On Synchronizing Colorings and the Eigenvectors of Digraphs*

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Abstract

An automaton is synchronizing if there exists a word that sends all states of the automaton to a single state. A coloring of a digraph with a fixed out-degree k is a distribution of k labels over the edges resulting in a deterministic finite automaton. The famous road coloring theorem states that every primitive digraph has a synchronizing coloring. We study recent conjectures claiming that the number of synchronizing colorings is large in the worst and average cases.

Our approach is based on the spectral properties of the adjacency matrix $\mathcal{A}(G)$ of a digraph G. Namely, we study the relation between the number of synchronizing colorings of G and the structure of the dominant eigenvector $\vec{v}$ of $\mathcal{A}(G)$. We show that a vector $\vec{v}$ has no partition of coordinates into blocks of equal sum if and only if all colorings of the digraphs associated with $\vec{v}$ are synchronizing. Furthermore, if for each b there exists at most one partition of the coordinates of $\vec{v}$ into blocks summing up to b, and the total number of partitions is equal to s, then the fraction of synchronizing colorings among all colorings of G is at least $\frac{k-s}{k}$. We also give a combinatorial interpretation of some known results concerning an upper bound on the minimal length of synchronizing words in terms of $\vec{v}$.

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1 Introduction

Let $\mathscr{A} = (Q, \Sigma, \delta)$ be a finite deterministic complete automaton with an alphabet Σ, a set of states Q and a transition function δ. The automaton $\mathscr{A}$ is *synchronizing* if there exist a word u and a state p such that for every state $q \in Q$ we have $q \cdot u = p$, where $q \cdot u$ denotes the image of q under the action of u. Any such word u is called *synchronizing* (or *reset*) word for $\mathscr{A}$. The length of the shortest synchronizing word $\mathrm{rt}(\mathscr{A})$ is called *the reset threshold* of $\mathscr{A}$. Synchronizing automata naturally appear in algebra, coding theory, industrial automation,

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discrete dynamical systems, etc. A brief survey of the theory of synchronizing automata may be found in [19].

Two fundamental problems about synchronizing automata that were intensively investigated in the last decades are the Černý conjecture and the road coloring problem. The former states that the reset threshold of an n-state automaton is at most $(n-1)^2$ [9]. Despite intensive research efforts it remains open for already half a century. The latter problem states a certain connection between primitive digraphs and synchronizing automata, which we will explain shortly, and was recently resolved by Trakhtman [18] after crucial insight by Culik, Karhumäki, and Kari [10]. Our paper is devoted to the generalizations of the road coloring theorem.

1.1 The road coloring theorem

The underlying digraph $\mathcal{G}(\mathscr{A})$ of an automaton $\mathscr{A}$ is a digraph with Q as a set of vertices, and for each $u \in Q, x \in \Sigma$ there is an edge $(u, u \cdot x)$. We allow loops and multiple edges, thus $\mathcal{G}(\mathscr{A})$ has a fixed out-degree equal to the cardinality of the alphabet Σ, i.e., $\mathcal{G}(\mathscr{A})$ is a $|\Sigma|$-out-regular digraph.

Vice versa, given a digraph G with a fixed out-degree k and a finite alphabet Σ with k letters, we can obtain a deterministic finite automaton by distributing the letters of Σ over the edges of G. Any automaton obtained in this way is called a *coloring* of G. A digraph is *primitive* if there exists a number t such that for any two vertices u and v there exists a path from u to v of length exactly t. An automaton is *strongly connected* if its underlying digraph is strongly connected.

▶ **Theorem 1** (Road coloring theorem). *A strongly connected digraph G with a fixed out-degree k has a synchronizing coloring if and only if it is primitive.*

This theorem was stated as a conjecture in 1977 [1]. The authors' original motivation comes from symbolic dynamics. Namely, synchronizing coloring defines a morphism from a shift of finite type given by G to a full shift over Σ with special properties, see [3].

The origin of the terminology is as follows. A digraph G represents a network of one-way roads. A coloring of G defines labels of the roads that can be perceived by drivers. If the coloring is synchronizing then the drivers who are unaware of their current location have the following strategy to relocate themselves: they can simply follow roads labelled by a synchronizing word and their final position will be well defined.

Although the road coloring theorem gives an answer for a principal connection between digraphs and synchronizing automata, there are still basic quantitative questions that remain unanswered. Namely, how many synchronizing colorings a primitive digraph G can have and what is the number of synchronizing colorings of an average (or random) digraph? These questions were addressed in [13] and two conjectures were formulated as a result of extensive computational experiments. In order to state them, we will need some definitions.

The *synchronizing ratio* of a digraph G is the number of synchronizing colorings divided by the total number of colorings. Note, that a coloring is a mapping from the set of edges to Σ with parallel edges being distinguished. Thus, the total number of colorings of a k-out-regular digraph with n states is always k^n.

▶ **Conjecture 2.** *The minimum value of the synchronizing ratio among all k-out-regular primitive digraphs with n vertices is equal to $\frac{k-1}{k}$, except for the case $k = 2$ and $n = 6$ when it is equal to $\frac{30}{64}$.*

We say that the digraph is *totally synchronizing* if its synchronizing ratio is equal to 1, i.e., every coloroing is synchronizing.

▶ **Conjecture 3.** *For every $k \geq 2$, the fraction of totally synchronizing digraphs among all k-out-regular primitive digraphs with n vertices tends to 1 as n goes to infinity.*

If both conjectures are true, then the road coloring theorem is a relatively weak statement that gives us just the first step towards satisfactory understanding of the synchronizing properties of automata and digraphs.

We want to mention another direction to strengthen the road coloring theorem.

▶ **Conjecture 4** (Hybrid Černý–Road Coloring Problem)**.** *Every primitive k-out-regular digraph with n vertices has a synchronizing coloring with the reset threshold at most $n^2 - 3n + 3$.*

This conjecture was made by M. V. Volkov and partial results were obtained in [17, 8].

1.2 Our contributions

One of the major obstacles in approaching conjectures 2 and 3 comes from the difficulty of proving that a coloring under consideration is synchronizing. A simple and straightforward proof of this fact tends to be tedious and technical even for relatively simple automata, see for example [12]. In order to overcome this difficulty we rely on spectral properties of the adjacency matrix $\mathcal{A}(G)$ of a primitive k-out-regular digraph G.

