Graph Properties in Node-Query Setting: Effect of Breaking Symmetry

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Abstract

The query complexity of graph properties is well-studied when queries are on the edges. We investigate the same when queries are on the nodes. In this setting a graph $G = (V, E)$ on n vertices and a property $\mathcal{P}$ are given. A black-box access to an unknown subset $S \subseteq V$ is provided via queries of the form "Does i belong to S?". We are interested in the minimum number of queries needed in the worst case in order to determine whether $G[S]$ – the subgraph of G induced on S – satisfies $\mathcal{P}$.

Our primary motivation to study this model comes from the fact that it allows us to initiate a systematic study of breaking symmetry in the context of query complexity of graph properties. In particular, we focus on the hereditary graph properties – properties that are closed under deletion of vertices as well as edges. The famous Evasiveness Conjecture asserts that even with a minimal symmetry assumption on G, namely that of vertex-transitivity, the query complexity for any hereditary graph property in our setting is the worst possible, i.e., n.

We show that in the absence of any symmetry on G it can fall as low as $O(n^{1/(d+1)})$ where d denotes the minimum possible degree of a minimal forbidden sub-graph for $\mathcal{P}$. In particular, every hereditary property benefits at least quadratically. The main question left open is: Can it go exponentially low for some hereditary property? We show that the answer is no for any hereditary property with finitely many forbidden subgraphs by exhibiting a bound of $\Omega(n^{1/k})$ for a constant k depending only on the property. For general ones we rule out the possibility of the query complexity falling down to constant by showing $\Omega(\log n/ \log \log n)$ bound. Interestingly, our lower bound proofs rely on the famous Sunflower Lemma due to Erdös and Rado.

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1 Introduction

1.1 The query model

The decision tree model (aka query model) has been extensively studied in the past and still remains a rich source of many fascinating questions. In this paper, we focus on Boolean functions, i.e., functions of the form $f : \{0,1\}^n \to \{0,1\}$ and their decision tree complexity. A deterministic decision tree D_f for f takes $x = (x_1, \ldots, x_n)$ as an input and determines the value of $f(x_1, \ldots, x_n)$ using queries of the form "is $x_i = 1$?". Let $C(D_f, x)$ denote the cost of the computation, that is the number of queries made by D_f on input x. The *deterministic decision tree complexity* (aka deterministic query complexity) of f is defined as

$$D(f) = \min_{D_f} \max_{x} C(D_f, x)$$

Randomized and the Quantum variants [6] of decision trees have also been extensively studied in the past. Several different variants such as parity decision trees have been studied in connection to communication complexity, learning, and property testing [25, 20, 4]. We refer the interested reader to the excellent survey by Buhrman and de Wolf [6] for more background on decision tree complexity.

Importance of query models

Variants of the decision tree model are fundamental for several reasons: Firstly, they occur naturally in connection to the other models of computation such as communication complexity [25], property testing [4], learning [20], circuit complexity [13] etc. Secondly, decision tree models are much simpler to analyse as compared to other models such as circuits. Thus one can actually hope to use them as a tool in the study of other models. Thirdly, these models are mathematically rich and beautiful – several connections to algebra, combinatorics, topology, Fourier analysis, and number theory [22, 2] make the decision tree models interesting in their own right. Finally, there remain some fascinating open questions [17] in query complexity that have attracted the attention of generations of researchers over the last few decades by their sheer elegance and notoriety.

1.2 Graph properties in node-query setting

In this paper , we investigate the query complexity of graph properties. In particular, we focus on the following setting: A graph $G = (V, E)$ and a property $\mathcal{P}$ are fixed. We have access to $S \subseteq V$ via queries of the form "Does i belong to S?". We are interested in the minimum number of queries needed in the worst case in order to determine whether $G[S]$ – the subgraph of G induced on S – satisfies $\mathcal{P}$, which we denote by $cost(\mathcal{P}, G)$. One may define a similar notion of cost for randomized and quantum models.

We call G the base graph for $\mathcal{P}$. We say that a vertex i of G is relevant for $\mathcal{P}$ if there exists some S containing i such that exactly one of $G[S]$ and $G[S - \{i\}]$ satisfies $\mathcal{P}$. We say that G is relevant for $\mathcal{P}$ if all its vertices are relevant for $\mathcal{P}$. The minimum possible cost of $\mathcal{P}$, denoted by[1] $min\text{-}cost(\mathcal{P})$, is defined as follows:

$$min\text{-}cost_n(\mathcal{P}) = \min_{G} \{cost(\mathcal{P}, G) \mid G \text{ is relevant for } \mathcal{P} \text{ \& } |V(G)| = n\}.$$

[1] We slightly abuse this notation by omitting the subscript n.

Note that in the node-query settings the notion of *relevance of a graph G for the property $\mathcal{P}$* is important because if any vertex $v \in G$ is *not* relevant then v cannot possibly influence the output of the function and hence any query algorithm does not need to query it.

