C}$.

Similarly, from Zhu's result one can derive a characterisation of nowhere dense classes of graphs, as presented in [14]. A class $\mathcal{C}$ of graphs is called *hereditary* if it is closed under induced subgraphs, that is, if H is an induced subgraph of $G \in \mathcal{C}$, then $H \in \mathcal{C}$.

▶ **Theorem 2** (Nešetřil and Ossona de Mendez [14]). *A hereditary class $\mathcal{C}$ of graphs is nowhere dense if and only if for every real $\epsilon > 0$ and every non-negative integer r, there is a positive integer n_0 such that if $G \in \mathcal{C}$ is an n-vertex graph with $n \ge n_0$, then $\mathrm{wcol}_r(G) \le n^\epsilon$.*

As shown in [4], for every non-negative integer r, computing $\mathrm{adm}_r(G)$ is fixed-parameter tractable on any class of bounded expansion (parameterized by $\mathrm{adm}_r(G)$). For $\mathrm{col}_r(G)$ and $\mathrm{wcol}_r(G)$ this is not known; however, by (1) we can use admissibility to obtain approximations of these numbers. On nowhere dense classes of graphs, for every $\epsilon > 0$ and every non-negative integer r, we can compute an order that witnesses $\mathrm{wcol}_r(G) \le n^\epsilon$ in time $\mathcal{O}(n^{1+\epsilon})$ if G is sufficiently large [8], based on Nešetřil and Ossona de Mendez's augmentation technique [12].

3 Uniform quasi-wideness and the splitter game

In this section we discuss the relation between weak r-colouring numbers and two notions that characterise nowhere dense classes: uniform quasi-wideness and the splitter game.

For a graph G, a vertex subset $A \subseteq V(G)$ is called *r-independent* in G, if $\mathrm{dist}_G(a,b) > r$ for all different $a, b \in V(G)$. A vertex subset is called *r-scattered*, if it is $2r$-independent, that is, if the r-neighbourhoods of different elements of A do not intersect.

Informally, uniform quasi-wideness means the following: in any large enough subset of vertices of a graph from $\mathcal{C}$, one can find a large subset that is r-scattered in G, possibly after removing from G a small number of vertices. Formally, a class $\mathcal{C}$ of graphs is *uniformly quasi-wide* if there are functions $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ such that for all $m, r \in \mathbb{N}$, if $W \subseteq V(G)$ for a graph $G \in \mathcal{C}$ with $|W| > N(m,r)$, then there is a set $S \subseteq V(G)$ of size at most $s(r)$ such that W contains a subset of size at least m that is r-scattered in $G - S$.

The notion of quasi-wideness was introduced by Dawar [2] in the context of homomorphism preservation theorems. It was shown in [13] that classes of bounded expansion are uniformly quasi-wide and that uniform quasi-wideness characterises nowhere dense classes of graphs.

▶ **Theorem 3** (Nešetřil and Ossona de Mendez [13])**.** *A hereditary class $\mathcal{C}$ of graphs is nowhere dense if and only if it is uniformly quasi-wide.*

It was shown by Atserias et al. in [1] that classes that exclude K_k as a minor are uniformly quasi-wide. In fact, in this case we can choose $s(r) = k - 1$, independent of r (if such a constant function for a class $\mathcal{C}$ exists, the class is called *uniformly almost wide*). However, the function $N(m,r)$ that was used in the proof is huge: it comes from an iterated Ramsey argument. The same approach was used in [13] to show that every nowhere dense class, and in particular, every class of bounded expansion, is uniformly quasi-wide. We present a new proof that every bounded expansion class is uniformly quasi-wide, which gives us a much better bound on $N(m,r)$ and which is much simpler than the previously known proof.

▶ **Theorem 4.** *Let G be a graph and let $r, m \in \mathbb{N}$. Let $c \in \mathbb{N}$ be such that $\mathrm{wcol}_r(G) \leq c$ and let $A \subseteq V(G)$ be a set of size at least $(c+1) \cdot 2^m$. Then there exists a set S of size at most $c(c-1)$ and a set $B \subseteq A$ of size at least m which is r-independent in $G - S$.*

Proof. Let $L \in \Pi(G)$ be such that $|\mathrm{WReach}_r[G,L,v]| \leq c$ for every $v \in V(G)$. Let H be the graph with vertex set $V(G)$, where we put an edge $uv \in E(H)$ if and only if $u \in \mathrm{WReach}_r[G,L,v]$ or $v \in \mathrm{WReach}_r[G,L,u]$. Then L certifies that H is c-degenerate, and hence we can greedily find an independent set $I \subseteq A$ of size 2^m in H. By the definition of the graph H, we have that $\mathrm{WReach}_r[G,L,v] \cap I = \{v\}$ for each $v \in I$.