More precisely, Perron-Frobenius theorem [15, Chapter 8] implies existence of entrywise positive eigenvector $\vec{v}$ of $\mathcal{A}(G)$ associated with the unique largest eigenvalue, which we will simply call *the eigenvector* of G. The vector $\vec{v}$ can also be seen as the unique stationary distribution of the Markov chain associated with G by assigning the probability $\frac{1}{k}$ for each of the outgoing edges.

The importance of the eigenvector of G in the context of synchronizing automata was demonstrated by Friedman [11]. We will require a few definitions to state his result. Let $Q = \{1, \dots, n\}$, and $\vec{v}[i]$ be the ith entry of $\vec{v}$. The *weight* of a subset $S \subseteq Q$ is given by $\mathrm{wg}(S) = \sum_{i \in S} \vec{v}[i]$. The subset $S \subseteq Q$ is *synchronizing* if there exists a word u and a state p such that for every $q \in S$ we have $q \cdot u = p$.

▶ **Theorem 5.** *Every coloring $\mathscr{A}$ of G has a partition of vertices into synchronizing subsets $Q_1, \dots, Q_\ell$ such that $\mathrm{wg}(Q_1) = \dots = \mathrm{wg}(Q_\ell)$ and for any other synchronizing subset S we have $\mathrm{wg}(S) \leq \mathrm{wg}(Q_1)$.*

A simple corollary of this statement allows us to easily identify a relatively large class of totally synchronizing digraphs. We will say that a vector $\vec{v}$ is *partitionable* if there exists a partition of $\vec{v}$ into blocks of equal weight b, i.e., a partition $Q_1, \dots, Q_\ell$ of Q with $\ell > 1$ such that $\sum_{i \in Q_1} \vec{v}[i] = \dots = \sum_{i \in Q_\ell} \vec{v}[i] = b$. Clearly, a digraph with non-partitionable eigenvector is totally synchronizing, otherwise maximal synchronizing subsets, i.e. synchronizing subsets with the largest weight, give rise to a partition by theorem 5. Our first contribution is the converse (in some sense) of this statement. Namely, let $\mathcal{G}(\vec{v})$ be the class of primitive digraphs of fixed out-degree with the eigenvector $\vec{v}$. We show that all digraphs in $\mathcal{G}(\vec{v})$ are totally synchronizing if and only if $\vec{v}$ is non-partitionable. We also formulate an algebraic conjecture that implies conjecture 3. These results are given in section 3.

Our second contribution is a lower bound on the synchronizing ratio of G depending on the structure of $\vec{v}$. We say that the partition $Q_1, \dots, Q_\ell$ of $\vec{v}$ into blocks of weight b is *unique* if for every partition $Q'_1, \dots, Q'_\ell$ of weight b there exists a permutation of $1, \dots, \ell$ such that $Q_i = Q'_{\sigma(i)}$ for all i. In section 4 we show that if all partitions of $\vec{v}$ into blocks of equal weight are unique and their number is bounded by s, then the synchronizing ratio of G is at least $\frac{k-s}{k}$. Note, that for $s = 1$ we obtain the bound of conjecture 2. To the best of our knowledge it is the first result that shows validity of the conjecture on a relatively large class of digraphs, e.g., this class contains all primitive Eulerian[1] digraphs with a prime number of states.

Let $\mathscr{A}$ be a coloring of G. We can consider an arbitrary probability distribution on the letters of $\mathscr{A}$ turning it into a Markov chain. Similarly to the previous uniform case we obtain an eigenvector $\vec{v'}$ corresponding to the unique stationary distribution. The vector $\vec{v'}$ played an important role in various proofs of the Černý conjecture in the special classes of automata, see e.g. [16, 5, 6]. Our third contribution is related to such approaches. First, in section 2 we generalize theorem 5 to the case of arbitrary probability distributions on the alphabet. Secondly, in section 5 we present a combinatorial reduction from an arbitrary synchronizing automaton $\mathscr{A}$ to an Eulerian automaton with possibly larger number of states, which has the same reset threshold as $\mathscr{A}$. This reduction gives a combinatorial view of results by Berlinkov [4] and Steinberg [16].

2 Partitions into synchronizing subsets

In the present section we will prove a generalization of Theorem 5. Let $\mathscr{A}$ be a strongly connected automaton with the set of states $\{1, 2, \dots, n\}$. Let $A_1, A_2, \dots, A_k$ be the adjacency matrices of the letters of $\mathscr{A}$, i.e., $A_\ell[i, j] = 1$ if i is mapped to j under the action of the ℓth letter, and $A_\ell[i, j] = 0$ otherwise.

Consider the matrix $A = \sum_{i=0}^{k} p_i A_i$, where $p_i > 0$ are rational for all i and $\sum_{i=0}^{k} p_i = 1$. Since the matrix A is row-stochastic the largest eigenvalue of A is equal to 1. By the Perron-Frobenius theorem [15, Chapter 8] there exists a positive left eigenvector $\vec{u}$ such that $\vec{u}A = \vec{u}$. Since the entries of A are rational, so are the entries of $\vec{u}$. Let $\vec{w} = \ell\vec{u}$, where ℓ is the least common multiple of the denominators of entries of $\vec{u}$. We will call the vector $\vec{w}$ *the eigenvector* of $\mathscr{A}$ in accordance with the distribution $p_1, \dots, p_k$. If the distribution is uniform, i.e., $p_1 = p_2 = \dots = p_k = \frac{1}{k}$, then we will usually omit its description. Since all colorings of a digraph G have the same eigenvector $\vec{w}$ in accordance with the uniform distribution we will call $\vec{w}$ obtained in this way *the eigenvector* of G.

The *kernel* of a word x with respect to an automaton $\mathscr{A}$ is an equivalence relation ρ on the set of states Q such that $i\rho j$ if and only if $i \cdot x = j \cdot x$. A subset S is *synchronizing* if there exists a word x such that the cardinality of $S \cdot x = \{q \cdot x \mid q \in S\}$ is equal to 1. By $S \cdot x^{-1}$ we denote the full preimage of the set S under the action of a word x, i.e., $S \cdot x^{-1} = \{q \in Q \mid q \cdot x \in S\}$. Let $\vec{w}$ be the eigenvector of the automaton $\mathscr{A}$. We define *the weight* $\mathrm{wg}(i)$ of a state i as $\vec{w}[i]$. The weight of a set S is defined as $\mathrm{wg}(S) = \sum_{i \in S} \mathrm{wg}(i)$.