Similarly one can define *max-cost*$(\mathcal{P})$, which is a more natural notion of complexity when one is interested in studying the universal upper bounds. Investigating the *max-cost* in our setting can indeed be a topic of an independent interest. However, for the purpose of this paper, the notion of *min-cost* will be more relevant as we are interested in finding how low can the universal lower bound on query complexity go under broken symmetry (Refer to Section 1.3 for more on symmetry). It turns out that in the presence of symmetry this bound is $\Omega(n)$ for most of the properties and it is conjectured to be $\Omega(n)$ for any hereditary property in our setting. Recall that a hereditary property is a property of graphs, which is closed under deletion of vertices as well as edges. For instance acyclicity, bipartiteness, planarity, and triangle-freeness are hereditary properties whereas connectedness and containing a perfect matching are not. Every hereditary property can be described by a (not necessarily finite) collection of its forbidden subgraphs.[2] [3]

It appears that the node-query setting is a natural abstraction of scenarios where one is interested in the properties of the subgraph induced by active nodes in a network. We discuss three such examples in the Appendix of the full version of this paper. To the best of our knowledge, no systematic study of node-query setting has been yet undertaken. Here we initiate such a line of inquiry for graph properties. In particular, we focus on the role of presence and absence of *symmetry*.

1.3 Effect of breaking symmetry

The primary reason why we are interested in the node-query model is that it allows us to study the effect of breaking symmetry on query complexities of graph properties. In particular, our setting provides a platform to compare the complexity of $\mathcal{P}$ when the base graph G has certain amount of symmetry with the complexity of $\mathcal{P}$ when G has no symmetry whatsoever. To formalize this, we define the notion of $\mathcal{G}$-*min-cost*$(\mathcal{P})$ for a class of graphs $\mathcal{G}$ by restricting ourselves only to graphs in $\mathcal{G}$.

$$\mathcal{G}\text{-}min\text{-}cost_n(\mathcal{P}) = \min_{G \in \mathcal{G}} \{cost(\mathcal{P}, G) \mid G \text{ is relevant for } \mathcal{P} \ \& \ |V(G)| = n\}.$$

When $\mathcal{G}$ has the highest amount of symmetry, i.e., when $\mathcal{G}$ is the class of complete graphs, then it is easy to see that for every hereditary $\mathcal{P}$, $\mathcal{G}$-*min-cost*$(\mathcal{P})$ is nearly the worst possible, i.e., $\Omega(n)$. It turns out that one does not require the whole symmetry of the complete graph to guarantee the $\Omega(n)$ bound. Even weaker symmetry assumptions on graphs in $\mathcal{G}$, for instance being Cayley graphs of some group, indeed suffices. Thus it is natural to ask how much symmetry is required to guarantee the $\Omega(n)$ bound. In fact, the famous Evasiveness Conjecture implies that even under the weakest form of symmetry on $\mathcal{G}$, i.e., when $\mathcal{G}$ is the class of transitive graphs, for any hereditary property $\mathcal{P}$ the $\mathcal{G}$-*min-cost*$(\mathcal{P})$ would remain the highest possible, i.e., n. So for the complexity to fall down substantially we might have to let go of the transitivity of $\mathcal{G}$. This is exactly what we do. In particular we take $\mathcal{G}$ to be the class of all graphs, i.e., we assume no symmetry whatsoever. Note that in this case

[2] In our setting, every hereditary property is a monotone Boolean function.

[3] We would like to highlight that although we didn't explicitly define *min-cost*$(\mathcal{P})$ or *max-cost*$(\mathcal{P})$ for randomized query model, all our lower bound proofs are based on sensitivity arguments and hence work even for randomized case.

$\mathcal{G}$-*min-cost*$(\mathcal{P})$ = *min-cost*$(\mathcal{P})$ that we defined earlier. Now a natural question is how low can *min-cost*$(\mathcal{P})$ go in the absence of any symmetry? This is the main question addressed by our paper. In particular, we show that for any hereditary property $\mathcal{P}$, the *min-cost*$(\mathcal{P})$ falls down at least quadratically, i.e, to $O(\sqrt{n})$. For some properties, it can go even further below (polynomially down) with polynomials of arbitrary constant degree, i.e. to $O(n^{1/k})$ where k is a constant depending only on the property. The main question left open by our work is: does there exist a hereditary property $\mathcal{P}$ for which *min-cost*$(\mathcal{P})$ is exponentially low? In other words:

▶ Question 1. Is it true that for every hereditary property $\mathcal{P}$ there exists an integer $k_{\mathcal{P}} > 0$ such that

$$\textit{min-cost}(\mathcal{P}) = \Omega(n^{1/k_{\mathcal{P}}})?$$

1.4 Related work

Understanding the effect of symmetry on computation is a very well-studied theme in the past. Perhaps its roots can also be traced back to the non-solvability of quintic equations by radicals – the legendary work of Galois [1]. In the context of query complexity, again there has been a substantial amount of effort invested in understanding the role of symmetry. A recurrent theme here is to exploit the symmetry and some other structure [19] of the underlying functions to prove good lower bounds on their query complexity. For instance the famous Andera-Rosenberge-Karp Conjecture [15] asserts that every non-trivial monotone graph property of n vertex graphs (in the edge-query model) must be evasive, i.e., its query complexity is $\binom{n}{2}$. While a weaker bound of $\Omega(n^2)$ is known, the conjecture remains widely open to this date. Several special cases of the conjecture have also been studied [7]. The randomized query complexity of monotone graph properties is also conjectured to be $\Omega(n^2)$ [10]. The generalizations of these conjectures for arbitrary transitive Boolean functions are also studied: In particular, recently Kulkarni [16] has formulated the Weak-Evasiveness Conjecture for monotone transitive functions, which vastly generalize monotone graph properties. In the past, Lovász had conjectured [14] the evasiveness of checking independence of S exactly in our setting. Sun,Yao, and Zhang [24] study query complexity of graph properties and several transitive functions including the circulant ones. Their motivation was to investigate how low can the query complexity go if one drops the assumption of monotonicity or lower the amount of symmetry. In this paper, we follow their footsteps and ask the same question under no symmetry assumption whatsoever. The main difference between the past works and this one is that most of the previous work exploit the symmetry to prove (or to conjecture) a good lower bound, whereas we investigate the consequences of breaking the symmetry for the query complexity.

1.5 Our main results

In this section we summarize our main results. Let $\mathcal{P}$ be a hereditary graph property and $d_{\mathcal{P}}$ denote the minimum possible degree of a minimal forbidden subgraph for $\mathcal{P}$.

▶ **Theorem 2.** *For any hereditary graph property $\mathcal{P}$:*

$$\textit{min-cost}(\mathcal{P}) = O(n^{1/(d_{\mathcal{P}}+1)}).$$

Table 1 Summary of Results for Finite/Infinite Forbidden Subgraphs.

	Properties		With Symmetry[4]	Without Symmetry
Finite	Independence/Emptiness	[Full Version]	$\Theta(n)$	$\Theta(\sqrt{n})$
	Bounded Degree	[Full Version]	$\Theta(n)$	$\Theta(\sqrt{n})$
	Triangle-freeness	[Full Version]	$\Theta(n)$	$\Theta(n^{1/3})$
	Containing K_t	[Thm. 2][Thm. 4]	$\Theta(n)$	$\Theta(n^{1/t})$
	Containing P_t	[Thm. 2][Thm. 4]	$\Theta(n)$	$O(\sqrt{n})$,$\Omega(n^{1/t})$
	Containing C_t	[Thm. 2][Thm. 4]	$\Theta(n)$	$O(n^{1/3})$, $\Omega(n^{1/t})$
	Containing H: $V(H) = k$	[Thm. 13][Thm. 2][Thm. 4]	$\Theta(n)$	$O(n^{1/(d_{min}+1)})$, $\Omega(n^{1/k})$
Infinite	Acyclicity	[Thm. 15]	$\Theta(n)$	$O(n^{1/3})$
	Bi-partiteness	[Thm. 2]	Open	$O(n^{1/3})$
	3-colorability	[Thm. 2]	Open	$O(n^{1/4})$
	Planarity	[Thm. 17]	$\Theta(n)^5$	$O(n^{1/4})$

▶ **Corollary 3.** *For any hereditary graph property $\mathcal{P}$:*

$$\textit{min-cost}(\mathcal{P}) = O(\sqrt{n}).$$

Theorem 2 and Corollary 3 show that in the absence of any symmetry on the graph G the query complexity can fall as low as $O(n^{1/(d+1)})$ where d denotes the minimum possible degree of a minimal forbidden sub-graph for $\mathcal{P}$. In particular, every hereditary property benefits at least quadratically.

We note that the above upper bound does not hold for general graph properties. For instance Connectivity has *min-cost* $\Theta(n)$, so does containment of a Perfect Matching, which are both non-hereditary properties (See Appendix of the full version of this paper).

As a partial answer to Question 1 we prove the following theorem.

▶ **Theorem 4.** *Let H be a fixed graph on k vertices and let $\mathcal{P}_H$ denote the property of containing H as a subgraph. Then,*

$$\textit{min-cost}(\mathcal{P}_H) = \Omega(n^{1/k}).$$

Interestingly our proof of Theorem 4 uses the famous Sunflower Lemma due to Erdös and Rado [9]. Moreover it generalizes to any fixed number of forbidden subgraphs each on at most k vertices. This implies that any hereditary property with *finitely many forbidden subgraphs* has a lower bound of $\Omega(n^{1/k})$, for a constant k depending only on the property.

We note that both Theorem 2 and Theorem 4 are not tight. However, we do prove tight bounds for several hereditary properties. We summarize a few such interesting bounds in the Table 1.

