Admissible Colourings of 3-Manifold Triangulations for Turaev-Viro Type Invariants*

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Abstract

Turaev-Viro invariants are amongst the most powerful tools to distinguish 3-manifolds. They are invaluable for mathematical software, but current algorithms to compute them rely on the enumeration of an extremely large set of combinatorial data defined on the triangulation, regardless of the underlying topology of the manifold.

In the article, we propose a finer study of these combinatorial data, called *admissible colourings*, in relation with the cohomology of the manifold. We prove that the set of admissible colourings to be considered is substantially smaller than previously known, by furnishing new upper bounds on its size that are aware of the topology of the manifold. Moreover, we deduce new topology-sensitive enumeration algorithms based on these bounds.

The paper provides a theoretical analysis, as well as a detailed experimental study of the approach. We give strong experimental evidence on large manifold censuses that our upper bounds are tighter than the previously known ones, and that our algorithms outperform significantly state of the art implementations to compute Turaev-Viro invariants.

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1 Introduction

In geometric topology, testing if two manifolds are equivalent is one of the most fundamental algorithmic problems. In fact, the task of comparing the topology of two given manifolds often stands at the very beginning of a question, and solving it is essential for conducting research in the field. In the active field of research of 3-manifold topology, this task is remarkably difficult. As a result, practitioners in computational topology rely on simpler invariants – properties of a topological space that can tell different spaces apart.

In the discrete setting, among the most useful invariants for 3-manifolds are the *Turaev-Viro invariants* [16]. They derive from quantum field theory but can be computed by purely combinatorial means – much like the famous Jones polynomial for knots. They are implemented in the major software packages *Regina* [4] and the *Manifold Recogniser* [12, 13],

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and they play a key role in developing census databases, which are analogous to the well-known dictionaries of knots [1, 12]. Their main difficulty is that they are slow to compute: the best implementations rely on the enumeration of exponentially large sets of combinatorial data defined on a triangulation.

The Turaev-Viro invariants are a family of invariants (TV_r) indexed by an integer $r \geq 3$. For a triangulation $\mathfrak{T}$, the Turaev-Viro invariant is based on *colourings* of the triangulation $\mathfrak{T}$, which are assignements of one of $r-1$ distinct colours to each of the edges of $\mathfrak{T}$. Only a subset of colourings satisfy *admissibility constraints* – which are of combinatorial nature –, and each *admissible colouring* defines a *weight*. The Turaev-Viro invariant $\mathrm{TV}_r(\mathfrak{T})$ is equal to the sum of these weights over all admissible colourings.

For any $r \geq 3$, a naive algorithm to compute $\mathrm{TV}_r(\mathfrak{T})$ on a triangulation $\mathfrak{T}$, with m edges, consists of a simple backtracking procedure enumerating all of the $(r-1)^m$ edge colourings, checking each of them for admissibility and summing the weights, resulting in a memory efficient but very slow implementation. More recently, Burton and the authors introduced a *fixed parameter tractable (FPT)* algorithm which is *linear* in the size of the input, and only singly exponential in the treewidth of the dual graph of $\mathfrak{T}$ [5]. This is possible by using the structure of the input to process large groups of admissible colourings simultaneously. Despite good performance in practice, this approach requires exponential memory and the running time is very sensitive to the combinatorial structure of the input triangulation, as opposed to the topology of the underlying manifold.

Algorithmic results exist for specific values of r. For $r = 3$, the Turaev-Viro invariant $\mathrm{TV}_3(\mathfrak{T})$ can be interpreted in terms of the cohomology of the manifold, which results in a polynomial time algorithm [5, 12]. For $r = 4$ however, the computation of the invariant is known to be hard for the *counting complexity class* #P [5, 11]. This gives evidence that a general efficient solution (for example polynomial) for computing TV_r is unlikely to exist.

In this article, we elaborate on the cohomology interpretation of the Turaev-Viro invariants, successful for the case $r = 3$, to design more efficient implementations for TV_r relying on an optimised enumeration of admissible colourings. More precisely, we use the admissibility constraints to connect colourings and cohomology classes of the manifold, and reduce a priori the number of colourings to be considered algorithmically in order to find all admissible colourings.

Using this technique, we study the structure of the set of admissible colourings for $r = 3$ and $r = 4$. We design new sharper upper bounds on the number of admissible colourings of a triangulation for $r = 4$, and deduce an algorithm to compute TV_4 which is linear in these new bounds. This is of particular interest considering the #P-hardness of this computation. We give experimental evidence on large censuses of triangulations that these upper bounds are sharp in many cases and significantly better than the naive ones.

We then study in more details admissible colourings that reduce to the trivial cohomology class. This is a special case of particular importance, as it allows the study of homology spheres – manifolds involved in the 3-sphere recognition problem – and later becomes a key ingredient for an improved algorithm to compute TV_r, with r odd, on any manifold. We deduce new sharp upper bounds on the number of colourings of homology spheres for $r \leq 7$.

Finally, building on this study at the trivial cohomology class, and work by Kirby and Melvin [10] and Matveev [12], we introduce an improved algorithm to compute the Turaev-Viro invariants for odd values of r. By embedding it within existing algorithms, our method allows a significant exponential speed-up on both backtracking algorithm and FPT algorithm to compute Turaev-Viro invariants. We provide large scale experiments to show the interest of the method. In particular, our new enumeration of colourings, combined with the FPT algorithm to compute Turaev-Viro, performs up to two orders of magnitude faster than state of the art implementation, hence opening notably the range of possible practical computations in 3-manifold topology.

These implementations will appear as features in the 3-manifold software `Regina` [4].

2 Background

Manifolds and generalised triangulations: Let M be a closed 3-manifold. A *generalised triangulation* $\mathfrak{T}$ of M is a collection of n abstract tetrahedra $\Delta_1, \ldots, \Delta_n$ together with $2n$ *gluing maps* identifying their $4n$ triangular faces in pairs, such that the underlying topological space is homeomorphic to M.

As a consequence of the gluings, vertices, edges or triangles of the same tetrahedron may be identified. It follows from an Euler characteristic argument, that any n-tetrahedra v-vertex triangulation of a closed 3-manifold must have $2n$ triangles and $n + v$ edges. It is common in practical applications to have a *one-vertex triangulation*, in which all vertices of all tetrahedra are identified to a single point. We refer to an equivalence class defined by the gluing maps as a single *face of the triangulation*. The number of tetrahedra n of $\mathfrak{T}$ is often referred to as the *size* of the triangulation. We denote by V, E, F and T the vertices, edges, triangles and tetrahedra, respectively, of a generalised triangulation.

Generalised triangulations are widely used in 3-manifold topology. They are more general than simplicial complexes, and can encode a wide range of manifolds, and very complex topologies, with very few tetrahedra. For instance, one can build 13 400 *distinct* prime manifolds with less than 11 tetrahedra [12], and the number of distinct manifolds represented by generalised triangulations with less than n tetrahedra grows super-exponentially with n.

We refer to [9] for more details on generalised triangulations.

Homology and cohomology: In the following section we give a very brief introduction to (co)homology theory. For more details see [7].