▶ **Theorem 6.** *Let $\vec{w}$ be the eigenvector of a strongly connected automaton $\mathscr{A}$ in accordance with a distribution $p_1, p_2, \dots, p_k$. There exists a partition of the states of $\mathscr{A}$ into synchronizing subsets of maximal weight. Furthermore, this partition is equal to the kernel of some word x.*

[1] A digraph is *Eulerian* if the outdegree and indegree of each vertex is equal to k for some constant k. The eigenvector of such digraph is equal to $(1, 1, \dots, 1)$.

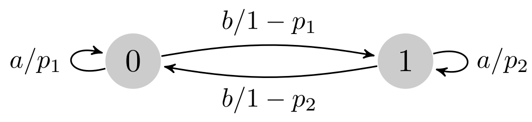

Figure 1 Automaton $\mathscr{F}$.

Proof. Let $\Sigma = \{a_1, a_2, \ldots, a_k\}$, and let S be an arbitrary subset of Q. Note the following equality:

$$\sum_{i=1}^{k} p_i \operatorname{wg}(S \cdot a_i^{-1}) = \operatorname{wg}(S)$$

(the incoming edges to S in total bring the weight equal to $\operatorname{wg}(S)$, and each preimage brings $p_i \operatorname{wg}(Sa_i^{-1})$; the weights are equal, since $\vec{w}$ is the eigenvector of $\mathscr{A}$). If S is a synchronizing subset of maximal weight, then the weights of preimages are bounded by $\operatorname{wg}(S)$, since every preimage is also a synchronizing subset. Moreover, every preimage has the weight equal to $\operatorname{wg}(S)$, otherwise the left-hand side would be strictly less than the right-hand side. Therefore, if S is a synchronizing subset of maximal weight, then every preimage of S is a synchronizing subset of maximal weight.

We will iteratively construct a partition of the set of states of $\mathscr{A}$ into synchronizing subsets of maximal weight. Let S_0 be a synchronizing subset of maximal weight. Let u be a word synchronizing S_0 to some state q: $S_0 \cdot u = q$. If $S_0 = Q$, then the automaton is synchronizing, and the proof is complete. Otherwise, let p be a state that doesn't belong to S_0. Since the automaton $\mathscr{A}$ is strongly connected, there exists a word v such that $q \cdot v = p$. Consider now the sets $S_1 = S_0 \cdot (uv)^{-1}$ and S_0. Note, that S_1 is also a maximal synchronizing subset by the preceding paragraph. Furthermore, both sets are synchronized by uvu. But their images are different, since q is not equal to $p \cdot u$ due to maximality of S_0. Continuing in the same manner we will eventually construct the desired partition of Q. ◀

From the matrix theory point of view, independent assignment of probabilities to the edges of the underlying digraph of $\mathscr{A}$ is more natural than the assignment of probabilities to the letters. Unfortunately, Theorem 6 does not hold in this case. Let $\mathscr{F}$ be the automaton depicted in Fig. 1. The notation ℓ/p means that the edge is labelled by ℓ and has the probability p. Note, that the eigenvector of $\mathscr{F}$ is equal to $(1-p_2, 1-p_1)$. Since every letter acts as a permutation, the automaton $\mathscr{F}$ is not synchronizing. Therefore, the partition of the states into synchronizing subsets should be of the form $\{\{0\},\{1\}\}$, but for $p_2 = \frac{1}{3}$ and $p_1 = \frac{1}{2}$ these subsets have different weight.

▶ **Corollary 7.** *Let $\vec{w}$ be the eigenvector of an automaton $\mathscr{A}$ in accordance with a distribution $p_1, p_2, \ldots, p_k$, and the weight of a subset of states S is given by $\sum_{i \in S} \vec{w}[i]$. If there is no partition of the states into subsets of equal weight, then the automaton $\mathscr{A}$ is synchronizing.*

Unfortunately, the converse of this corollary does not hold. Let $\mathscr{B}$ be an automaton depicted in Fig. 2. It is synchronized by the word $bbaab$ to the state 1. If p and $1-p$ are the probabilities of the letters a and b respectively, then the eigenvector of $\mathscr{B}$ is equal to $(1, 1, p, p)$. Thus, the subsets $\{0, 2\}$ and $\{1, 3\}$ form a partition of the states of $\mathscr{B}$ for any p, in other words, there is no witness of the fact that $\mathscr{B}$ is synchronizing.

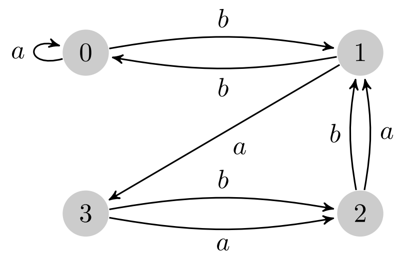

Figure 2 Automaton $\mathscr{B}$.

3 The eigenvectors of totally synchronizing digraphs

Let $\vec{w}$ be an entrywise positive integer vector. We denote by $\mathcal{G}(\vec{w})$ the class of primitive digraphs with the eigenvector $\vec{w}$ such that every digraph in this class has a fixed out-degree (which can be different for two different digraphs from the class). In this section we will characterize in terms of $\vec{w}$ the classes $\mathcal{G}(\vec{w})$ consisting of only totally synchronizing digraphs.

Let $\mathscr{A}$ be an automaton with the set of states Q and an alphabet Σ. Recall that an equivalence relation $\sim$ on Q is a *congruence* if $i \sim j$ implies $i \cdot x \sim j \cdot x$ for all $i, j \in Q$ and $x \in \Sigma$. The *factor automaton* $\mathscr{A}/\sim$ of $\mathscr{A}$ with respect to $\sim$ is defined as follows. The set of states of $\mathscr{A}/\sim$ is equal to the equivalence classes of $\sim$, and its alphabet is equal to Σ. The action of a letter x on an equivalence class C defined in accordance with the representative $c \in C$, i.e., $C \cdot x$ is equal to the class of $c \cdot x$ in $\mathscr{A}$. Since $\sim$ is a congruence, this definition is correct and does not depend on the representative c.