Finally we note a non-constant lower bound, which holds for *any* hereditary property. Our proof again relies on the Sunflower Lemma.

▶ **Theorem 5.** *For any hereditary graph property $\mathcal{P}$*

$$\textit{min-cost}(\mathcal{P}) = \Omega\left(\frac{\log n}{\log\log n}\right).$$

As we use sensitivity arguments all our lower bounds work for randomized case as well.

[4] assuming Weak Evasiveness
[5] when $d(G) \geq 7$

1.6 Organization

The rest of the paper is organized as follows: We introduce some preliminary notions in Section 2. We revisit some results on Weak Evasiveness under symmetry in Section 3. In Section 4, we provide proofs of Theorem 2 and Theorem 4. Proof of some tight bounds for Theorem 2 are deferred to Appendix. In Section 5 we state some results on restricted graph classes and their proofs are deferred to Appendix. Finally in Section 6 we discuss questions and directions that are naturally raised by our work.

The whole Appendix section of this paper can be found in the full version, which is available on the arXiv [3].

2 Preliminaries

▶ **Definition 6** (Randomized query complexity)**.** A randomized decision tree $\mathcal{T}$ is simply a probability distribution on the deterministic decision trees $\{T_1, T_2, \ldots\}$ where the tree T_i occurs with probability p_i. We say that $\mathcal{T}$ computes f correctly if for every input x: $\Pr_i[T_i(x) = f(x)] \geq 2/3$. The depth of $\mathcal{T}$ is the maximum depth of a T_i. The (bounded error) randomized query complexity of f, denoted by $R(f)$, is the minimum possible depth of a randomized tree computing f correctly on all inputs.

▶ **Definition 7** (Monotone, Transitive and Evasive Boolean functions)**.** A Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ is said to be *monotone* increasing if for any $x \leq y$, we have $f(x) \leq f(y)$, where $x \leq y$ means $x_i \leq y_i$ for all $i \in [n]$. Similarly one can define a monotone decreasing function. A Boolean function $f(x_1, \ldots, x_n)$ is said to be *transitive* if there exists a group G that acts transitively on the variables x_is such that f is invariant under this action, i.e., for every $\sigma \in G$: $f(x_{\sigma_1}, \ldots, f_{\sigma_n}) = f(x_1, \ldots, x_n)$. A Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ is said to be *evasive* if $D(f) = n$.

▶ **Definition 8** (Hereditary graph properties)**.** A property $\mathcal{P}$ of graphs is simply a collection of graphs. The members of $\mathcal{P}$ are said to satisfy $\mathcal{P}$ and non-members are said to fail $\mathcal{P}$. A property is hereditary if it is closed under deletion of vertices as well as edges[6]. For instance: acyclicity, planarity, and 3-colorability are hereditary properties, whereas connectivity and containing a perfect matching are not. Every hereditary property $\mathcal{P}$ can be uniquely expressed as a (possibly infinite) family $\mathcal{F}_{\mathcal{P}}$ of its forbidden subgraphs. For instance: acyclicity can be described as forbidding all cycles. Given a graph G, a hitting set $S_{G,\mathcal{P}}$ for $\mathcal{P}$ is a subset of $V(G)$ such that removing $S_{G,\mathcal{P}}$ from G would make the property $\mathcal{P}$ present[7]. Hereditary graph properties in node-query setting are monotone decreasing Boolean functions. Sometimes we refer hereditary properties by their negation. For instance: containing triangle.

▶ **Definition 9** (Sensitivity and block-sensitivity [12])**.** The i^{th} bit of an input $x \in \{0,1\}^n$ is said to be sensitive for $f : \{0,1\}^n \rightarrow \{0,1\}$ if $f(x_1, \ldots, x_i, \ldots, x_n) \neq f(x_1, \ldots, 1-x_i, \ldots, x_n)$. The sensitivity of f on x, denoted by $s_{f,x}$ is the total number of sensitive bits of x for f. The sensitivity of f, denoted by $s(f)$, is the maximum of $s_{f,x}$ over all possible choices of x. A block $B \subseteq [n]$ of variables is said to be sensitive for f on input x, if flipping the values of all x_i such that $i \in B$ and keeping the remaining x_i the same, results in flipping the output of f. The block sensitivity of f on an input x, denoted by $bs_{f,x}$ is the maximum

[6] on the other hand, vertex-hereditary is closed only under vertex-deletion (e.g. being chordal).
[7] such that every graph in $\mathcal{F}_{\mathcal{P}}$ shares a node with $S_{\mathcal{G},\mathcal{P}}$.

number of *disjoint* sensitive blocks for f on x. The block sensitivity of a function f, denoted by $bs(f)$, is the maximum value of $bs_{f,x}$ over all possible choices of x. It is known that $D(f) \geq R(f) \geq bs(f) \geq s(f)$. For monotone functions, $bs(f) = s(f)$.