▶ **Claim 5.** *Let $v \in I$. Then deleting $\mathrm{WReach}_r[G,L,v] \setminus \{v\}$ from G leaves v at a distance greater than r (in $G - (\mathrm{WReach}_r[G,L,v] \setminus \{v\})$) from all the other vertices of I.*

Proof. Let $u \in I$ and let P be a path in G that has length at most r and connects u and v. Let $z \in V(P)$ be minimal with respect to L. Then $z <_L v$ or $z = v$. If $z <_L v$, then $z \in \mathrm{WReach}_r[G,L,v]$ and hence the path P no longer exists after the deletion of $\mathrm{WReach}_r[G,L,v] \setminus \{v\}$ from G. On the other hand, if $z = v$, then $v \in \mathrm{WReach}_r[G,L,u]$, contradicting the fact that both $u, v \in I$. ⌟

We iteratively find sets $B_0 \subseteq \ldots \subseteq B_m \subseteq I$, sets $I_0 \supseteq \ldots \supseteq I_m$, and sets $S_0 \subseteq \ldots \subseteq S_m$ such that B is r-independent in $G - S$, where $B := B_m$ and $S := S_m$. We maintain the invariant that sets B_i, I_i, and S_i are pairwise disjoint for each i. Let $I_0 = I$, $B_0 = \emptyset$ and $S_0 = \emptyset$. In one step $i = 1, 2, \ldots, m$, we delete some vertices from I_i (thus obtaining I_{i+1}), shift one vertex from I_i to B_i (obtaining B_{i+1}) and, possibly, add some vertices from $V(G) \setminus I_i$ to S_i (obtaining S_{i+1}). More precisely, let v be the vertex of I_i that is the largest in the order L. We set $B_{i+1} = B_i \cup \{v\}$, and now we discuss how I_{i+1} and S_{i+1} are constructed.

We distinguish two cases. First, suppose v is connected by a path of length at most r in $G - S_i$ to at most half of the vertices of I_i (including v). Then we remove these reachable vertices from I_i, and set I_{i+1} to be the result. We also set $S_{i+1} = S_i$. Note that $|I_{i+1}| \geq |I_i|/2$.

Second, suppose v is connected by a path of length at most r in $G - S_i$ to more than half of the vertices of I_i (including v). We proceed in two steps. First, we add the at most $c - 1$ vertices of $\mathrm{WReach}_r[G, L, v] \setminus \{v\}$ to S_{i+1}, that is, we let $S_{i+1} = S_i \cup (\mathrm{WReach}_r[G, L, v] \setminus \{v\})$. (Recall here that $\mathrm{WReach}_r[G, L, v] \cap I = \{v\}$.) By Claim 5, this leaves v at a distance greater than r from every other vertex of I_i in $G - S_{i+1}$. Second, we construct I_{i+1} from I_i by removing the vertex v and all the vertices of I_i that are not connected to v by a path of length at most r in $G - S_i$, hence we have $|I_{i+1}| \geq \lfloor |I_i|/2 \rfloor$.

Observe the construction above can be carried out for m steps, because in each step, we remove at most half of the vertices of I_i (rounded up) when constructing I_{i+1}. As $|I_0| = |I| = 2^m$, it is easy to see that the set I_i cannot become empty within m iterations. Moreover, it is clear from the construction that we end up with a set $B = B_m$ that has size m and is r-scattered in $G - S$, where $S = S_m$. It remains to argue that $|S_m| \leq c(c-1)$. For this, it suffices to show that the second case cannot apply more than c times in total.

Suppose the second case was applied in the ith iteration, when considering a vertex v. Every vertex $u \in I_i$ with $u <_L v$ that was connected to v by a path of length at most r in $G - S_i$ satisfies $\mathrm{WReach}_r[G, L, v] \cap \mathrm{WReach}_r[G, L, u] \neq \emptyset$. Thus, every remaining vertex $u \in I_{i+1}$ has at least one of its weakly r-reachable vertices deleted (that is, included in S_{i+1}). As the number of such vertices is at most $c - 1$ at the beginning, and it can only decrease during the construction, this implies that the second case can occur at most c times. ◀

As shown in [16], if $K_k \not\preccurlyeq G$, then $\mathrm{wcol}_r(G) \in \mathcal{O}(r^{k-1})$. Hence, for such graphs we have to delete only a polynomial (in r) number of vertices in order to find an r-independent set of size m in a set of vertices of size single exponential in m.

We now implement the same idea to find a very simple strategy for splitter in the splitter game, introduced by Grohe et al. [8] to characterise nowhere dense classes of graphs. Let $\ell, r \in \mathbb{N}$. The *simple ℓ-round radius-r splitter game* on G is played by two players, *connector* and *splitter*, as follows. We let $G_0 := G$. In round $i + 1$ of the game, connector chooses a vertex $v_{i+1} \in V(G_i)$. Then splitter picks a vertex $w_{i+1} \in N_r^{G_i}(v_{i+1})$. We let $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) \setminus \{w_{i+1}\}]$. Splitter wins if $G_{i+1} = \emptyset$. Otherwise the game continues at G_{i+1}. If splitter has not won after ℓ rounds, then connector wins.