We will call an equivalence relation β on the coordinates of $\vec{w}$ a *partition* if it has at least two classes and satisfies the following property: there exists a constant b such that for every class B of β we have $\sum_{i \in B} \vec{w}[i] = b$. We will refer to the classes of partition β as *blocks*. If $\vec{w}$ is the eigenvector of an automaton $\mathscr{A}$, then every coordinate corresponds to a state of $\mathscr{A}$. Thus, we can naturally obtain an equivalence relation β' on the states of $\mathscr{A}$ from the partition β. Abusing notation, we will refer to β' as β. A vector $\vec{w}$ is called *partitionable* if it possesses a partition.

▶ **Theorem 8.** *An entrywise positive integer vector $\vec{w}$ is not partitionable if and only if all digraphs from $\mathcal{G}(\vec{w})$ are totally synchronizing.*

Proof. Let G be a digraph from $\mathcal{G}(\vec{w})$. If G has a non-synchronizing coloring, then by Theorem 6 it admits a partition of the states into synchronizing subsets of equal weight. Since the coloring is not synchronizing such partition has at least two blocks. Thus, the vector $\vec{w}$ is also partitionable.

Assume now that $\vec{w}$ is partitionable, i.e., there are sets $B_1, B_2, \ldots, B_\ell$ such that for every i we have $\sum_{j \in B_i} \vec{w}[j] = b$. Let n be the number of entries of $\vec{w}$. We will construct a digraph G belonging to $\mathcal{G}(\vec{w})$ on the set of vertices $V = \{0, 1, \ldots, n-1\}$ that is not totally synchronizing as follows: for every pair of vertices i and j there is an edge (i, j) of multiplicity $\vec{w}[j]$.

First, let us show that $G \in \mathcal{G}(\vec{w})$. Note, that the out-degree of every vertex is equal to the sum of entries of $\vec{w}$, i.e., $b\ell$. Furthermore, the digraph G is primitive since there is a path of length 1 between every two vertices. It remains to show that $\vec{w}$ is the eigenvector of G corresponding to the eigenvalue 1. Let $c_{ij} = \vec{w}[j]$ be the multiplicity of the edge from i to j, and $c = b\ell$ be the out-degree of G. We have

$$\sum_{i \in V} \frac{c_{ij}}{c} \vec{w}[i] = \sum_{i \in V} \frac{\vec{w}[j]}{c} \vec{w}[i] = \vec{w}[j] \sum_{i \in V} \frac{\vec{w}[i]}{c} = \vec{w}[j]$$

(the incoming and the outgoing weights are equal). Therefore, $\vec{w}$ is the eigenvector of G.

Now we are going to construct a non-synchronizing coloring of G. We will write $i\,\beta\,j$ for $i, j \in V$ if both i and j belong to B_s for some s. Let $\mathcal{A}$ be the set of colorings of G that have β as a congruence, i.e., for every letter x and for every pair of states i, j such that $i\,\beta\,j$ we necessarily have that $(i \cdot x)\,\beta\,(j \cdot x)$. The set $\mathcal{A}$ is not empty, since it contains the following coloring: the action of the first $\vec{w}[1]$ letters brings all states to the state 1, the action of the next $\vec{w}[2]$ letters brings all states to the state 2, and so on.

Let us fix some automaton $\mathscr{A} \in \mathcal{A}$. Recall that an automaton over k-letter alphabet is Eulerian if the indegree (and the out-degree) of every state is equal to k. Clearly, an automaton is Eulearian if and only if its eigenvector is equal to $(1, 1, \ldots, 1)$. We will show now that the factor automaton $\mathscr{A}'$ of $\mathscr{A}$ with respect to β is Eulerian. Let Σ be the alphabet of $\mathscr{A}$ and $\mathscr{A}'$. Relying on the fact that $\vec{w}$ is the eigenvector of $\mathscr{A}$ we have the following equalities for every block B_t:

$$\sum_{\substack{i \in Q, x \in \Sigma \\ i \cdot x = j}} \frac{1}{c} \vec{w}[i] = \vec{w}[j] \quad \Rightarrow \sum_{\substack{i \in Q, x \in \Sigma \\ i \cdot x \in B_t}} \frac{1}{c} \vec{w}[i] = b \quad \Rightarrow \sum_{\substack{B_s, x \in \Sigma \\ B_s \cdot x = B_t}} \frac{1}{c} b = b$$

The last equality ensures that $(b, b, \ldots, b)$ is the eigenvector of $\mathscr{A}'$, thus, it is Eulerian.

Lemma 1 from [14] states that every Eulerian automaton has a non-synchronizing coloring[2]. Thus, we can recolor an automaton $\mathscr{A}'$ into a non-synchronizing automaton $\mathscr{B}'$. Such recoloring procedure can be seen as a sequence of basic flips, i.e., for a fixed B_t, $x_1, x_2 \in \Sigma$ we change the label from x_1 to x_2 and vice versa on the outgoing edges of B_t. Therefore, this recoloring can be applied to $\mathscr{A}$ leading to an automaton $\mathscr{B}$ in the following manner: a basic flip is applied simultaneously to all states of B_t. The latter ensures that β is a congruence of $\mathscr{B}$ and $\mathscr{B}'$ is the factor automaton of $\mathscr{B}$ with respect to β. Note that the automaton $\mathscr{B}$ is not synchronizing, since any synchronizing word of the automaton $\mathscr{B}$ will synchronize the automaton $\mathscr{B}'$ leading to a contradiction. ◀

Theorem 8 allows us to obtain very simple proofs for otherwise non-obvious statements. Recall that the Černý automaton $\mathscr{C}_n$ [9] can be defined as $\langle\{0, \ldots, n-1\}, \{a, b\}, \delta\rangle$, where $\delta(i, a) = i + 1$ for $i < n - 1$, $\delta(n - 1, a) = 0$, $\delta(n - 1, b) = 0$, and $\delta(i, b) = i$ for $i < n - 1$.