3 Presence of symmetry in node-query setting: Does it guarantee weak-evasiveness?

In edge-query setting, Aanderaa-Rosenberg-Karp Conjecture [15, 7] asserts that any non-trivial monotone graph property must be evasive, i.e., one must query all $\binom{n}{2}$ edges in worst-case. The following generalization of the ARK Conjecture asserts that only monotonicity and modest amount of symmetry, namely transitivity, suffices to guarantee the evasiveness [21].

▶ Conjecture 10 (Evasiveness Conjecture). Any non-constant monotone transitive function f on n variables has $D(f) = n$.

This conjecture appears to be notoriously hard to prove even in several interesting special cases. Recently Kulkarni [16] formulates:

▶ Conjecture 11 (Weak Evasiveness Conjecture). If f_n is a sequence of monotone transitive functions on n variables then for every $\epsilon > 0$:

$$D(f_n) = \Omega(n^{1-\epsilon}).$$

Although Weak EC appears to be seemingly weaker, Kulkarni [16] observes that it is equivalent to the EC itself. His results hint towards the possibility that disproving Weak EC might be as difficult as separating TC^0 from NC^1. However: proving special cases of Weak EC appears to be relatively less difficult. In fact, Rivest and Vuillemin [23] confirm the Weak EC for graph properties and recently Kulkarni, Qiao, and Sun [18] confirm Weak EC for 3-uniform hyper graphs and Black [5] extends this result to k-uniform hyper graphs. All these results are studied in the edge-query setting. It is natural to ask whether the Weak EC becomes tractable in node-query setting. The monotone functions in node-query setting translate precisely to the hereditary graph properties. Here we show that it does become tractable for several hereditary graph properties. But first we need the following lemma [8, 24]:

▶ **Lemma 12.** *Let f be a non-trivial monotone transitive function. Let k be the size of a 1-input with minimal number of 1s. Then: $D(f) = \Omega(n/k^2)$.*

Let $\mathcal{G}_{\mathcal{T}}$ denote the class of transitive graphs. Let H be a fixed graph. Let $\mathcal{P}_H$ denote the property of containing H as a subgraph. The following theorem directly follows from Lemma 12.

▶ **Theorem 13.**

$$\mathcal{G}_{\mathcal{T}}\text{-}min\text{-}cost(\mathcal{P}_H) = \Omega(n).$$

The above result can be generalized for any finite family of forbidden subgraphs. We do not yet know how to prove it for infinite family in general. However below we illustrate a proof for one specific case when the infinite family is the family of cycles. First we need the following lemma:

▶ **Lemma 14.** *Let G be a graph on n vertices, m edges, and maximum degree d_{max}. Let $\mathcal{C}$ denote the property of being acyclic. Then,*

$$cost(\mathcal{C}, G) \geq (m - n)/d_{max}.$$

Proof. To make G acyclic one must remove at least $m - n$ edges. Removing one vertex can remove at most d_{max} edges. Thus the size of minimum feedback vertex set (FVS) is at least $(m-n)/d_{\max}$. The adversary answers all vertices outside this FVS to be present. Now the algorithm must query every vertex in the minimum FVS. ◀

▶ **Theorem 15.**

$$\mathcal{G}_{\mathcal{T}}\text{-}min\text{-}cost(\mathcal{C}) = \Omega(n).$$

Proof. Since G is transitive, G is d regular for some d [11]. Therefore $m = dn/2$ and $d_{max} = d$. Hence from Lemma 14 we get the desired bound. ◀

We also show similar bound for the property of being planar:

▶ **Lemma 16.** *Let G be a graph on n vertices, m edges, and maximum degree d_{max}. Let $\mathcal{P}'$ denote the property of being planar. Then,*

$$cost(\mathcal{P}', G) \geq (m - 3n + 6)/d_{max}.$$

Proof. To make G planar one has to remove at least $(m - 3n + 6)$ edges from the graph G. Removing one vertex can remove at most d_{max} edges. Thus the size of minimum hitting set of G is at least $(m - 3n + 6)/d_{\max}$. The adversary answers all vertices outside this minimum hitting set to be present. Now the algorithm must query every vertex in the minimum hitting set. ◀

▶ **Theorem 17.**

$$\mathcal{G}_{\mathcal{T}}\text{-}min\text{-}cost(\mathcal{P}') = \Omega(n).$$

Proof. Since G is transitive, G is d regular for some d [11]. Therefore $m = dn/2$ and $d_{max} = d$. Hence for $d \geq 7$ using Lemma 16 we get the desired bound[8]. ◀

Following special case of Weak EC remains open:

▶ Conjecture 18. For any hereditary property $\mathcal{P}$, for any $\epsilon > 0$:

$$\mathcal{G}_{\mathcal{T}}\text{-}min\text{-}cost(\mathcal{P}) = \Omega(n^{1-\epsilon}).$$

4 Absence of symmetry in node-query setting: How low can query complexity go?

4.1 A general upper bound

Let $\mathcal{P}$ be a hereditary graph property and $d_{\mathcal{P}}$ denote the minimum possible degree of a minimal forbidden subgraph for $\mathcal{P}$.