Let $\mathfrak{T}$ be a generalised 3-manifold triangulation. For the field of coefficients $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, the *group of p-chains*, $0 \leq p \leq 3$, denoted $\mathbf{C}_p(\mathfrak{T}, \mathbb{Z}_2)$, of $\mathfrak{T}$ is the group of formal sums of p-faces with $\mathbb{Z}_2$ coefficients. The *boundary operator* is a linear operator $\partial_p : \mathbf{C}_p(\mathfrak{T}, \mathbb{Z}_2) \to \mathbf{C}_{p-1}(\mathfrak{T}, \mathbb{Z}_2)$ such that $\partial_p \sigma = \partial_p \{v_0, \cdots, v_p\} = \sum_{j=0}^{p} \{v_0, \cdots, \widehat{v_j}, \cdots, v_p\}$, where σ is a face of $\mathfrak{T}$, $\{v_0, \ldots, v_p\}$ represents σ as a face of a tetrahedron of $\mathfrak{T}$ in local vertices $v_0, \ldots, v_p$, and $\widehat{v_j}$ means v_j is deleted from the list. Denote by $\mathbf{Z}_p(\mathfrak{T}, \mathbb{Z}_2)$ and $\mathbf{B}_{p-1}(\mathfrak{T}, \mathbb{Z}_2)$ the kernel and the image of ∂_p respectively. Observing $\partial_p \circ \partial_{p+1} = 0$, we define the *$p$-th homology group* $\mathbf{H}_p(\mathfrak{T}, \mathbb{Z}_2)$ of $\mathfrak{T}$ by the quotient $\mathbf{H}_p(\mathfrak{T}, \mathbb{Z}_2) = \mathbf{Z}_p(\mathfrak{T}, \mathbb{Z}_2) / \mathbf{B}_p(\mathfrak{T}, \mathbb{Z}_2)$. These structures are vector spaces.

The concept of *cohomology* is in many ways dual to homology, but more abstract and endowed with more algebraic structure. It is defined in the following way: The *group of p-cochains* $\mathbf{C}^p(\mathfrak{T}, \mathbb{Z}_2)$ is the formal sum of linear maps of p-faces of $\mathfrak{T}$ into $\mathbb{Z}_2$. The *coboundary operator* is a linear operator $\delta^p : \mathbf{C}^{p-1}(\mathfrak{T}, \mathbb{Z}_2) \to \mathbf{C}^p(\mathfrak{T}, \mathbb{Z}_2)$ such that for all $\phi \in \mathbf{C}^{p-1}(\mathfrak{T}, \mathbb{Z}_2)$ we have $\delta^p(\phi) = \phi \circ \partial_p$. As above, *$p$-cocycles* are the elements in the kernel of δ^{p+1}, *p-coboundaries* are elements in the image of δ^p, and the *p-th cohomology group* $\mathbf{H}^p(\mathfrak{T}, \mathbb{Z}_2)$ is defined as the p-cocycles factored by the d-coboundaries.

We denote by $\beta_1(\mathfrak{T}, \mathbb{Z}_2)$ the dimension of $\mathbf{H}_1(\mathfrak{T}, \mathbb{Z}_2)$, called the first *Betti number* of the manifold. By duality, this is also the dimension of homology and cohomology groups of dimension $p \in \{1, 2\}$, with $\mathbb{Z}_2$ coefficients.

In Section 3.1 we discuss how 1-cocycles correspond to (sets of) *admissible colourings* of the edges of $\mathfrak{T}$ used in the definition of Turaev-Viro invariants.

Turaev-Viro invariants: In this section we briefly describe *invariants of Turaev-Viro type* TV_r, parameterised by an integer $r \geq 3$. We then have a closer look at the more specialised original *Turaev-Viro invariants* $\mathrm{TV}_{r,q}$, which also depend on a second integer $0 < q < 2r$.

Let $\mathfrak{T}$ be a generalised triangulation of a closed 3-manifold M, and let $r \geq 3$, be an integer. Let V, E, F and T denote the set of vertices, edges, triangles and tetrahedra of the triangulation $\mathfrak{T}$ respectively. Let $I = \{0, 1/2, 1, 3/2, \ldots, (r-2)/2\}$ be the set of the first $r-1$ non-negative half-integers. A *colouring* of $\mathfrak{T}$ is defined to be a map $\theta : E \to I$; that is, θ "colours" each edge of $\mathfrak{T}$ with an element of I. A colouring θ is *admissible* if, for each triangle of $\mathfrak{T}$, the three edges e_1, e_2, and e_3 bounding the triangle satisfy the

- *parity condition* $\theta(e_1) + \theta(e_2) + \theta(e_3) \in \mathbb{Z}$;
- *triangle inequalities* $\theta(e_i) \leq \theta(e_j) + \theta(e_k)$, $\{i, j, k\} = \{1, 2, 3\}$; and
- *upper bound constraint* $\theta(e_1) + \theta(e_2) + \theta(e_3) \leq r - 2$.

For a triangulation $\mathfrak{T}$ and $r \geq 3$, its set of admissible colourings is denoted by $\mathrm{Adm}(\mathfrak{T}, r)$.

For each admissible colouring θ and for each vertex $w \in V$, edge $e \in E$, triangle $f \in F$ or tetrahedron $t \in T$ we define *weights* $|w|_\theta, |e|_\theta, |f|_\theta, |t|_\theta \in \mathbb{C}$. The weights of vertices are constant, and the weights of edges, triangles and tetrahedra only depend on the colours of edges they are incident to. Using these weights, we define the *weight of the colouring* to be

$$|\mathfrak{T}|_\theta = \prod_{w \in V} |w|_\theta \times \prod_{e \in E} |e|_\theta \times \prod_{f \in F} |f|_\theta \times \prod_{t \in T} |t|_\theta, \tag{1}$$

Invariants of Turaev-Viro types of $\mathfrak{T}$ are defined as sums of the weights of all admissible colourings of $\mathfrak{T}$, that is $\mathrm{TV}_r(\mathfrak{T}) = \sum_{\theta \in \mathrm{Adm}(\mathfrak{T},r)} |\mathfrak{T}|_\theta$.

In [16], Turaev and Viro show that, when the weighting system satisfies some identities, $\mathrm{TV}_r(\mathfrak{T})$ is indeed an invariant of the manifold; that is, if $\mathfrak{T}$ and $\mathfrak{T}'$ are generalised triangulations of the same closed 3-manifold M, then $\mathrm{TV}_r(\mathfrak{T}) = \mathrm{TV}_r(\mathfrak{T}')$ for all r. We thus sometimes abuse notation and write $\mathrm{TV}_r(M)$, meaning the Turaev-Viro type invariant computed for a triangulation of M.

We refer to [5] for a precise definition of the weights of the original Turaev-Viro invariant at $sl_2(\mathbb{C})$, which not only depend on r but also on a second integer $0 < q < 2r$. The exact definition of these weights is rather involved, but not at all important in order to understand the findings presented in this article, we thus continue to denote these weights by $|\cdot|_\theta$ despite the fact that they not only depend on θ, but also on r and q. We use these weights in our experiments in Section 4.

For an n-tetrahedra triangulation $\mathfrak{T}$ with v vertices there is a simple backtracking algorithm to compute $\mathrm{TV}_{r,q}(\mathfrak{T})$ by testing the $(r-1)^{v+n}$ possible colourings for admissibility and computing their weights. The case $r = 3$ can however be computed in polynomial time, due to a connection between $\mathrm{Adm}(\mathfrak{T}, 3)$ and cohomology, see Section 3.1 and [5, 12].

Classical results about Turaev-Viro invariants: Note that the Turaev-Viro invariants $\mathrm{TV}_{r,q}$ are closely related to the more general invariant of Witten and Reshetikhin-Turaev $\tau_{r,q}$ ($\in \mathbb{C}$), due to the following result.