▶ **Proposition 9.** *[13, Proposition 2] The underlying digraph of the Černý automaton $\mathscr{C}_n$ is totally synchronizing.*

Proof. It is easy to verify that the eigenvector $\vec{w}$ of the underlying digraph of the n-state Černý automaton is equal to $(2, 2, \ldots, 2, 1)$. Since in every partition exactly one block will have an odd sum, we conclude that $\vec{w}$ is not partitionable. Thus, the digraph is totally synchronizing. ◀

A similar proof can be presented for many other examples in [2].

Another application of theorem 8 is related to conjecture 3. We believe that significant progress on this conjecture can be made through the study of the eigenvectors of digraphs.

[2] It is also a relatively simple corollary of the Birkhoff-von Neumann theorem

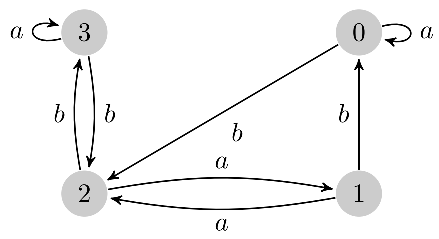

Figure 3 Automaton $\mathscr{D}$.

Despite the fact that the statement of Theorem 8 gives only a necessary condition for a digraph to be totally synchronizing we expect it to hold in most cases. More formally, we state the following conjecture:

▶ **Conjecture 10.** *The eigenvector of a random primitive k-out-regular digraph with n vertices has no partition into blocks of equal sum with probability 1 as n goes to infinity.*

This conjecture has the following interpretation in terms of Markov chains theory. A primitive k-out-regular digraph G correspond to Markov chain via the distribution of the probability $\frac{1}{k}$ for each edge. Furthermore, this chain is mixing, i.e., irreducible and aperiodic, and its stationary distribution is equal to the eigenvector of the the digraph. Informally speaking, a partition of the eigenvector corresponds to a partition of states of the Markov chain into classes such that an infinitely long random walk will spend equal amount of time in each of the classes. Conjecture 10 states that the fraction of Markov chains with this property goes to 0 as the number of states grows.

▶ **Corollary 11.** *If the eigenvector of G is not partitionable, then G is totally synchronizing.*

There are classes of digraphs $\mathcal{G}(\vec{w})$ that contain both totally synchronizing and not totally synchronizing digraphs. Let $\vec{w}$ be $(1, 1, 2, 2)$. The underlying digraph of the automaton $\mathscr{D}$, see fig. 3, belongs to $\mathcal{G}(\vec{w})$. It is not totally synchronizing, since the pair $\{2, 3\}$ is not synchronizable in the coloring $\mathscr{D}$. At the same time, it is easy to see that the underlying digraph of the automaton $\mathscr{B}$, see fig. 2, belongs to $\mathcal{G}(\vec{w})$ and it is totally synchronizing.

There are also classes of digraphs $\mathcal{G}(\vec{w})$ which do not contain totally synchronizing digraphs at all. Namely, if $\vec{w} = (1, 1, \ldots, 1)$ then every digraph in $\mathcal{G}(\vec{w})$ is Eulerian, thus it possesses a non-synchronizing coloring [14, Lemma 1].

4 Partitions of the eigenvectors and the synchronizing ratios

In this section we will present a bound on the synchronizing ratio of a digraph G depending on the structure of its eigenvector. It can be seen as the the first theoretical statement supporting conjecture 2. In order to obtain our result we will rely on the following key lemma:

▶ **Lemma 12.** *Let $\mathscr{A}$ be a non-synchronizing automaton with the eigenvector $\vec{w}$. A partition into maximal synchronizing subsets is unique if and only if it is a congruence.*

Proof. Assume first that the partition into maximal synchronizing subsets is unique. We will denote the block containing a state p by $[p]$. If the partition is not a congruence, then there exists a letter ℓ such that $[p] = [q]$ and $[p\ell] \neq [q\ell]$ for some states p and q. Note, that the preimage of a maximal synchronizing subset by any letter is also a maximal synchronizing subset (see the proof of Theorem 6). Hence, the preimage of a partition into maximal synchronizing subsets is also a partition into maximal synchronizing subsets. Thus, $[p\ell]\ell^{-1}$ is a maximal synchronizing subset and $[p\ell]\ell^{-1} \cap [p] \neq \varnothing$. We also have $[p\ell]\ell^{-1} \neq [p]$, otherwise we get $[p\ell] = [p]\ell$ which implies $[p\ell] = [q\ell]$. Therefore, the preimage of the partition by the letter ℓ is a different partition into maximal synchronizing subsets. A contradiction.

Let τ be a partition into maximal synchronizing subsets. Let us assume that the partition τ is a congruence. Assume to the contrary that there is another partition σ into synchronizing subsets of maximal weight. Note, that there are states p and q such that $p \sim_\sigma q$ and $p \nsim_\tau q$, otherwise σ is a refinement of τ, and σ is not a partition into synchronizing subsets of maximal weight. Since $p \sim_\sigma q$ there exists a word u such that $pu = qu$. Let $[p]$ and $[q]$ be the blocks of the partition τ of p and q respectively. Since τ is a congruence both $[p]u$ and $[q]u$ are subsets of the same block $[r]$ for some state r. The subset $[r]$ is synchronizing. Therefore, the subset $[p] \cup [q]$ is also synchronizing, which contradicts maximality of $[p]$ and $[q]$. ◀

▶ **Corollary 13.** *A digraph G with the eigenvector $\vec{w}$ is totally synchronizing if the following conditions hold:*

1. *if there exists a partition of $\vec{w}$ into blocks of weight b, then it is unique;*
2. *every partition of $\vec{w}$ is not a congruence for every coloring.*

Let Q be a set of states of an automaton $\mathscr{A}$ with the eigenvector $\vec{w}$. A partition of $\vec{w}$ into blocks of weight b is a partition $Q_1, \dots, Q_\ell$ of Q with $\ell > 1$ such that $\sum_{i \in Q_1} \vec{w}[i] = \dots = \sum_{i \in Q_\ell} \vec{w}[i] = b$. For simplicity in this section we will sometimes say "a partition" meaning a partition into blocks of equal weight. A partition $Q_1, \dots, Q_\ell$ of $\vec{w}$ into blocks of weight b is unique if for every partition $Q'_1, \dots, Q'_\ell$ of weight b there exists a permutation of $1, \dots, \ell$ such that $Q_i = Q'_{\sigma(i)}$ for all i.