Proof of Theorem 2: Let $k = c \cdot n^{1/(d_{\mathcal{P}}+1)}$ where we choose the constant c appropriately. Construct a graph G on n vertices as follows (See Figure 1):

- Start with a clique on vertices $v_1, \ldots, v_k$.
- For every $S \subseteq [k]$ such that $|S| = d_{\mathcal{P}}$
  - add k new vertices $u_1^S, \ldots, u_k^S$ and
  - connect every vertex $v_i : i \in S$ to each of these new k vertices $u_1^S, \ldots, u_k^S$.

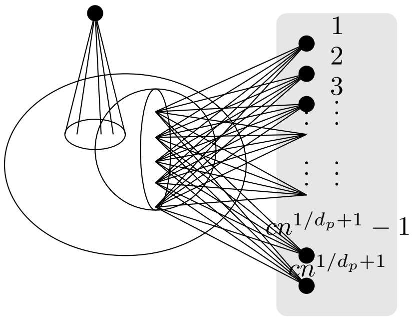


Figure 1 Construction of G for a general upper bound .

Algorithm 1:

- Query $v_1, \ldots, v_k$.
- If at least $c_{\mathcal{P}}$ of these vertices are present then $\mathcal{P}$ must fail.
- Otherwise there are at most $c_{\mathcal{P}} - 1$ vertices present (wlog: $v_1, \ldots, v_{c_{\mathcal{P}}-1}$).
  - For every subset $S \subseteq [c_{\mathcal{P}} - 1]$ such that $|S| = d_{\mathcal{P}}$, query $u_1^S, \ldots, u_k^S$.
  - If the graph induced on the nodes present (after all these $\binom{c_{\mathcal{P}}-1}{d_{\mathcal{P}}} \times k$ queries) satisfies $\mathcal{P}$ then answer Yes. Otherwise answer No.

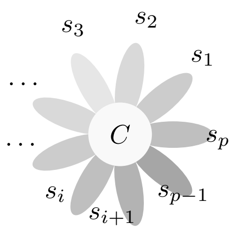
Now we describe an algorithm (See Algorithm 1) to determine $\mathcal{P}$ in $O(n^{1/(d_{\mathcal{P}}+1)})$ queries. Let $c_{\mathcal{P}}$ denote the smallest integer such that the clique on $c_{\mathcal{P}}$ vertices satisfies $\mathcal{P}$.

Note that any vertex that is not queried by the above algorithm can have at most $d_{\mathcal{P}} - 1$ edges to the vertices in the clique $v_1, \ldots, v_k$. Since $d_{\mathcal{P}}$ is the minimum degree of a minimal forbidden subgraph for $\mathcal{P}$, these vertices now become irrelevant for $\mathcal{P}$. Thus the algorithm can correctly declare the answer based on only the queries it has made. It is easy to check that the query complexity of the above algorithm is $O(k)$ which is $O(n^{1/(d_{\mathcal{P}}+1)})$. $\square$

This completes the proof of Theorem 2. Corollary 3 follows from this by observing that $d_{\mathcal{P}} \geq 2$ for any non-trivial $\mathcal{P}$.

4.2 General lower bounds

Now we show that any hereditary property with *finitely many forbidden subgraphs* has a lower bound of $\Omega(n^{1/k})$, for a constant k depending only on the property.



[8] Currently our proof works only when $d \geq 7$, but we believe that it can be extended for any degree d.

▶ **Definition 19** (Sunflower). A sunflower with core set C and p petals is a collection of sets $S_1, \ldots, S_p$ such that for all $i \neq j$: $S_i \cap S_j = C$.

We use the following lemma due to Erdös and Rado [9].

▶ **Lemma 20** (Sunflower Lemma). *Let $\mathcal{F}$ be a family of sets of cardinality k each. If $|\mathcal{F}| > k!(p-1)^k$ then $\mathcal{F}$ contains a sunflower with p petals.*

Proof of Theorem 4: Let G be a graph on n vertices such that every vertex of G is relevant for the property of containing H. Let

$$\mathcal{F} := \{S \mid |S| = k \ \&\ H \text{ is a subgraph of } G[S]\}.$$

Since every vertex of G is relevant for $\mathcal{P}_H$, we have: $|\mathcal{F}| \geq n/k$. Now from Lemma 20 we can conclude that $\mathcal{F}$ contains a sunflower on at least $|\mathcal{F}|^{1/k}/k = \Omega(n^{1/k})$ petals. Let C be the core of this sunflower. We consider the restriction of $\mathcal{P}_H$ on G where every vertex in C is present. Since $|C| < k$, $G[C]$ does not contain H. Now it is easy to check that one must query at least one vertex from each petal in order to determine $\mathcal{P}_H$.