▶ **Theorem 1** (Turaev [15], Roberts [14]). *For the invariants of Witten and Reshetikhin-Turaev $\tau_{r,q}$, and the Turaev-Viro invariants, the following equality holds*

$$\mathrm{TV}_{r,q} = | \tau_{r,q} |^2 \ .$$

Theorem 1 enables us to translate a number of key results about the Witten and Reshetikhin-Turaev invariants in terms of Turaev-Viro invariants. Namely, the following statement holds.

▶ **Theorem 2** (Based on Kirby and Melvin [10]). *Let M and N be closed compact 3-manifolds, and let $r \geq 3$, $1 \leq q \leq r-1$. Then there exist $\gamma_r \in \mathbb{C}$, such that for $\mathrm{TV}'_{r,1} = \gamma_r \mathrm{TV}_{r,1}$ we have*

$$\mathrm{TV}'_{r,1}(M\#N) = \mathrm{TV}'_{r,1}(M) \cdot \mathrm{TV}'_{r,1}(N).$$

Additionally, when a manifold M is represented by a triangulation with n tetrahedra, the normalising factor γ_r can be computed in polynomial time in n.

Using Turaev-Viro invariants at the trivial cohomology class we have the following identity for odd degree r.

▶ **Theorem 3** (Based on Kirby and Melvin [10]). *Let M be a closed compact 3-manifold, and let $r \geq 3$ be an odd integer. Then*

$$\mathrm{TV}_{r,1}(M) = \mathrm{TV}_{3,1}(M) \cdot \mathrm{TV}_{r,1}(M,[0]).$$

3 Reduction of colourings at cohomology classes

Let $\mathfrak{T}$ be a 3-manifold triangulation with v vertices, $n+v$ edges, $2n$ triangles and n tetrahedra. Following the definitions in Section 2 above, there are at most $(r-1)^{n+v}$ admissible colourings for $\mathrm{Adm}(\mathfrak{T},r)$. Due to the admissibility constraints for colourings, as described in Section 2, this bound is usually far from being sharp. However, current enumeration algorithms for admissible colourings do not try to capitalise on this fact (including the parameterised algorithm from [5]).

In this section we discuss methods that incorporate these constraints in a controlled fashion when enumerating admissible colourings. More precisely, we present improved upper bounds on the number of admissible colourings in important special cases (thus reducing a priori the number of options an enumeration algorithm needs to consider). Moreover, we give a number of examples where these new upper bounds are actually attained. The bounds are then used to construct a structure sensitive algorithm to enumerate $\mathrm{Adm}(\mathfrak{T},4)$, and to achieve a significant exponential speed-up for the computation of the Turaev-Viro invariants $\mathrm{TV}_{r,1}$ where r is odd.

3.1 Turaev-Viro invariants for r=3 and cohomology

There is a close connection between the first cohomology group of a 3-manifold triangulation $\mathfrak{T}$ and the admissible colourings of the Turaev-Viro invariants for $r=3$. We discuss this connection under the viewpoint of triangulations which helps setting the scene for improved bounds on the number of admissible colourings for higher values of r, as presented in Sections 3.2 and 3.4 below.

▶ **Proposition 4.** *Let $\mathfrak{T}$ be a 3-manifold triangulation with v vertices. Then there is a bijection between $\mathrm{Adm}(\mathfrak{T},3)$ and the 1-cocycles of $\mathfrak{T}$, and we have $|\mathrm{Adm}(\mathfrak{T},3)| = 2^{v+\beta_1(\mathfrak{T},\mathbb{Z}_2)-1}$.*

Proof. An edge colouring $\theta : E \to \{0,1/2\}$ defines a 1-cochain α_θ with coefficients in $\mathbb{Z}_2$ evaluating to 1 on edges coloured $1/2$ and to 0 otherwise. The parity condition on θ is then equivalent to the boundary of α_θ (which is a 2-chain) vanishing over $\mathbb{Z}_2$. Moreover, note that every colouring $\theta : E \to \{0, 1/2\}$ satisfying the parity condition is admissible for $r = 3$. Thus θ is admissible if and only if α_θ is a cocycle. This proves the first statement. The second statement follows from the observation that $\mathfrak{T}$ has exactly $2^{v+\beta_1(\mathfrak{T},\mathbb{Z}_2)-1}$ cocycles. ◀

Let $\mathbf{H}^1(\mathfrak{T}, \mathbb{Z}_2) = (\mathbb{Z}_2)^{\beta_1(\mathfrak{T},\mathbb{Z}_2)}$ be the first cohomology group of $\mathfrak{T}$. Since every 1-cocycle α in $\mathfrak{T}$ is a representative of a cohomology class $[\alpha] \in \mathbf{H}^1(\mathfrak{T}, \mathbb{Z}_2)$, every admissible colouring in $\mathrm{Adm}(\mathfrak{T}, 3)$ can be associated to a cohomology class. This correspondence can be generalised to arbitrary $r \geq 3$ with the help of the following observation.

▶ **Proposition 5.** *Let $\mathfrak{T}$ be a 3-manifold triangulation with edge set E, $r \geq 3$, and $\theta \in \mathrm{Adm}(\mathfrak{T}, r)$. Then the* reduction of θ, *defined by $\theta' : E \to \{0, 1/2\};\ e \mapsto \theta(e) - \lfloor\theta(e)\rfloor$, is an admissible colouring in $\mathrm{Adm}(\mathfrak{T}, 3)$.*

Proof. Let f be a triangle of $\mathfrak{T}$ with edges e_1, e_2, and e_3. Since $\theta \in \mathrm{Adm}(\mathfrak{T}, r)$ is admissible, we have $\theta(e_1) + \theta(e_2) + \theta(e_3) \in \mathbb{Z}$. Thus, there are either no or two half-integers amongst the colours of the edges of f and $\theta' \in \mathrm{Adm}(\mathfrak{T}, 3)$. ◀

We have seen that every colouring $\theta \in \mathrm{Adm}(\mathfrak{T}, r)$ can be associated to a 1-cohomology class of $\mathfrak{T}$ via its reduction $\theta' \in \mathrm{Adm}(\mathfrak{T}, 3)$ and Proposition 4. We know from [12, 16] that this construction can be used to split $\mathrm{TV}_r(\mathfrak{T})$ (and thus also $\mathrm{TV}_{r,q}(\mathfrak{T})$) into simpler invariants indexed by the elements of $\mathbf{H}^1(\mathfrak{T}, \mathbb{Z}_2)$. More precisely, let $[\alpha] \in \mathbf{H}^1(\mathfrak{T}, \mathbb{Z}_2)$ be a cohomology class, then

$$\mathrm{TV}_r(\mathfrak{T}, [\alpha]) = \sum_{\substack{\theta \in \mathrm{Adm}(\mathfrak{T}, r) \\ \theta \bmod 2 \in [\alpha]}} |\mathfrak{T}|_\theta,$$

where $\theta \bmod 2$ denotes the reduction of θ, is an invariant of $\mathfrak{T}$. The special case $\mathrm{TV}_r(\mathfrak{T}, [0])$ is of particular importance for computations as explained in further detail in Section 3.4.

3.2 Admissible colourings for r=4

We have seen in Proposition 4 that admissible colourings for $r = 3$ are in one-to-one correspondence to the 1-cocycles of a triangulated 3-manifold $\mathfrak{T}$. This basic but very useful observation has consequences for the structure of $\mathrm{Adm}(\mathfrak{T}, 4)$. This is particularly interesting as computing $\mathrm{TV}_{4,1}$ is known to be $\#P$-hard [5, 11]. More precisely, the following statement holds.