▶ **Theorem 14.** *If all partitions of the eigenvector $\vec{w}$ are unique and their number is equal to s, then the synchronizing ratio of every k-out-regular digraph in $\mathcal{G}(\vec{w})$ is at least $\frac{k-s}{k}$.*

Proof. Every non-synchronizing coloring is associated with a partition of $\vec{w}$ according to theorem 6. We will show that with every partition at most $\frac{1}{k} \cdot k^n$ such colorings can be associated. Thus, the total number of non-synchronizing colorings will be bounded by $\frac{s}{k} \cdot k^n$, and the theorem will follow.

Let G be a digraph in $\mathcal{G}(\vec{w})$, and let β be one of the partitions of $\vec{w}$. In order to show that the fraction of non-synchronizing colorings associated with β is at most $\frac{1}{k}$ we will consider two cases depending on the structure of G and β.

Case I: there are two distinct vetrices q, p belonging to the same block B of β with the following property: there are edges (q, q') and (p, p') such that q' and p' belong to different blocks of β. Let $\mathscr{A}$ be a non-synchronizing coloring associated with β (if there is no such coloring, then the proof is complete). By lemma 12 the partition β is a congruence for $\mathscr{A}$. Thus, for every block B' there is the same number of letters (and edges) going from q to B' and from p to B'. Let k_1 be the number of edges going from q to B_1, k_2 be the number of edges going from q to B_2, …, k_ℓ be the number of edges going from q to B_ℓ, where $B_1, \dots, B_\ell$ are blocks of β and $k_1, \dots, k_\ell$ are positive integers such that $\sum_{i=1}^{\ell} k_i = k$.

Now we will divide all colorings of G into classes and show that the fraction of non-synchronizing colorings in each class is at most $\frac{1}{k}$. Let us fix a coloring C of all edges, except for the outgoing edges of q and p. Let $\mathcal{A}(C)$ be the set of automata obtainable from C by all possible colorings of the remaining edges. We will show that the fraction of non-synchronizing automata in $\mathcal{A}(C)$ is at most $\frac{1}{k}$. By lemma 12 every non-synchronizing coloring of G associated with β must be a congruence. Note, that there are at most

$\binom{k}{k_1}(k_1!)^2\binom{k-k_1}{k_2}(k_2!)^2\ldots\binom{k_\ell}{k_\ell}(k_\ell!)^2$ automata in $\mathcal{A}(C)$ that have β as a congruence. Whereas the total number of automata in $\mathcal{A}(C)$ is $(k!)^2$. Thus, the fraction of non-synchronizing automata is at most $\frac{k_1!k_2!\ldots k_\ell!}{k!}$. It is not hard to see, that this value is bounded by $\frac{1}{k}$. Since every coloring of G belongs to $\mathcal{A}(C)$ for some C, the result will follow.

Case II: for all **distinct** vetrices q, p belonging to the same block B of β and for all edges (q, q') and (p, p') we have that q' and p' belong to the same block of β. Thus, there is at least one singleton, i.e. a block of β consisting of a single vertex. Indeed, if it is not the case, then each of the blocks has a unique successor. It implies that there are no paths of the same length leading from vertices belonging to different blocks to some fixed vertex, so G is not primitive. Note, that β is a congruence for every coloring of G.

First, we will show that a coloring $\mathscr{A}$ of G is synchronizing if and only if the factor automaton $\mathscr{A}/\beta$ is synchronizing. Clearly, if $\mathscr{A}$ is synchronizing, then $\mathscr{A}/\beta$ is synchronizing too. Assume now that $\mathscr{A}/\beta$ has a synchronizing word u that brings it to a state i. Let j be the state belonging to a singleton block of β. Since G is primitive, there is a word v that brings i to j. It is not hard to see that the word uv is synchronizing for the automaton $\mathscr{A}$.

Secondly, we note that the factor automaton $\mathscr{A}/\beta$ is Eulerian with a prime number of states. Indeed, since β is a partition into blocks of equal weight, we conclude that $\mathscr{A}/\beta$ is Eulerian (see the proof of Theorem 8). If the number of states of $\mathscr{A}/\beta$ is not prime, then the partition β will not be unique. Now it remains to show that the synchronizing ratio of G/β is at least $\frac{k-1}{k}$. The proof of this fact is reminiscent of case I.

Note, that there exist a vertex q and edges $(q, r), (q, s)$ for $r \neq s$, otherwise G is not primitive. Let all the k outgoing edges of q be (q, p_1) of multiplicity k_1, (q, p_2) of multiplicity k_2, …, (q, p_ℓ) of multiplicity k_ℓ. Let us fix a coloring C of all edges, except for the outgoing edges of q. Let $\mathcal{A}(C)$ be the set of automata obtainable from C by all possible colorings of the remaining edges. In order to show that the synchronizing ratio of G is at least $\frac{k-1}{k}$ we will demonstrate that the fraction of non-synchronizing automata in $\mathcal{A}(C)$ is at most $\frac{1}{k}$.

If all automata in $\mathcal{A}(C)$ are synchronizing, then the statement holds true. Otherwise, let $\mathscr{A} \in \mathcal{A}(C)$ be a non-synchronizing automaton. Since the number of states is prime and the eigenvector of $\mathscr{A}$ is equal to $(1, 1, \ldots, 1)$, by Theorem 6 we conclude that every letter of $\mathscr{A}$ acts as a permutation on the set of states. Note, that if edges (q, p_1) and (q, p_2) are labelled by x and y respectively, then the automaton $\mathscr{A}' \in \mathcal{A}(C)$ obtained by flipping the labels on these edges, i.e., assigning letter y to (q, p_1) and letter x to (q, p_2), is synchronizing. Indeed, either p_1 or p_2 is not equal to q. Without loss of generality we will assume that $p_1 \neq q$. Since every letter in $\mathscr{A}$ acts as a permutation, there exists a state r such that $r \cdot y = p_1$. Thus, $r \cdot y = q \cdot y$ for the automaton $\mathscr{A}'$ and it is synchronizing by Theorem 6. More generally, there are at most $k_1!k_2!\ldots k_\ell!$ permutations of labels on the outgoing edges of q that keep the resulting automaton non-synchronizing. Since the value of the fraction $\frac{k_1!k_2!\ldots k_\ell!}{k!}$ is bounded by $\frac{1}{k}$ we obtain the desired statement. ◀

5 The eigenvectors and the reset thresholds

The structure of the eigenvector an automaton $\mathscr{A}$ in accordance with some distribution can be utilized to bound the reset threshold of $\mathscr{A}$. To the best of our knowledge, the first such result was obtained by Kari [14]. He bounded the reset threshold of automata with the eigenvector $(1, 1, \ldots, 1)$ in accordance with the uniform distribution, i.e., Eulerian automata.