A *strategy* for splitter is a function σ that maps every partial play $(v_1, w_1, \ldots, v_s, w_s)$, with associated sequence $G_0, \ldots, G_s$ of graphs, and the next move $v_{s+1} \in V(G_s)$ of connector, to a vertex $w_{s+1} \in N_r^{G_s}(v_{s+1})$ that is the next move of splitter. A strategy σ is a *winning strategy* for splitter if splitter wins every play in which she follows the strategy f. We say that splitter *wins* the simple ℓ-round radius-r splitter game on G if she has a winning strategy.

▶ **Theorem 6** (Grohe et al. [8]). *A class $\mathcal{C}$ of graphs is nowhere dense if and only if there is a function $\ell : \mathbb{N} \to \mathbb{N}$ such that splitter wins the simple $\ell(r)$-round radius-r splitter game on every graph $G \in \mathcal{C}$.*

More precisely, it was shown in [8] that $\ell(r)$ can be chosen as $N(2s(r), r)$, where N and s are the functions that characterise $\mathcal{C}$ as a uniformly quasi-wide class of graphs. We present a proof that on bounded expansion classes, splitter can win much faster.

▶ **Theorem 7.** *Let G be a graph, let $r \in \mathbb{N}$ and let $\ell = \mathrm{wcol}_{2r}(G)$. Then splitter wins the ℓ-round radius-r splitter game.*

Proof. Let L be a linear order that witnesses $\mathrm{wcol}_{2r}(G) = \ell$. Suppose in round $i+1 \leq \ell$, connector chooses a vertex $v_{i+1} \in V(G_i)$. Let w_{i+1} (splitter's choice) be the minimum vertex of $N_r^{G_i}(v_{i+1})$ with respect to L. Then for each $u \in N_r^{G_i}(v_{i+1})$ there is a path between u and w_{i+1} of length at most $2r$ that uses only vertices of $N_r^{G_i}(v_{i+1})$. As w_i is minimum in $N_r^{G_i}(v_{i+1})$, w_{i+1} is weakly $2r$-reachable from each $u \in N_r^{G_i}(v_{i+1})$. Now let $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) \setminus \{w_{i+1}\}]$. As w_{i+1} is not part of G_{i+1}, in the next round splitter will choose another vertex which is weakly $2r$-reachable from every vertex of the remaining r-neighbourhood. As $\mathrm{wcol}_{2r}(G) = \ell$, the game must stop after at most ℓ rounds. ◀

4 Uniform orders for graphs excluding a topological minor

If $\mathcal{C}$ is a class of bounded expansion such that $\mathrm{wcol}_r(G) \leq f(r)$ for all $G \in \mathcal{C}$ and all $r \in \mathbb{N}$, the order L that witnesses this inequality for G may depend on the value r. We say that a class $\mathcal{C}$ *admits uniform orders* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for each $G \in \mathcal{C}$, there is a linear order $L \in \Pi(G)$ such that $|\mathrm{WReach}_r[G, L, v]| \leq f(r)$ for all $v \in V(G)$ and all $r \in \mathbb{N}$. In other words, there is one order that simultaneously certifies the inequality $\mathrm{wcol}_r(G) \leq f(r)$ for all r.

It is implicit in [16] that every class that excludes a fixed minor admits uniform orders, which can be efficiently computed. We are going to show that the same holds for classes that exclude a fixed topological minor. Our construction is similar to the construction of [16], in particular, our orders can be computed quickly in a greedy fashion. The proof that we find an order of high quality is based on the decomposition theorem for graphs with excluded topological minors, due to Grohe and Marx [9]. Note however, that for the construction of the order we do not have to construct a tree decomposition according to Grohe and Marx [9].

Construction. Let G be a graph. We present a construction of an order of $V(G)$ of high quality. We iteratively construct a sequence $H_1, \ldots, H_\ell$ of pairwise disjoint and connected subgraphs of G such that $\bigcup_{1 \leq i \leq \ell} V(H_i) = V(G)$. For $0 \leq i < \ell$, let $G_i := G - \bigcup_{1 \leq j \leq i} V(H_j)$. We say that a component C of G_i is *connected* to a subgraph H_j, $j \leq i$, if there is a vertex $u \in V(H_j)$ and a vertex $v \in V(C)$ such that $uv \in E(G)$. For all i, $1 \leq i < \ell$, we will maintain the following invariant. If C is a component of G_i, then the subgraphs $H_{i_1}, \ldots, H_{i_s} \in \{H_1, \ldots, H_i\}$ that are connected to C form a minor model of the complete graph K_s, where s is their number.