□

Using similar technique we prove Theorem 5 showing that *min-cost*$(\mathcal{P})$ for any hereditary $\mathcal{P}$ can not fall to a constant.

▶ Theorem 5. (Restated) For any hereditary graph property $\mathcal{P}$

$$\textit{min-cost}(\mathcal{P}) = \Omega\left(\frac{\log n}{\log\log n}\right).$$

Proof. Let G be a graph on n vertices such that every vertex of G is relevant for $\mathcal{P}$. Let k be the largest integer such that G contains a minimal forbidden subgraph for $\mathcal{P}$ on k vertices. Note that we are concerned with vertex minimal certificates.

Case 1: $k \geq \frac{\log n}{2\log\log n}$.

Since one must query all the vertices of a minimal forbidden subgraph, we obtain a lower bound of $k = \Omega(\log n/\log\log n)$.

Case 2: $k < \frac{\log n}{2\log\log n}$.

Since every vertex of G is relevant for $\mathcal{P}$ and all the minimal forbidden subgraphs of $\mathcal{P}$ present in G are of size at most k, every vertex of G must belong to some minimal forbidden subgraph of size at most k. Consider the property $\mathcal{P}_k$ obtained from $\mathcal{P}$ by omitting the minimal forbidden subgraphs of $\mathcal{P}$ on k or more vertices. Our simple but crucial observation is that $\mathcal{P}$ and $\mathcal{P}_k$ are equivalent as far as G is concerned. Therefore, they have the same complexity. Now we define $\mathcal{F}_i$ for $i \leq k$ as follows:

$$\mathcal{F}_i := \{S \mid |S| = i \ \&\ G[S] \notin \mathcal{P} \ \&\ \forall T \subset S : G[T] \in \mathcal{P}\}.$$

Since every vertex of G is relevant for $\mathcal{P} \equiv \mathcal{P}_k$, we have: $|\bigcup_{i=1}^{k} \mathcal{F}_i| \geq n/k$. Since $\mathcal{F}_i$ and $\mathcal{F}_j$ are disjoint when $i \neq j$, we have $\sum_{i=1}^{k} |\mathcal{F}_i| \geq n/k$. Therefore one of the $\mathcal{F}_i$s must be of size at least n/k^2. We denote that $\mathcal{F}_i$ by $\mathcal{F}'$.

Now from Lemma 20 we can conclude that $\mathcal{F}'$ contains a sunflower on at least $|\mathcal{F}'|^{1/k}/k$ petals. Let C be the core of this sunflower. We consider the restriction of $\mathcal{P}$ on G where every vertex in C is present. Since $|C| < i$, by definition of $\mathcal{F}_i$ we must have $G[C] \in \mathcal{P}$. Now it is easy to check that one must query at least one vertex from each petal in order to determine $\mathcal{P}$. A simple calculation yields that one can obtain a lower bound of $\min\{k, \frac{2^{\Omega(\log n/k)}}{k}\}$. When $k = \log n/(2\log\log n)$, this gives us $\Omega(\log n/\log\log n)$ bound. ◀

4.3 Some tight bounds

We manage to show that Theorem 2 is tight for several special properties like Independence, Triangle-freeness, Bounded-degree etc. In the Appendix of the full version of this paper we present them in detail. In order to prove the tight bounds, we show several inequalities which might be of independent interest combinatorially. We present one such inequality below.

Lower bound based on the chromatic number

▶ **Theorem 21.** *Let $\mathcal{I}$ denote the property of being an independent subset of nodes (equivalently the property of being an empty graph). Then,*

$$\mathcal{G}\text{-}min\text{-}cost(\mathcal{I}) \geq n/\chi$$

where χ is the maximum chromatic number of a graph $G \in \mathcal{G}$.

Proof. Let $G \in \mathcal{G}$ be a graph on n vertices such that every vertex of G is relevant for $\mathcal{I}$, i.e., G does not contain any isolated vertices. Consider a coloring of vertices of G with χ colors. Let C_i denote the set of vertices colored with color i. We pick a coloring that maximizes $\max_{i \leq \chi}\{|C_i|\}$. Let C_{max} denote such a color class with maximum number of vertices in this coloring. Thus $|C_{\max}| \geq n/\chi$.

When $|C_{\max}| \leq (1 - \frac{1}{\chi})n$, the adversary answers all the vertices in $C_{\max}$ to be present. Since $C_{\max}$ is maximal and G does not contain any isolated vertices, every vertex outside $C_{\max}$ must be connected to some vertex in $C_{\max}$. As long as any of these outside vertices are present there will be an edge. Hence we get a lower bound of $n - |C_{\max}| \geq n/\chi$.