▶ **Theorem 15.** *The reset threshold of an Eulerian automaton with n states is at most $n^2 - 3n + 3$.*

Afterwards, Steinberg noticed that the same bound holds true for automata with the eigenvector $(1, 1, \dots, 1)$ in accordance with *some* distribution [16]. Both of these results were later subsumed by the following theorem[3] of Berlinkov [4, Corollary 1].

▶ **Theorem 16.** *Let* $\mathbf{w}$ *be the sum of the coordinates of the integer eigenvector* $\vec{w}$ *of a strongly connected automaton* $\mathscr{A}$ *in accordance with some distribution. If* $\mathscr{A}$ *is synchronizing, then the reset threshold of* $\mathscr{A}$ *is at most* $1 + (n-1)(\mathbf{w} - 2)$.

Note, that the eigenvector of $\mathscr{A}$ in accordance with the uniform distribution depends only on $\mathcal{G}(\mathscr{A})$. Therefore, the bound given in this theorem will be valid for every recoloring of $\mathscr{A}$.

In this section we will present a simple reduction from an automaton $\mathscr{A}$ with the eigenvector $\vec{w}$ to an Eulerian automaton $\mathscr{B}$ with $\mathbf{w} = \sum_i \vec{w}[i]$ states such that $\mathrm{rt}(\mathscr{A}) \leq \mathrm{rt}(\mathscr{B}) \leq \mathrm{rt}(\mathscr{A}) + 1$. Thus, we will be able to utilize results of Kari about Eulerian automata to analyze $\mathscr{A}$. This reduction also gives a combinatorial interpretation of the aforementioned results by Steinberg and, to some extent, of Berlinkov.

▶ **Theorem 17.** *Let* $\mathbf{w}$ *be the sum of the coordinates of the integer eigenvector* $\vec{w}$ *of a strongly connected automaton* $\mathscr{A}$ *in accordance with a distribution* $p_1, p_2, \dots, p_k$*. If* $\mathscr{A}$ *is synchronizing, then there exists a synchronizing Eulerian automaton* $\mathscr{B}$ *with* $\mathbf{w}$ *states such that* $\mathscr{A}$ *is the factor automaton of* $\mathscr{B}$ *and* $\mathrm{rt}(\mathscr{A}) \leq \mathrm{rt}(\mathscr{B}) \leq \mathrm{rt}(\mathscr{A}) + 1$*.*

Proof. Let $\Sigma = \{a_1, a_2, \dots, a_k\}$ and $p_i = \frac{m_i}{\ell}$ for $1 \leq i \leq k$, where m_i, ℓ are positive integers. If there exists p_i such that $p_i \neq \frac{1}{k}$, then we will perform the next step, otherwise we proceed to step II.

Step I. We are going to duplicate certain letters of $\mathscr{A}$ in order to obtain an automaton $\mathscr{A}'$ such that its eigenvector in accordance with the uniform distribution is equal to $\vec{w}$. The alphabet of $\mathscr{A}'$ is equal to $\Sigma' = \{a_1^1, a_1^2, \dots, a_1^{m_1}, a_2^1, a_2^2, \dots, a_2^{m_2}, \dots, a_k^1, a_k^2, \dots, a_k^{m_k}\}$. The actions of these letters are as follows: for every i and j the action of the letter a_i^j in $\mathscr{A}'$ coincides with the action of the letter a_i in $\mathscr{A}$. It is easy to see that $\mathscr{A}'$ is synchronizing and $\mathrm{rt}(\mathscr{A}') = \mathrm{rt}(\mathscr{A})$. Furthermore, the eigenvector of $\mathscr{A}'$ in accordance with the uniform distribution coincides with $\vec{w}$. Thus, if $\vec{w} = (1, 1, \dots, 1)$, then $\mathrm{rt}(\mathscr{A}) = \mathrm{rt}(\mathscr{A}') \leq n^2 - 3n + 3$ by Theorem 15. Therefore, this simple reduction gives an alternative way to obtain the result of Steinberg.

Step II. Now we are going to construct an Eulerian automaton $\mathscr{B}$ on a larger set of states and on a larger alphabet such that $\mathrm{rt}(\mathscr{A}) \leq \mathrm{rt}(\mathscr{B}) \leq \mathrm{rt}(\mathscr{A}) + 1$. Let $Q = \{1, \dots, n\}$ be the set of states of $\mathscr{A}'$ and Σ' be the alphabet of $\mathscr{A}'$. The set of states of $\mathscr{B}$ is equal to $\{(i, j) \mid i \in Q, 1 \leq j \leq \vec{w}[i]\}$. The alphabet of $\mathscr{B}$ is equal to $\Sigma' \cup \Lambda$, where Λ is a set of letters that we will define shortly. We will denote the set $\{(i, j) \mid 1 \leq j \leq \vec{w}[i]\}$ by S_i. Note that the number of states in $\mathscr{B}$ is equal to $\mathbf{w}$. Our construction of $\mathscr{B}$ will ensure the following properties:

1. $\mathscr{B}$ is Eulerian, and for every $i \in Q, x \in \Lambda$ we have $S_i \cdot x \subseteq S_i$;
2. for every i and $x \in \Sigma'$ we have $S_i \cdot x \subseteq S_{i \cdot x}$, where $i \cdot x$ is an image of i under the action of x in $\mathscr{A}'$.