To start, we choose an arbitrary vertex $v \in V(G)$ and let H_1 be the connected subgraph $G[\{v\}]$. Clearly, H_1 satisfies the above invariant. Now assume that for some i, $1 \leq i < \ell$, the sequence $H_1, \ldots, H_i$ has already been constructed. Fix some component C of G_i and, by the invariant, assume that the subgraphs $H_{i_1}, \ldots, H_{i_s} \in \{H_1, \ldots, H_i\}$ with $1 \leq i_1 < \ldots < i_s \leq i$ that have a connection to C form a minor model of K_s. For a vertex $v \in V(C)$, let $m(v)$ be the maximum cardinality of a family $\mathcal{P}$ of paths with the following properties: each path of $\mathcal{P}$ connects v with a different subgraph H_{i_j}, the internal vertices of each path from $\mathcal{P}$ belong to G_i, and the paths of $\mathcal{P}$ are pairwise disjoint apart from sharing v. Note that $m(v)$ can be computed in polynomial time using any maximum flow algorithm. Pick v to be a vertex of C with maximum $m(v)$. Let T be the tree of the breadth-first search in $G[C]$ that starts in v; thus, T is rooted at v. We choose H_{i+1} to be a minimal connected subtree of T that contains v and, for each j with $1 \leq j \leq s$, at least one neighbour of H_{i_j} in C.

From the construction it is easy to see that for every component C' of G_{i+1}, the subgraphs $H'_{i_1}, \ldots, H'_{i_{s'}} \in \{H_1, \ldots, H_{i+1}\}$ that are connected to C' form the minor model of a complete graph, hence the invariant is again established. Having chosen H_{i+1}, we proceed to the next iteration. The construction stops when all vertices are part of some H_i, $1 \leq i \leq \ell$.

We construct an order L of $V(G)$ as follows. Let $v <_L u$ if $v \in V(H_i)$ and $u \in V(H_j)$ for some $i < j$. Furthermore, we order the vertices within each H_i arbitrarily. Obviously, the construction does not depend on r, hence the produced order is uniform for G.

Analysis. From now on we assume that G excludes K_k as a topological minor, for some constant k. Furthermore, assume that the graphs $H_1, \dots, H_\ell$ and a corresponding order L have been constructed, as described above. We now show that the constructed order has good qualities. Our proof is based on the following two key lemmas. The first lemma states that for every component C of G_i arising after the construction of $H_1, \dots, H_i$, every vertex v of C can reach only a bounded number of subgraphs among $H_1, \dots, H_i$ by disjoint paths.

▶ **Lemma 8.** *There is a constant α (depending only on k) such that for all integers i, $1 \le i < \ell$, if C is a component of G_i, then for every vertex $v \in V(C)$, we have $m(v) \le \alpha$, where $m(v)$ is defined as in the construction.*

The second lemma states that from a vertex of H_{i+1}, we can reach only a bounded number of vertices of each H_j, $1 \le j \le i+1$, by short disjoint paths in G_i.

▶ **Lemma 9.** *There is a constant β (depending only on k) such that for all integers i, j, where $1 \le j \le i \le \ell$, and all positive integers r, the following holds. Suppose $v \in V(H_i)$, and let $\mathcal{P}$ be any family of paths of length at most r with the following properties: each path from $\mathcal{P}$ connects v with a different vertex of H_j, the internal vertices of $\mathcal{P}$ belong to G_j, and paths from $\mathcal{P}$ are internally vertex disjoint. Then $\mathcal{P}$ has size not larger than $\beta \cdot r$.*

It is easy to show that the above two lemmas guarantee that L has the required properties. The proof of this fact, as well as all the other facts marked with $*$, is in the appendix.

▶ **Corollary 10** ($*$). *If $K_k \not\preccurlyeq^t G$, then there exists a constant γ (depending only on k) and a uniform order L that witnesses $\mathrm{adm}_r(G) \le \gamma \cdot r$ for all non-negative integers r.*

The proof of Lemma 8 is based on the decomposition theorem for graphs with excluded topological minors of Grohe and Marx [9]. Recall that a *tree decomposition* of a graph G is a pair (T, β), where T is a tree and $\beta : V(T) \to 2^{V(G)}$, such that for every vertex $v \in V(G)$ the set $\beta^{-1}(v) = \{t \in V(T) : v \in \beta(t)\}$ is non-empty and connected in T, and for every edge $e \in E(G)$ there is a node $t \in V(T)$ such that $e \subseteq \beta(t)$. The *width* of (T, β) is $\max\{|\beta(t)| - 1 : t \in V(T)\}$ and the *adhesion* of (T, β) is $\max\{|\beta(s) \cap \beta(t)| : st \in E(T)\}$.