▶ **Theorem 6.** *Let $\mathfrak{T}$ be an n-tetrahedron 3-manifold triangulation with v vertices, and let $\theta \in \mathrm{Adm}(\mathfrak{T}, 3)$. Furthermore, let $\ker_\theta$ be the number of edges coloured 0 by θ. Then*

$$\begin{aligned} |\mathrm{Adm}(\mathfrak{T}, 4)| &\leq \left(\Sigma_{\theta \in \mathrm{Adm}(\mathfrak{T},3)\setminus\{\mathbf{0}\}} 2^{\ker_\theta}\right) + 2^{v+\beta_1(\mathfrak{T},\mathbb{Z}_2)-1} && (2)\\ &\leq (|\mathrm{Adm}(\mathfrak{T}, 3)| - 1)(2^{n+v-1} + 1) + 1, && (3)\end{aligned}$$

where $\mathbf{0}$ *denotes the all zero colouring. Moreover, both bounds are sharp.*

Proof. Let $\theta \in \mathrm{Adm}(\mathfrak{T}, 4)$, and let θ' be its reduction, as defined in Proposition 5. If θ' is the trivial colouring (that is, if no colour of θ is coloured by $1/2$) the colouring $\theta/2$, obtained by dividing all of the colours of θ by two, must be in $\mathrm{Adm}(\mathfrak{T}, 3)$. It follows from Proposition 4 that exactly $2^{v+\beta_1(\mathfrak{T},\mathbb{Z}_2)-1}$ colourings in $\mathrm{Adm}(\mathfrak{T}, 4)$ reduce to the trivial colouring.

If θ' is not the trivial colouring then θ colours some edges by $1/2$. In particular it is not the trivial colouring. Since the only colours in θ are 0, $1/2$, and 1, all edges coloured by $1/2$

in θ are coloured by $1/2$ in θ' and vice versa. Thus, $\ker_{\theta'}$ denotes all edges coloured by 0 or 1 in θ. Naturally, there are at most $2^{\ker_{\theta'}}$ such colourings. The result now follows by adding these upper bounds $2^{\ker_{\theta'}}$ over all non-trivial reductions $\theta' \in \mathrm{Adm}(\mathfrak{T}, 3)$, and adding the $2^{v+\beta_1(\mathfrak{T},\mathbb{Z}_2)-1}$ extra colourings with trivial reduction.

For Equation (3) note that each non-trivial colouring in $\mathrm{Adm}(\mathfrak{T}, 3)$ has at least one edge coloured $1/2$ and thus $\ker_{\theta'}$ is at most the number of edges minus one.

It follows that for $\beta_1(\mathfrak{T}, \mathbb{Z}_2)$ or v sufficiently large this bound cannot be tight. For 1-vertex triangulations $\mathfrak{T}$ with $\beta_1(\mathfrak{T}, \mathbb{Z}_2) = 0$ this bound is sharp as explained in Proposition 7. Looking at all 1-vertex triangulations with $\beta_1(\mathfrak{T}, \mathbb{Z}_2) = 1$ up to six tetrahedra, the cases of equality in Inequality (3) are summarised in Table 2. See Table 1 for a large number of cases of equality for Inequality (2). ◀

3.3 A structure-sensitive algorithm to compute $\mathrm{Adm}(\mathfrak{T}, 4)$

In this section we describe an algorithm to compute $\mathrm{TV}_{4,q}$ – a problem known to be $\#P$-hard – exploiting the combinatorial structure of the input triangulation. The algorithm is a direct consequence of the proof of Theorem 6.

Input. A v-vertex n-tetrahedra triangulation of a closed 3-manifold $\mathfrak{T}$ with set of edges E

1. Compute $\mathrm{Adm}(\mathfrak{T}, 3)$. Following the proof of Proposition 4, it is enough to compute a basis of the 1-cohomology of $\mathfrak{T}$ with coefficients in the field with two elements $\mathbb{Z}_2$. Then every cocycle naturally defines an admissible colouring and vice versa. This can be done in polynomial time by solving a linear system of equations. $\mathrm{Adm}(\mathfrak{T}, 3)$ can then be enumerated using the cohomology basis.
2. For all $\theta \in \mathrm{Adm}(\mathfrak{T}, 3)$, enumerate the set of edges $\ker_\theta \subset E$ of $\mathfrak{T}$ coloured zero in θ.
3. For each non-trivial $\theta \in \mathrm{Adm}(\mathfrak{T}, 3)$, for each subset $A \subseteq \ker_\theta$: Let θ' be the edge colouring that colours (i) all edges in A by 1, (ii) all edges in $(E \setminus \ker_\theta)$ by $1/2$, and (iii) all edges in $(\ker_\theta \setminus A)$ by 0. For each non-trivial θ, set up a backtracking procedure to check all such θ' for admissibility. Add the admissible colourings θ' to $\mathrm{Adm}(\mathfrak{T}, 4)$.
4. For all colourings $\theta \in \mathrm{Adm}(\mathfrak{T}, 3)$, double all colours of θ and add the result to $\mathrm{Adm}(\mathfrak{T}, 4)$.

Correctness of the algorithm and running time. Due to Theorem 6 we know that the above procedure enumerates all colourings in $\mathrm{Adm}(\mathfrak{T}, 4)$. Computing $\mathrm{TV}_{4,q}(\mathfrak{T})$ thus runs in

$$O\left(\left(\Sigma_{\theta \in \mathrm{Adm}(\mathfrak{T},3)\setminus\{\mathbf{0}\ \}} 2^{\ker_\theta}\right) \cdot n + 2^{v+\beta_1(\mathfrak{T},\mathbb{Z}_2)-1}\right)$$

arithmetic operations. This upper bound is much smaller than the worst case running time $(r-1)^{n+v}$ of the naive backtracking procedure.

In Section 4.3, we provide experimental evidence on a large census of triangulations that the new upper bounds on the number of admissible colourings from Theorem 6 are tight in many cases and close to being tight in average, and that our new algorithm to enumerate the colourings of $\mathrm{Adm}(\mathfrak{T}, 4)$ experimentally exhibits an output-sensitive nature.

3.4 Computing Turaev-Viro invariants at the zero cohomology class

Following Proposition 4 the complexity of enumerating admissible colourings of a 3-manifold triangulation $\mathfrak{T}$ not only depends on the size n of $\mathfrak{T}$, but also on (i) the number of vertices, and (ii) the first Betti number of $\mathfrak{T}$.

Regarding (i) we show in Section 3.5 that, given $\mathfrak{T}$, we can efficiently find a triangulation $\mathfrak{T}'$ of the same 3-manifold of same or smaller size with only one vertex. Regarding (ii) the first Betti number of $\mathfrak{T}$ is a topological invariant and hence an unchangeable part of the input. Thus, when computing $\mathrm{TV}_{r,q}(\mathfrak{T})$ by enumerating colourings, this layer of complexity can not be avoided. However, this observation does not hold for the invariant $\mathrm{TV}_{r,q}(\mathfrak{T},[0])$ which is a useful tool for various reasons.