Note, that these conditions imply that the partition $S_1, \dots, S_n$ is a congruence for the automaton $\mathscr{B}$, and the factor automaton with respect to this congruence is equal to the automaton $\mathscr{A}'$.

[3] In the original formulation of the theorem the bound is given in terms of the least common multiple L of the coordinates' denominators of the eigenvector $\vec{v}$ associated with the eigenvalue 1 such that $\sum_i \vec{v}[i] = 1$. Clearly, $\mathbf{w} = L$.

Let $k' = |\Sigma'|$ be the cardinality of Σ' and c_{ij} be the number of letters in Σ' that bring i to j in the automaton $\mathscr{A}'$. By the definition of $\vec{w}$ we have $\sum_{i \in Q} c_{ij}\vec{w}[i] = k'\vec{w}[j]$ for every j. Since $\vec{w}[i] = |S_i|$ we derive the equality $\sum_{i \in Q} c_{ij}|S_i| = k'|S_j|$ for every j. Due to the second property, $c_{ij}|S_i|$ is the total number of edges labelled by Σ' going from S_i to S_j in $\mathscr{B}$. Since the total number of incoming edges to S_j labelled by Σ' is equal to $k'|S_j|$, we can arrange them in such a way that every state of S_j has exactly k' incoming edges labelled by Σ'. We fix any such arrangement to define the action of Σ' on $\mathscr{B}$. Note, that the automaton $\mathscr{B}$ restricted to the alphabet Σ' is Eulerian.

The additional set of letters Λ is defined as follows. For every $i \in Q$ and every $j \in S_i$ we add a letter u_i^j. The action of u_i^j brings all states from S_i to j, and all the remaining states are fixed. Note, that the automaton $\mathscr{B}$ restricted to the alphabet Λ is Eulerian. Therefore, the first property is satisfied.

Step III. We will show now that the automaton $\mathscr{B}$ is synchronizing and $\mathrm{rt}(\mathscr{A}) \leq \mathrm{rt}(\mathscr{B}) \leq \mathrm{rt}(\mathscr{A}) + 1$. Let u be the shortest synchronizing word of $\mathscr{A}'$, and the action of u brings it to a state i. Since $\mathscr{A}'$ is the factor automaton of $\mathscr{B}$, we conclude that the automaton $\mathscr{B}$ is brought to S_i under the action of u. Thus, uu_i^i is a synchronizing word of $\mathscr{B}$ and $\mathrm{rt}(\mathscr{B}) \leq \mathrm{rt}(\mathscr{A}) + 1$.

Let u be a synchronizing word of $\mathscr{B}$, and the action of u brings it to a state in S_i for some i. Let v be a word over the alphabet Σ' obtained from u by removing all the letters from Λ. Since the action of every letter x from Λ of the automaton $\mathscr{B}$ satisfies $S_i \cdot x \subseteq S_i$ and $\mathscr{A}'$ is the factor automaton of $\mathscr{B}$, we conclude that u is a synchronizing word for the automaton $\mathscr{A}'$ and $\mathrm{rt}(\mathscr{A}) \leq \mathrm{rt}(\mathscr{B})$. ◀

▶ **Corollary 18.** *Let* $\mathbf{w}$ *be the sum of the coordinates of the integer eigenvector* $\vec{w}$ *of a strongly connected automaton* $\mathscr{A}$ *in accordance with a distribution* $p_1, p_2, \ldots, p_k$*. If* $\mathscr{A}$ *is synchronizing, then* $\mathrm{rt}(\mathscr{A}) \leq \mathbf{w}^2 - 3\mathbf{w} + 3$.

Clearly, this corollary is much weaker than Theorem 16. Nevertheless, both of these statements give $O(n^2)$ bound when $\mathbf{w} = O(n)$. This case is the most typical application of Theorem 16. At the same time, we believe that such simple reduction to an Eulerian automaton is of interest by itself.

To conclude this section we will estimate entries of the eigenvector of an arbitrary digraph. In general, they can be exponential in terms of the number of vertices. Consider the following k-out-regular digraph $U_{n,k}$. The set of vertices is equal to $\{0, 1, 2, \ldots, n-1\}$. For each $0 \leq i \leq n-1$ there is an edge $(i, 0)$ of multiplicity $k-1$ and an edge $(i, i+1 \mod n)$. It is easy to verify that the integer eigenvector of $U_{n,k}$ is $(k^{n-1}, k^{n-2}, \ldots, k, 1)$. Thus, the upper bound given by Theorem 16 can be exponential in n.

▶ **Proposition 19.** *Let* G *be a primitive* k*-out-regular digraph with* n *vertices. The entries of the eigenvector* $\vec{w}$ *are at most* $(2k^2)^{\frac{n-1}{2}}$.

Proof. Let A be the adjacency matrix of G, i.e., $A[i,j] = 1$ if there exists an edge going from i to j and $A[i,j] = 0$ otherwise. According to the definition of the eigenvector $\vec{w}$ in Section 2 we have the equality $\vec{w}(\frac{1}{k}A) = \vec{w}$. After rearrangement we get $\vec{w}(A - kI) = 0$, where I is the identity matrix. By Perron-Frobenius theorem we conclude that the rank of $A - kI$ is equal to $n-1$, since the eigenspace associated with $\vec{w}$ is one-dimensional. The main result of [7] states that for every integer matrix M of rank r if a system of linear equations $Mx = 0$ admits a nontrivial non-negative integer solution, then there exists such solution with entries bounded by the maximum of the absolute values of the $r \times r$ minors of M.

Thus, we conclude that there exists a non-negative integer vector $\vec{w'}$ such that $\vec{w'}(A - kI) = 0$ and entries of $\vec{w}'$ are bounded by the maximum of the absolute values of the $(n-1) \times (n-1)$

minors of $A - kI$. Note, that the Eucledean norm of each row of every minor is at most $\sqrt{2k^2}$, since the absolute of the entries is at most k and their sum is at most $2k$. Thus, by the Hadamard's inequality for the determinant we obtain an upper bound $(2k^2)^{\frac{n-1}{2}}$ on the minors. Since the non-negative vector $\vec{w'}$ is an eigenvector of A associated with the largest eigenvalue, we immediately conclude that $\vec{w'}$ is positive by the Perron-Frobenius theorem. ◀

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