For a node $t \in T$, we call $\beta(t)$ the *bag at t*. If $T' \subseteq T$, we write $\beta(T')$ for $\bigcup_{t' \in V(T')} \beta(t')$ and if $M \subseteq V(G)$, we write $\beta^{-1}(M)$ for $\bigcup_{v \in M} \beta^{-1}(v)$. Denote by $K[X]$ the complete graph on a vertex set X. The *torso at t* is the graph $\tau(t) := G[\beta(t)] \cup \bigcup_{st \in E(T)} K[\beta(s) \cap \beta(t)]$.

▶ **Theorem 11** ([9]). *For every $k \in \mathbb{N}$, there exist constants $a(k), c(k), d(k)$ and $e(k)$ such that the following holds. Let H be a graph on k vertices. Then for every graph G with $H \not\preccurlyeq^t G$ there is a tree decomposition (T, β) of adhesion at most $a(k)$ such that for all $t \in V(T)$ one of the following two alternatives hold.*

1. *The torso $\tau(t)$ has at most $c(k)$ vertices of degree larger than $d(k)$, which we call the* apex vertices *of $\tau(t)$. Such a node t will be called a* bounded degree node*.*
2. *The torso $\tau(t)$ excludes the complete graph $K_{e(k)}$ as a minor. Such a node t will be called an* excluded minor node*.*

We will need the following well-known properties of trees and tree decompositions.

▶ **Lemma 12** (Helly-property for trees). *Let T be a tree and let $(T_i)_{i \in I}$ be a family of subtrees of T. If $V(T_i) \cap V(T_j) \neq \emptyset$, for all $i, j \in I$, then $\bigcap_{i \in I} V(T_i) \neq \emptyset$.*

▶ **Lemma 13.** *Let (T,β) be a tree decomposition of a graph G. Let $e = st$ be an edge of T and let T_1, T_2 be the components of $T - e$. Then $\beta(s) \cap \beta(t)$ separates $\beta(T_1)$ from $\beta(T_2)$, that is, every path from a vertex of $\beta(T_1)$ to a vertex of $\beta(T_2)$ traverses a vertex of $\beta(s) \cap \beta(t)$.*

▶ **Lemma 14.** *If $H \subseteq G$ is a connected subgraph of G, then $\beta^{-1}(V(H))$ is connected in T.*

For the proof of Lemma 8, assume that G is decomposed as described by Theorem 11. Assume that $H_1, \ldots, H_i$ have been constructed and let C be a component of G_i that has a connection to the subgraphs $H_{i_1}, \ldots, H_{i_s}$. Recall that throughout the construction we guarantee that the subgraphs $H_{i_1}, \ldots, H_{i_s}$ form the minor model of a complete graph K_s. We first identify one bag of the decomposition as a bag which intersects many distinct branch sets of this minor model. The following lemma follows easily from the separator properties of tree decompositions, in particular Lemma 13.

▶ **Lemma 15** ($*$)**.** *There can be at most one node t such that $\beta(t)$ intersects strictly more than $a(k)$ of the branch sets H_{i_j}, for $1 \leq j \leq s$.*

We now show that there is a bag that intersects every branch set. The proof is a simple application of the Helly property of trees (Lemma 12) and Lemma 14.

▶ **Lemma 16** ($*$)**.** *There is a node t such that $\beta(t)$ intersects each H_{i_j}, for $1 \leq j \leq s$.*

Hence, provided $s > a(k)$, there is a node t with $\beta(t)$ intersecting at least $a(k)+1$ branch sets H_{i_j}. By Lemma 15, this node is unique. We call it the *core node* of the minor model. Next we show that if the model is large, then its core node must be a bounded degree node. Shortly speaking, this is because the model $H_{i_1}, \ldots, H_{i_s}$ trimmed to the torso of the core node is already a minor model of K_s in this torso.

▶ **Lemma 17** ($*$)**.** *If $s > \max\{a(k), e(k)\}$, then the core node of the minor model is a bounded degree node.*

For vertices outside the bag of the core node, the bound promised in Lemma 8 can be proved similarly as Lemma 15.

▶ **Lemma 18** ($*$)**.** *Let C be a component of G_i that has a connection to the subgraphs $H_{i_1}, \ldots, H_{i_s}$. If $s > a(k)$, then for every vertex $v \in V(C) \setminus \beta(t)$, where t is the core node of the model, we have that $m(v) \leq a(k)$.*

We now complete the proof of Lemma 8 by looking at the vertices inside the core bag.