1. First of all and most prominently, in order to compute $\mathrm{TV}_{r,q}(\mathfrak{T},[0])$ we only need to consider admissible colourings which correspond to the zero cohomology class. Following Proposition 4 for a one-vertex triangulation $\mathfrak{T}$, the colourings corresponding to the zero cohomology class are precisely the ones which reduce to the all zero colouring and thus can only have integer colours. A similar statement for the case of special spines can be found in [12, Remark 8.1.2.2].
2. One of the most important tasks of 3-manifold invariants is to distinguish between a 3-manifold triangulation $\mathfrak{T}$ and the 3-sphere (this task is known as the 3-*sphere recognition problem*). Whenever the homology groups of $\mathfrak{T}$ and the 3-sphere are different, this distinction can efficiently be made (i.e., in polynomial time). Hence, 3-sphere recognition is most interesting when homology fails, that is, when $\mathfrak{T}$ has the (trivial) homology of the 3-sphere $\mathbf{H}^1(\mathfrak{T},\mathbb{Z}_2)=\{[0]\}$. In this important case we have $\mathrm{TV}_{r,q}(\mathfrak{T})=\mathrm{TV}_{r,q}(\mathfrak{T},[0])$.
3. There are several non-trivial further cases when $\mathrm{TV}_{r,q}(\mathfrak{T})$ can be obtained from $\mathrm{TV}_{r,q}(\mathfrak{T},[0])$ in polynomial time, see Section 3.5 for details.

For the remainder of this section, instead of considering $\mathrm{TV}_{r,q}(\cdot,[0])$, we follow the related approach of considering $\mathrm{TV}_{r,q}(\cdot)$ and triangulations with vanishing first Betti number. We will use this study in the next section to derive a faster algorithm to compute $\mathrm{TV}_{r,q}(\cdot)$ on *all* manifold triangulations for r odd and $q=1$. The following facts follow from the observations made in Sections 3.1 and 3.2.

▶ **Proposition 7.** *Let $\mathfrak{T}$ be a 1-vertex triangulation such that $\beta_1(\mathfrak{T},\mathbb{Z}_2)=0$. Then*
(i) $|\mathrm{Adm}(\mathfrak{T},r)|=1$ *for $r\le 4$;*
(ii) *Let $\theta\in\mathrm{Adm}(\mathfrak{T},r)$, then all colours in θ must be integers;*
(iii) $|\mathrm{Adm}(\mathfrak{T},r)|\le\lfloor\frac{r}{2}\rfloor^{n+1}$.

In particular, $\mathrm{TV}_{r,q}(\mathfrak{T})$, $r\le 4$, must be trivial, and manifolds with trivial $\mathbb{Z}_2$-cohomology (a large group of 3-manifolds) can never be distinguished from the 3-sphere by $\mathrm{TV}_{r,q}$, $r\le 4$.

Proof.
(i) It follows from Proposition 4 that $\mathrm{Adm}(\mathfrak{T},3)=\{\mathbf{0}\}$ and the statement follows from Theorem 6.
(ii) Since $\mathrm{Adm}(\mathfrak{T},3)=\{\mathbf{0}\}$ all colourings must reduce to the all zero colouring.
(iii) Since all colours in θ must be (a) integers, (b) sum to at most $r-2$ on each triangle, and (c) satisfy the triangle inequality. It follows that all colours must be integers between 0 and $\lfloor\frac{r-2}{2}\rfloor$. The statement now follows from the fact that $\mathfrak{T}$ has $n+1$ edges. ◀

The bound from Proposition 7 cannot be sharp since not all triangle colourings $(a,b,c)\in\{0,1,\dots,\lfloor\frac{r-2}{2}\rfloor\}^3$ are admissible. For $5\le r\le 7$ we have the following situation.

▶ **Theorem 8.** *Let $\mathfrak{T}$ be a 1-vertex n-tetrahedron triangulation such that $\beta_1(\mathfrak{T},\mathbb{Z}_2)=0$, then*

$$|\mathrm{Adm}(\mathfrak{T},5)|\le 2^n+1;\qquad |\mathrm{Adm}(\mathfrak{T},6)|\le 3^n+1;\qquad |\mathrm{Adm}(\mathfrak{T},7)|\le 3^n+1.$$

Moreover, all these upper bounds are sharp.

Proof. For $r = 5$ the admissible triangle colourings are $(0,0,0)$, $(1/2,1/2,0)$, $(1,1,0)$, $(1,1/2,1/2)$, $(1,1,1)$, $(3/2,3/2,0)$, $(3/2,1,1/2)$, up to permutations. By Proposition 5, no colouring in $\mathrm{Adm}(\mathfrak{T},5)$ can contain an edge colour $1/2$ or $3/2$: Otherwise the reduction of such a colouring would be a non-trivial colouring in $\mathrm{Adm}(\mathfrak{T},3)$, which does not exist (cf. Proposition 4 and Corollary 7 with $v = 1$ and $\beta_1(\mathfrak{T},\mathbb{Z}_2) = 0$). Hence, all edge colours must be 0 or 1, leaving triangle colourings $(0,0,0)$, $(1,1,0)$, and $(1,1,1)$.

By an Euler characteristic argument, a 1-vertex n-tetrahedron 3-manifold has $n+1$ edges. Hence the number of colourings of $\mathrm{TV}_{5,q}$ is trivially bounded above by 2^{n+1}. Furthermore, let $\theta \in \mathrm{Adm}(\mathfrak{T},5)$, then either θ is constant 0 on the edges, constant 1 on the edges, or θ contains a triangle coloured $(1,1,0)$. In the last case, the complementary colouring θ', obtained by flipping the colour on all the edges, contains a triangle coloured $(0,0,1)$ and thus $\theta' \notin \mathrm{Adm}(\mathfrak{T},5)$. It follows that $|\mathrm{Adm}(\mathfrak{T},5)| \leq 2^n + 1$.

For $r = 6$ the admissible triangle colourings are the ones from the case $r = 5$ above plus $(3/2,3/2,1)$, $(2,1,1)$, $(2,2,0)$, $(2,3/2,1/2)$. Again, due to Proposition 5, no half-integers can occur in any colouring. Thus, the only admissible triangle colourings are $(0,0,0)$, $(1,1,0)$, $(1,1,1)$, $(2,1,1)$, and $(2,2,0)$.

We trivially have $|\mathrm{Adm}(\mathfrak{T},6)| \leq 3^{n+1}$. Let $\theta \in \mathrm{Adm}(\mathfrak{T},6)$. We want to show, that at most a third of all non-constant assignment of colours 0, 1, 2 to the edges of $\mathfrak{T}$ can be admissible. For this, let $\theta \in \mathrm{Adm}(\mathfrak{T},6)$ and let θ' be defined by adding 1 (mod 3) to every edge colour. For θ' to be admissible, all triangles of θ must be of type $(0,0,0)$ and $(2,1,1)$. If at least one triangle has colouring $(0,0,0)$, θ must be the trivial colouring. Hence, all triangles are of type $(2,1,1)$ in θ. Replacing 2 by 0 and 1 by $1/2$ in θ yields a non-trivial admissible colouring in $\mathrm{Adm}(\mathfrak{T},3)$, a contradiction by Corollary 7. Hence, for every non-trivial admissible colouring θ, the colouring θ' cannot be admissible.

Analogously, let θ'' be defined by adding 2 (mod 3) to every edge colour of θ. For θ'' to be admissible, all triangles of θ must be of the type $(1,1,1)$, or $(2,2,0)$. A single triangle of type $(1,1,1)$ in θ forces θ to be constant. Hence, all triangles must be of type $(2,2,0)$. Dividing θ by four defines a non-trivial colouring in $\mathrm{Adm}(\mathfrak{T},3)$, a contradiction.