Now when $|C_{\max}| > (1 - \frac{1}{\chi})n$, since there are no isolated vertices in G, every vertex in $C_{\max}$ must have an edge to some vertex in $C_i \neq C_{\max}$. Furthermore as $|C_{\max}| > (1 - \frac{1}{\chi})n$, there are at least $(1 - \frac{1}{\chi})n$ edges incident on $C_{\max}$.

Now the vertices outside $C_{\max}$ are colored with $(\chi - 1)$ colors. Thus there must exists a C_i such that at least $\frac{(1-\frac{1}{\chi})n}{\chi-1} = n/\chi$ edges incident on $C_{\max}$ are also incident on C_i. Now the adversary answers all the vertices in that C_i to be present. Then one must check at least n/χ vertices from $C_{\max}$ because as soon as any one of them is present we have an edge in the graph. ◀

5 Results on restricted graph classes

5.1 Triangle-freeness in planar graphs

A graph G is called *inherently sparse* if every subgraph of G on k nodes contains $O(k)$ edges.

▶ **Theorem 22.** *Let $\mathcal{G}_s$ be a family of inherently sparse graphs on n vertices and $\mathcal{T}$ denote the property of being triangle-free. Then,*

$$\mathcal{G}_s\text{-}min\text{-}cost(\mathcal{T}) = \Omega(\sqrt{n}).$$

The proof of Theorem 22 is deferred to the Appendix of the full version of this paper. As a consequence we obtain the same for the class of planar graphs.

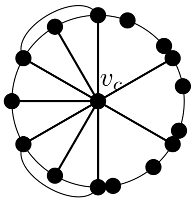


Figure 2 A wheel with d_{max} spokes.

5.2 Acyclicity in planar graphs

▶ **Theorem 23.** *Let $\mathcal{G}_{\mathcal{P}_3}$ be a family of* 3*-connected planar graphs and $\mathcal{C}$ denote the property of being acyclic. Then,*

$$\mathcal{G}_{\mathcal{P}_3} - \textit{min-cost}(\mathcal{C}) = \Omega(\sqrt{n}).$$

Proof. Let $G \in \mathcal{G}_{\mathcal{P}_3}$ be a graph on n vertices and m edges such that every vertex is relevant for the acyclicity property. Let d_{max} denote the maximum degree of G.

Case 1: $d_{\max} > \sqrt{n}$: We use the following fact: In 3-connected planar graphs, removing any vertex leaves a facial cycle around it. We apply this for the maximum degree vertex. In other words, we have a (not necessarily induced) wheel with d_{max} spokes (some spokes might be missing). See Figure 2. The adversary answers the central vertex of the wheel to be present. We can find a matching of size $\Omega(n)$ among the vertices of the cycle. Hence we have $\Omega(n)$ sensitive blocks of length 2 each, which can not be left un-queried.

Case 2: $d_{\max} \leq \sqrt{n}$: We use the fact that every 3-connected graph must have at least $3n/2$ edges. Now using Lemma 14 we obtain a lower bound of $(m - n)/d_{max} \geq \Omega(\sqrt{n})$. ◀

We can generalize the above proof to any planar graph (See the Appendix of the full version of this paper).

6 Conclusion & open directions

- **Weak-evasiveness in the presence of symmetry:** Is it true that every hereditary graph property $\mathcal{P}$ in the node-query setting is weakly-evasive under symmetry, i.e., $\mathcal{G}_{\mathcal{T}}$-*min-cost*$(\mathcal{P}) = \Omega(n)$? What about the randomized case?
- **Polynomial lower bound in the absence of symmetry:** How low can *min-cost*$(\mathcal{P})$ go for a hereditary $\mathcal{P}$ in the absence of symmetry? Is it possible to improve the $\Omega(\log n/\log\log n)$ bound substantially?
- **Further restrictions on graphs:** How low can $\mathcal{G}$-*min-cost*$(\mathcal{P})$ go for hereditary properties $\mathcal{P}$ on restricted classes of graphs $\mathcal{G}$ such as social-network graphs, planar graphs, bipartite graphs, bounded degree graphs etc?
- **Tight bounds on *min-cost*:** What are the tight bounds for natural properties such as acyclicity, planarity, containing a cycle of length t, path of length t?
- **Extension to hypergraphs:** What happens for hereditary properties of (say) 3-uniform hypergraphs in node-query setting? We note that *min-cost*$(\mathcal{I}) = \Theta(n^{1/3})$ for 3-uniform hypergraphs. What about other properties?

- **Global vs local:** We note (See the Appendix of the full version of this paper) that global connectivity requires $\Theta(n)$ queries whereas the cost of s-t connectivity for fixed s and t can go as low as $O(1)$. What about other properties such as min-cut?
- **How about *max-cost* upper bounds? :** From algorithmic point of view, it might be interesting to obtain good upper bounds on the *max-cost*$(\mathcal{P})$ for some natural properties. It might also be interesting to investigate $\mathcal{G}$-*max-cost*$(\mathcal{P})$ for several restricted graph classes such as social-network graphs, planar graphs, bipartite graphs etc.

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