Proof of Lemma 8. We set $\alpha := a(k) + c(k) + d(k) + e(k)$. Assume towards a contradiction that for some i, $1 \leq i < \ell$, we have that some component C of G_i contains a vertex v_1 with $m(v_1) > \alpha$. Denote the branch sets that have a connection to C by $H_{i_1}, \ldots, H_{i_s}$, where $i_1 < i_2 < \ldots < i_s$. Let $\mathcal{P}$ be a maximum-size family of paths that pairwise share only v_1 and connect v_1 with different branch sets H_{i_j}. As $m(v_1) > \alpha$, we have that $|\mathcal{P}| > \alpha$, and in particular $s > \alpha$. As $\alpha > a(k)$, by Lemmas 15 and 16 we can identify the unique core node t of the minor model. As $s > \max\{a(k), e(k)\}$, by Lemma 17 the core node is a bounded degree node. As $m(v_1) > a(k)$, by Lemma 18 we have $v_1 \in \beta(t)$. As $\mathcal{P}$ contains more than $d(k)$ disjoint paths from v to distinct branch sets, the degree of v_1 in G must be greater than $d(k)$, hence v_1 is an apex vertex of $\tau(t)$.

Since $i_1 < i_2 < \ldots < i_s$, we have that the component C was created when H_{i_s} was removed from $G_{i_s - 1}$. Let C' be the component of G_{i_s-1} that contains C and H_{i_s} (and thus v_1). Observe that C' is still connected to $H_1, \ldots, H_{i_{s-1}}$, and possibly to some other branch sets. Recall that H_{i_s} was constructed as a subtree of the breadth-first search tree in G_{i_s} that started in a vertex $v_2 \in V(C')$ which, at this point of the construction, had maximum $m(v_2)$ among vertices in C'. However, at this point vertex v_1 was also present in C', and $\mathcal{P}$ certifies that it could send at least $\alpha - 1$ disjoint paths to different branch sets among $H_1, \ldots, H_{i_{s-1}}$ (in $\mathcal{P}$, at most one path leads to H_{i_s}, and all the other paths are also present in C'). We infer that it held that $m(v_2) \geq \alpha - 1$ at the moment v_2 was taken. Since $\alpha > a(k) + c(k) + d(k) + e(k) \geq a(k) + d(k) + e(k) + 1$, the same reasoning as above shows that t is also the core vertex of the minor model formed by branch sets connected to C'. Thus, by exactly the same reasoning we obtain that v_2 is also an apex vertex of $\tau(t)$.

Since $\alpha > a(k) + c(k) + d(k) + e(k)$, we can repeat this reasoning $c(k) + 1$ times, obtaining vertices $v_1, \ldots, v_{c(k)+1}$, which are all apex vertices of $\tau(t)$. This contradicts the fact that $\tau(t)$ contains at most $c(k)$ apex vertices. ◀

At last, we come to the proof of Lemma 9

Proof of Lemma 9. We set β so that $\beta \cdot r \geq (2r + 1) \cdot \alpha$, where α is the constant given by Lemma 8. For the sake of contradiction, suppose there is a family of paths $\mathcal{P}$ as in the statement, whose size is larger than $(2r + 1) \cdot \alpha$.

Recall that H_j was chosen as a subtree of a breadth-first search tree in G_{j-1}; throughout the proof, we treat H_j as a rooted tree. As H_j is a subtree of a BFS tree, every path from a vertex w of the tree to the root v' of the tree is an isometric path in G_{j-1}, that is, a shortest path between w and v' in the graph G_{j-1}. If P is an isometric path in a graph H, then $|N_r^H(v) \cap V(P)| \leq 2r + 1$ for all $v \in V(H)$ and all $r \in \mathbb{N}$. As the paths from $\mathcal{P}$ are all contained in G_{j-1}, and they have lengths at most r, this implies that the path family $\mathcal{P}$ cannot connect v with more than $2r + 1$ vertices of H_j which lie on the same root-to-leaf path in H_j. Since $|\mathcal{P}| > (2r + 1) \cdot \alpha$, we can find a set $X \subseteq V(H_j)$ such that $|X| > \alpha$, each vertex of X is connected to v by some path from $\mathcal{P}$, and no two vertices of X lie on the same root-to-leaf path in H_j. Recall that, by the construction, each leaf of H_j is connected to a different branch set $H_{j'}$ for some $j' < j$. Consequently, we can take the paths of $\mathcal{P}$ leading to X and extend them within H_j to obtain a family of more than α disjoint paths in G_{j-1} that connect v with different branch sets $H_{j'}$ for $j' < j$. This contradicts Lemma 8. ◀

Observe that the order can be computed in time $\mathcal{O}(n^5)$: for each vertex, we compute by a standard flow algorithm in time $\mathcal{O}(n^3)$ whether it should be chosen as the next tree root to form a subgraph H_{i_j}. This choice has to be made at most n times.

Finally, we state one property of the construction that follows immediately from Lemma 8.