Combining these observations, at most every third non-trivial assignment of colours 0, 1, 2 to the edges of θ can be admissible. Adding the two admissible constant colourings yields $|\mathrm{Adm}(\mathfrak{T},6)| \leq 3^n + 1$.

The proof for $r = 7$ follows from a slight adjustment of the proof for $r = 6$. Admissible triangle colourings for colourings in $\mathrm{Adm}(\mathfrak{T},7)$ are the ones from $r = 6$ plus $(2,2,1)$. Again, we want to show that at most every third non-trivial assignment of colours 0, 1, 2 to the edges of $\mathfrak{T}$ can be admissible. For this let $\theta \in \mathrm{Adm}(\mathfrak{T},7)$ and let θ' and θ'' be defined as above. For θ' to be admissible θ must consist of triangle colourings of type $(0,0,0)$, $(1,1,0)$ and $(2,1,1)$. Whenever θ is non-constant replacing 2 by 0, and 1 by $1/2$ yields a non-trivial colouring in $\mathrm{Adm}(\mathfrak{T},3)$ which is not possible. The argument for θ'' is the same as in the case $r = 6$. It follows that $|\mathrm{Adm}(\mathfrak{T},7)| \leq 3^n + 1$.

All of the above bounds are attained by a number of small 3-sphere triangulations. See Table 2 for more details about 1-vertex triangulations $\mathfrak{T}$ with $\beta_1(\mathfrak{T},\mathbb{Z}_2) = 0$ with up to six tetrahedra and their average number of admissible colourings $|\mathrm{Adm}(\mathfrak{T},r)|$, $5 \leq r \leq 7$. ◀

There are 27,202 1-vertex triangulations with vanishing first Betti number and up to 6 tetrahedra. Exactly 142 of them attain equality in all three bounds. For more details about these cases of equality and the average number of colourings for $5 \leq r \leq 7$ in the census, see Table 2.

Note that the sharp bounds from Theorem 8 suggest that the over count of the general bound from Proposition 7(iii) is only linear in r.

3.5 An algorithm to compute $\mathrm{TV}_{r,1}$, r odd

In this section we describe a significant exponential speed-up for computing $\mathrm{TV}_{r,1}(\mathfrak{T})$ in the case where r is odd and $\mathfrak{T}$ does not contain any two-sided projective planes[1]. Note that the case of r odd is of importance for 3-sphere recognition problem. The main ingredients for this speed-up are:

- The *crushing and expanding procedure* for closed 3-manifolds as described by Burton, and Burton and Ozlen, which turns an arbitrary v-vertex triangulation into a number of smaller 1-vertex triangulations in polynomial time [3, 6];
- A classical result about Turaev-Viro invariants due to Turaev [15], Roberts [14], and Kirby and Melvin [10] stating that there exists a scaled version $\mathrm{TV}'_{r,1} = \gamma_r \mathrm{TV}_{r,1}$ which is multiplicative under taking *connected sums*[2], i.e., $\mathrm{TV}'_{r,1}(M \# N) = \mathrm{TV}'_{r,1}(M)\mathrm{TV}'_{r,1}(N)$ (see Theorem 2);
- Another classical result due to the same authors and publications stating that, for r odd, we have

  $$\mathrm{TV}_{r,1}(\mathfrak{T}) = \mathrm{TV}_{3,1}(\mathfrak{T}) \cdot \mathrm{TV}_{r,1}(\mathfrak{T}, [0]),$$

  and thus $\mathrm{TV}_{r,1}(\mathfrak{T}, [0])$ and $\mathrm{TV}_{3,1}(\mathfrak{T})$ are sufficient to compute $\mathrm{TV}_{r,1}(\mathfrak{T})$ (see Theorem 3);
- Proposition 7(ii) stating that computing $\mathrm{TV}_{r,1}(\mathfrak{T}, [0])$ of a 1-vertex closed 3-manifold triangulation can be done by only enumerating colourings with all integer colours.

Input. A v-vertex n-tetrahedra triangulation of a closed 3-manifold $\mathfrak{T}$

1. If $\mathfrak{T}$ has more than one vertex, apply the crushing and expanding procedure to $\mathfrak{T}$ as described in [3] and [6] respectively. It is not necessary to understand this procedure in detail. We only need this step to efficiently transform an arbitrary v-vertex n-tetrahedra triangulation $\mathfrak{T}$ into a number of triangulations $\mathfrak{T}_i$, $1 \leq i \leq s$, such that the following properties hold.
   - For $m \leq s$, every triangulation $\mathfrak{T}_i$, $1 \leq i \leq m$, is a 1-vertex n_i-tetrahedron triangulation;
   - For $m < \ell \leq s$, the topological type of every triangulation $\mathfrak{T}_\ell$ can be detected in polynomial time, and must be one of only three types (for which Turaev-Viro invariants can be pre-computed in constant time);
   - We have $(s - m) + \sum\limits_{i=1}^{m} n_i \leq n$;
   - We have $\mathfrak{T} \cong \mathfrak{T}_1 \# \ldots \# \mathfrak{T}_s$, i.e., $\mathfrak{T}$ is the connected sum of the $\mathfrak{T}_i$, $1 \leq i \leq s$.

   If $\mathfrak{T}$ contains a two-sided projective plane the crushing procedure will detect this fact and the computation is cancelled. The total running time of this step is polynomial.
2. For $1 \leq i \leq m$, compute $\mathrm{TV}_{r,1}(\mathfrak{T}_i, [0])$. This is the only step of this algorithm with an exponential running time. All other steps can be completed in polynomial time.
3. For all $\mathfrak{T}_i$, compute $\mathrm{TV}_{3,1}(\mathfrak{T}_i)$ – a polynomial time procedure, due to the one-to-one correspondence between admissible colourings in $\mathrm{Adm}(\mathfrak{T}, 3)$ and 1-cocycles of $\mathfrak{T}$.
4. Use Theorem 3 (for r odd we have $\mathrm{TV}_{r,1}(\cdot) = \mathrm{TV}_{3,1}(\cdot)\ \mathrm{TV}_{r,1}(\cdot, [0])$) to obtain $\mathrm{TV}_{r,1}(\mathfrak{T}_i)$.
5. Scale all values from the previous step to $\mathrm{TV}'_{r,1}$, multiply them and re-scale the product. The result equals $\mathrm{TV}_{r,1}(\mathfrak{T})$, by Theorem 2.

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[1] This is a technical pre-condition for the procedure to succeed. Triangulations not satisfying this pre-condition are extremely rare.

[2] Building the connected sum $M \# N$ of two manifolds M and N consists of removing a small ball from M and N respectively, and glue them together along their newly created boundaries.

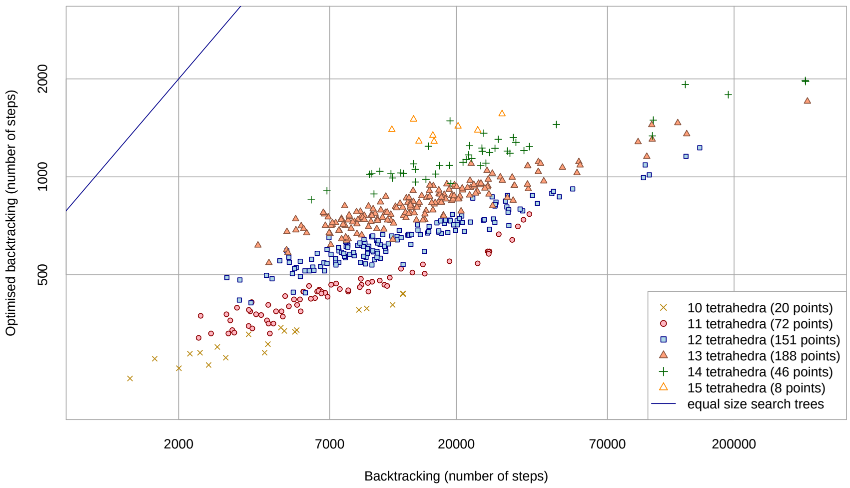

Figure 1 Number of nodes in the search tree visited by the naive algorithm and the optimised backtracking procedure for the 500 first 1-vertex triangulations of the Hodgson-Weeks census.