▶ **Lemma 19.** *Each constructed subgraph H_i has maximum degree at most $\alpha + 1$, where α is the constant given by Lemma 8.*

5 Model-checking for successor-invariant first-order formulas

A *finite and purely relational signature* τ is a finite set $\{R_1, \ldots, R_k\}$ of relation symbols, where each relation symbol R_i has an associated arity a_i. A finite τ-structure $\mathfrak{A}$ consists of a finite set A, the universe of $\mathfrak{A}$, and a relation $R_i(\mathfrak{A}) \subseteq A^{a_i}$ for each relation symbol $R_i \in \tau$. If $\mathfrak{A}$ is a finite τ-structure, then the *Gaifman graph* of $\mathfrak{A}$, denoted $G(\mathfrak{A})$, is the graph with $V(G(\mathfrak{A})) = A$ and there is an edge $uv \in E(G(\mathfrak{A}))$ if and only if $u \neq v$ and u and v appear together in some relation $R_i(\mathfrak{A})$ of $\mathfrak{A}$. We say that a class $\mathcal{C}$ of finite τ-structures has *bounded expansion* if the graph class $G(\mathcal{C}) := \{G(\mathfrak{A}) : \mathfrak{A} \in \mathcal{C}\}$ has bounded expansion. Similarly, for $r \in \mathbb{N}$, we write $\mathrm{adm}_r(\mathfrak{A})$ for $\mathrm{adm}_r(G(\mathfrak{A}))$ etc.

Let V be a set. A successor relation on V is a binary relation $S \subseteq V \times V$ such that (V, S) is a directed path of length $|V| - 1$. Let τ be a finite relational signature. A formula $\varphi \in \mathrm{FO}[\sigma \cup \{S\}]$ is *successor-invariant* if for all τ-structures $\mathfrak{A}$ and for all successor relations S_1, S_2 on $V(\mathfrak{A})$ it holds that $(\mathfrak{A}, S_1) \models \varphi \iff (\mathfrak{A}, S_2) \models \varphi$.

Successor-invariant logics have been studied in database theory and finite model theory in the past. It was shown by Rossman [15] that successor-invariant FO is more expressive than FO without access to a successor relation. It is known that successor-invariant FO (in fact even order-invariant FO) can express only local queries [10], however, the proof does not translate formulas into local FO-formulas which could be evaluated algorithmically. It was shown in [7] that the model-checking problem for successor-invariant first-order formulas is fixed-parameter tractable on any proper minor closed class of graphs. Very recently, the same result was shown for classes with excluded topological minors [6]. We give a new proof of the model-checking result of [6] which is based on the nice properties of the order we have constructed for graphs that exclude a topological minor.

Eickmeyer et al. [7] showed that on well-behaved classes of graphs one can apply the following reduction from the model-checking problem for successor-invariant formulas to the model-checking problem for plain first-order formulas.

▶ **Lemma 20** (Eickmeyer et al. [7]). *Let $\mathcal{C}$ be a class of τ-structures such that for each $\mathfrak{A} \in \mathcal{C}$ one can compute in polynomial time a graph $H(\mathfrak{A})$ such that*

1. *$V(H(\mathfrak{A})) = V(G(\mathfrak{A}))$ and $E(H(\mathfrak{A})) \supseteq E(G(\mathfrak{A}))$.*
2. *H contains a spanning tree T which can be computed in polynomial time and which is of maximum degree d for some fixed integer d depending on $\mathcal{C}$ only.*
3. *The model-checking problem for first-order formulas on the graph class $\{H(\mathfrak{A}) : \mathfrak{A} \in \mathcal{C}\}$ is fixed-parameter tractable.*

Then the model-checking problem for successor-invariant first-order formulas is fixed-parameter tractable on $\mathcal{C}$.

We remark that the original lemma from [7] refers to k-walks in H, which are easily seen to be equivalent to spanning trees of maximum degree k. In our view, spanning trees are more intuitive to handle in our graph theoretic context.

▶ **Lemma 21.** *Let $k \in \mathbb{N}$. There is a constant δ, depending only on k, and a function $f \colon \mathbb{N} \to \mathbb{N}$ such that the following holds. For every graph G with $K_k \not\preccurlyeq^t G$ we can compute in polynomial time a supergraph H with $V(H) = V(G)$ and $E(H) \supseteq E(G)$ such that $\mathrm{adm}_r(H) \le f(r)$ for all $r \in \mathbb{N}$ and such that H contains a spanning tree T with maximum degree at most δ; furthermore, such a spanning tree T can be also computed in polynomial time.*

Proof. Without loss of generality, we assume that G is connected. Otherwise, we may apply the construction in each connected component separately, and then connect the components arbitrarily using single edges (added to H) in a path-like manner. It is easy to see that including the additional edges to the spanning tree increases its maximum degree by at most 2, while the admissibility of the graph also increases by at most 2.