Running time of the proposed algorithm. The crushing and expanding procedure, computing $\mathrm{TV}_{3,1}(\mathfrak{T})_i$, $1 \leq i \leq s$, computing $\mathrm{TV}_{r,1}(\mathfrak{T}_i, [0])$, $m < i \leq s$, and scaling and multiplying the invariants are all polynomial time procedures [3, 6]. Following Proposition 7(iii) the running time to compute $\mathrm{TV}_{r,1}(\mathfrak{T}_i, [0])$, $1 \leq i \leq m$, is $O(\lfloor r/2 \rfloor^{n_i+1})$ (remember, $\mathfrak{T}_i$ is a 1-vertex triangulation). The overall running time is thus $O(\lfloor r/2 \rfloor^{n+1})$. The same procedure can be applied to improve the fixed parameter tractable algorithm as presented in [5] – which is also based on enumerating colourings – to get the running time $O(n \lfloor r/2 \rfloor^{6(k+1)} k^2 \log r)$, where k is the treewidth of the dual graph of $\mathfrak{T}$.

In Sections 4.1 and 4.2 we show that this improvement has also strong practical implications. In particular, the proposed algorithms allow computations up to several orders of magnitude faster than state of the art procedures to compute Turaev-Viro invariants.

4 Experiments

Here we run large scale experiments to illustrate the interest and performance of the methods introduced above. Implementations will appear within the 3-manifold software `Regina` [4].

We use two data sets for our experiments, both taken from large "census databases" of 3-manifolds to ensure that the experiments are comprehensive and not cherry-picked. The first census contains all 50 817 closed minimal triangulations that can be formed from $n \leq 11$ tetrahedra [2, 12]. This simulates "real-world" computation – the Turaev-Viro invariants were used to build this census. The second data set contains the triangulations from the Hodgson-Weeks census of closed hyperbolic manifolds [8]. This shows performance on larger triangulations, with n ranging from 9 to 20.

The admissible colouring weights may be computed symbolically or numerically, which acts substantially on running times. In the following, we either avoid this difficulty by measuring "discrete data" (like size of search spaces) to represent performance, or we indicate which weight representation we use.

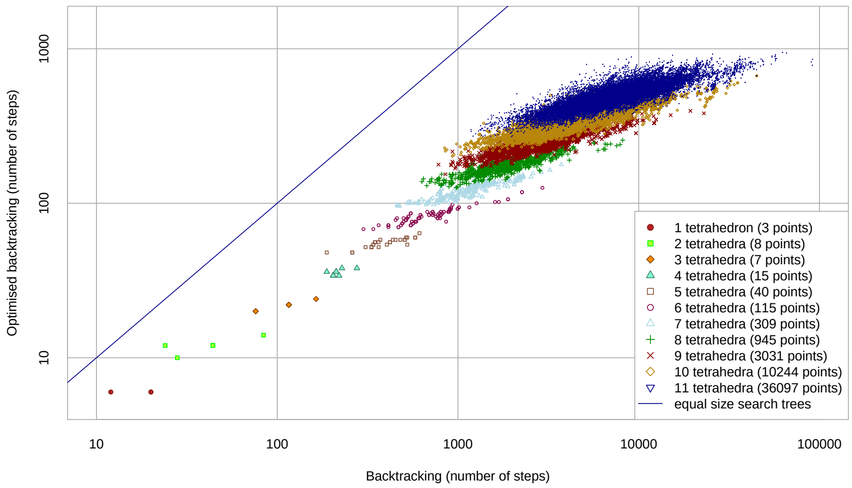

Figure 2 Number of nodes in the search tree visited by the optimised backtracking procedure over the naive algorithm for the 1-vertex minimal closed triangulations.

4.1 Computing $\mathrm{TV}_{r,1}$ with the backtracking method, r odd

We compare experimentally the performance of the naive backtracking algorithm with our proposed backtracking algorithm (Section 3.5) enumerating only colouring at the cohomology class [0]. To do so, we count the number of nodes in the backtracking search tree visited by both algorithms for computing $\mathrm{TV}_{5,1}$ (i) for all triangulations with ≤ 11 tetrahedra in the census of closed minimal triangulations [2] (see Figure 2), and (ii) for the first 500 triangulations of the Hodgson-Weeks census, with $10 \leq n \leq 15$, [8] (see Figure 1). These triangulations all have one vertex, and the improvement is solely due to the reduction of the space of colourings studied above (in particular, the crushing step is not applied).

Because a colouring may be declared non-admissible before colouring all edges of the triangulation, the standard backtracking algorithm visits generally fewer nodes than the $O((r-1)^{n+1})$, for a triangulation with n tetrahedra, predicted by the worst case complexity analysis. Despite this fact, the improvements of our algorithm for the minimal triangulations census range from factors 2 to 117. Improvements in the Hodgson-Weeks census, which contains much larger triangulations, range from factors 5.6 to 215. On both data sets, the range of improvements rapidly grows larger as the size of the triangulations increase. We confirm this observation below.

4.2 Computing $\mathrm{TV}_{r,1}$ with the FPT algorithm, r odd

As demonstrated in [5], the fixed parameter tractable (FPT) algorithm is the most efficient procedure to compute Turaev-Viro invariants experimentally. Improving the running time of this implementation is thus highly significant in practice.

We compare the running times of the FPT algorithm from [5] with the optimised FPT algorithm relying on the enumeration of colourings at the trivial cohomology class, as presented in Section 3.5. Here, the enumeration of colourings is done within the bags of the tree decomposition of the dual graph of the triangulation [5]. Turaev-Viro invariants are computed with floating point arithmetic. Figure 4 represents the running time of both algorithms on the census of closed minimal triangulations of up to 11 tetrahedra, for $r = 5$. All triangulations only have one vertex. We have removed from the timings triangulations with

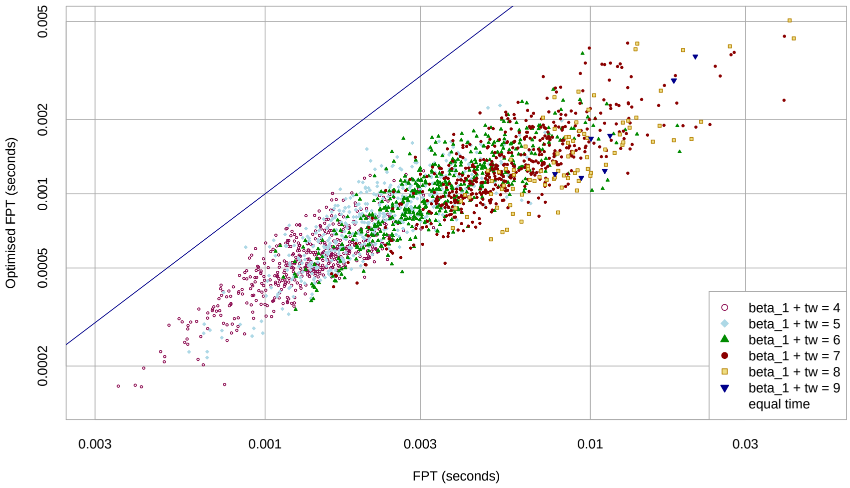

Figure 3 Comparison of the running times of the FPT algorithm presented in [5] and the FPT algorithm with the enumeration procedure introduced in Section 3.5, to compute $\mathrm{TV}_{5,1}$ on the Hodgson-Weeks census with $n \leq 20$ tetrahedra (with one vertex and $\mathrm{TV}_{3,1} \neq 0$). We use the parameter $\mathrm{tw} + \beta_1$ as a measure of "difficulty" for the computation. For readability, the plot presents a sparsified cloud (500 uniform random samples for each parameter value).