We perform the construction of the subgraphs $H_1, \ldots, H_\ell$ almost exactly as in Section 4. However, when constructing the H_i's and the order L, we put some additional restrictions that do not change the quality of L. First, recall that when we defined H_{i+1}, for some $0 \le i < \ell$, we considered a tree of breadth-first search starting at v_{i+1} in a connected component C of G_i. Suppose that the subgraphs that C is connected to are $H_{i_1}, \ldots, H_{i_s}$, where $1 \le i_1 < \ldots < i_s \le i$. Then H_{i+1} was defined as a minimal subtree of the considered BFS tree that contained, for each $1 \le j \le s$, some vertex of H_{i_j} that is adjacent to C.

Observe that in the construction we were free to choose which neighbour of H_{i_j} will be picked to be included in H_{i+1}. For $j < s$ we make an arbitrary choice as before, but the neighbour of H_{i_s} (if exists; note that this is the case for $i > 0$) is chosen as follows. We first select the vertex $w'_{i+1} \in V(H_{i_s})$ that is the largest in the order L among those vertices of H_{i_s} that are adjacent to C (the vertices of H_j for $j \leq i$ are already ordered by L at this point). Then, we select any its neighbour w_{i+1} in C as the vertex that is going to be included in H_{i+1} in its construction. Finally, recall that in the construction of L, we could order the vertices of H_{i+1} arbitrarily. Hence, we fix an order of H_{i+1} so that w_{i+1} is the smallest among $V(H_{i+1})$. This concludes the description of the restrictions applied to the construction.

We now construct H by taking G and adding some edges. During the construction, we will mark some edges of H as *spanning edges*. We start by marking all the edges of all the trees H_i, for $1 \leq i \leq \ell$, as spanning edges. At the end, we will argue that the spanning edges form a spanning tree of H with maximum degree at most δ.

For each i with $1 \leq i < \ell$, let us examine the vertex w_{i+1}, and let us *charge* it to w'_{i+1}. Note that in this manner every vertex w_{i+1} is charged to its neighbour that lies before it in the order L. For any $w \in V(G)$, let $D(w)$ be the set of vertices charged to w. Now examine the vertices of G one by one, and for each $w \in V(G)$ do the following. If $D(w) = \emptyset$, do nothing. Otherwise, if $D(w) = \{u_1, u_2, \ldots, u_h\}$, mark the edge wu_1 as a spanning edge, and add edges $u_1u_2, u_2u_3, \ldots, u_{h-1}u_h$ to H, marking them as spanning edges as well.

▶ **Claim 22** (∗)**.** *The spanning edges form a spanning tree of H of maximum degree at most $\alpha + 4$, where α is the constant given by Lemma 8.*

It remains to argue that H has small admissibility. For this, it suffices to prove the following claim. The proof uses the additional restrictions we introduced in the construction.

▶ **Claim 23** (∗)**.** *Let r be a positive integer. If the order L certifies that $\mathrm{col}_{2r}(G) \leq m$, that is, $\max_{v\in V(G)} |\mathrm{SReach}_{2r}[G, L, v]| \leq m$, then $\mathrm{adm}_r(H) \leq m + 2$.*

The statement of the lemma now directly follows from Claims 22 and 23. ◀

Given a graph G that excludes K_k as a topological minor, let us write $H(G)$ for a graph constructed according to Lemma 21.

▶ **Corollary 24.** *The class $\{H(G) : K_k \not\preccurlyeq^t G\}$ has bounded expansion.*

We can now use Theorem 20 to combine the following result of Dvořak et al. [5] with Lemma 21, to prove fixed-parameter tractability of successor-invariant FO on classes that exclude a fixed topological minor.

▶ **Lemma 25** (Dvořák et al. [5])**.** *The model-checking problem for first-order formulas is fixed-parameter tractable on any class of bounded expansion.*

▶ **Corollary 26.** *The model-checking problem for successor-invariant first-order formulas is fixed parameter tractable on any class of graphs that excludes a fixed topological minor.*

6 Conclusions

In this work we gave several new applications of the generalised colouring numbers on classes of bounded expansion. In particular, we have shown that whenever a graph class $\mathcal{C}$ excludes some fixed topological minor, then any graph from $\mathcal{C}$ admits one ordering of vertices that certifies the boundedness of the generalised colouring numbers for all radii r at once. It is tempting to conjecture that such an ordering exists for any graph class of bounded expansion.

Our construction of the uniform ordering proved to be useful in showing that model-checking successor-invariant FO is FPT on any graph class that excludes a fixed topological minor. We believe that our construction may be helpful in extending this result to any graph class of bounded expansion, since both the construction of the order, and the reasoning of Section 5, are oblivious to the fact that the graph class excludes some topological minor. The only place where we used this assumption is the analysis of the constructed order.

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