$\mathrm{TV}_{3,1} = 0$, as we can conclude in polynomial time, in our implementation, that $\mathrm{TV}_{r,1} = 0$ using the formula involving $\mathrm{TV}_{3,1}$ in Section 3.5. Consequently, Figure 4 illustrates the improvement solely due to the enumeration of colourings at the trivial cohomology class. For our implementation of the FPT algorithm, we include the timings of *all* steps of the algorithm presented in Section 3.5; the dominating step is naturally the computation of $\mathrm{TV}_{r,1}(\mathfrak{T}, [0])$ (step 2), which is the only exponential step of the procedure.

Figure 4 shows a clear improvement of the running time of our algorithm. Most interestingly, this range of improvement seems to increase (points getting further away from the diagonal) for triangulations on which the standard FPT algorithm is slower.

To confirm this tendency on larger scales, we run the computational power-intensive computation of $\mathrm{TV}_{11,1}$ on the first 1000 triangulations of the Hodgson-Weeks census (Figure 3), with triangulations with up to 20 tetrahedra. We observe that 40% of the total running time of the standard FPT algorithm over the 1000 triangulations is spent on only 10 of them. On these 10 inputs, our implementation is up to 130 times faster, and 29 times faster in average, reducing the total running time for these "hardest" 10 triangulations from several hours to a few minutes of computation.

4.3 Experiments for computing $\mathrm{Adm}(\mathfrak{T}, 4)$

In this section, we study experimentally the bounds on the number of admissible colourings for $r = 4$, and the efficiency of the algorithm for $\mathrm{TV}_{4,1}$, introduced in Sections 3.2 and 3.3, depending on them. Table 1 gives details on the bounds given by Theorem 6, and hence the worst case number of steps of our algorithm to compute $\mathrm{TV}_{4,1}$, and the average number of steps the backtracking algorithm requires to compute $\mathrm{TV}_{4,1}$. We run the experiments on all minimal triangulations with up to 6 tetrahedra, sorted by Betti number β_1.

We note that the actual number of nodes visited by the backtracking algorithm is smaller than the worst case bound, but it is significantly larger than the upper bound of Equation (2) in Theorem 6. Additionally, the bound given by Equation (2) is very close

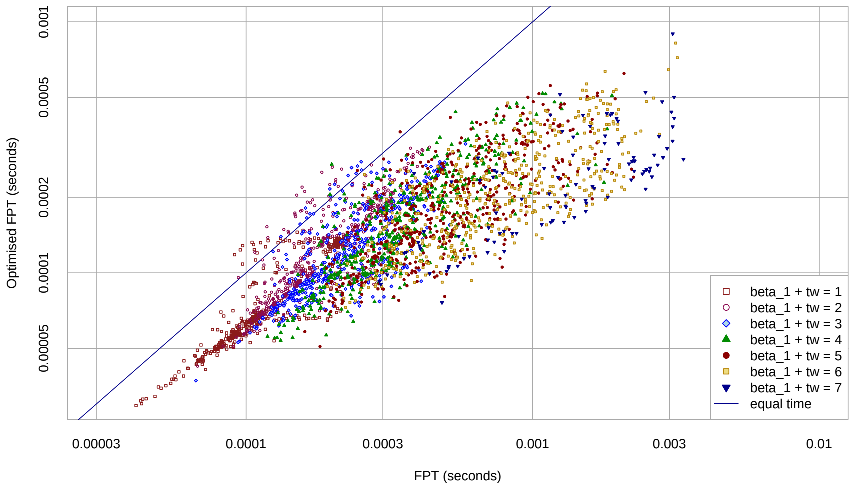

Figure 4 Comparison of the running times of the FPT algorithm from [5] and the FPT algorithm with the enumeration procedure introduced in Section 3.5, to compute $\mathrm{TV}_{5,1}$ on the census of minimal triangulations with $n \leq 11$ tetrahedra (with one vertex and $\mathrm{TV}_{3,1} \neq 0$). We use the parameter $\mathrm{tw} + \beta_1$ as a measure of "difficulty" for the computation. For readability, the plot presents sparsified cloud (500 uniform random samples for each parameter value).

Table 1 "$\#\mathfrak{T}$" lists the number of triangulations contained in the n-tetrahedra census of minimal triangulations with first Betti number $\beta_1(\mathfrak{T}, \mathbb{Z}_2)$, "# eq. (2)" lists the number of triangulations satisfying equality in Inequality (2). Below, the average number of nodes of the search tree visited by the backtracking algorithm ("# tree"), the bound "Eqn. (2)" given by Inequality (2), and the average number "Av." of admissible colourings in $\mathrm{Adm}(\mathfrak{T}, 4)$ are listed.

(n, β_1)	$(1,1)$	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(4,1)$	$(4,2)$	$(5,1)$	$(5,2)$	$(6,1)$	$(6,2)$	$(6,3)$
$\#\mathfrak{T}$	1	5	1	27	3	205	19	1858	184	21459	2516	34
#eq. (2)	1	5	1	14	1	67	4	261	10	1574	47	0
#tree	12.0	33.0	39.0	46.4	69.0	75.2	110.1	93.1	159.2	120.4	214.5	413.2
Eqn. (2)	4.0	6.0	10.0	7.7	14.7	13.1	21.8	20.4	35.8	34.6	58.0	94.5
Av.	4.0	6.0	10.0	6.3	11.3	8.7	15.3	9.3	18.6	10.7	22.0	41.4

to the average number of admissible colourings of the triangulations, which underlines the fact that our algorithm for $\mathrm{TV}_{4,1}$ has a practical output-sensitive behaviour in the number of admissible colourings. Finally, our bounds on $|\mathrm{Adm}(\mathfrak{T}, 4)|$ are sharp on 1985 out of the $26,312$ triangulations of the experiment.

5 Experiments on triangulations with vanishing β_1

In this section we provide experimental details on the number of admissible colourings of triangulations with $\beta_1 = 0$, and the ability of $\mathrm{TV}_{r,1}$ to distinguish between these manifolds and the 3-sphere, which is of particular importance in 3-manifold topology.

In Table 2 we provide details on the number of admissible colourings of triangulations with up to 6 tetrahedra and $\beta_1 = 0$. In particular, the table shows that the bounds from Proposition 8 are tight, and much finer in general than the naive bound $(r-1)^{n+v}$.

As evidence for the interest of the algorithm to compute $\mathrm{TV}_{r,1}$, r odd, introduced in Section 3.5, we analyse the ability of $\mathrm{TV}_{r,1}$, $r \in \{3, 5, 7, 9\}$, to distinguish 3-manifolds

Table 2 Number "# trigs." of 1-vertex triangulations $\mathfrak{T}$ of manifolds with $\beta_1(\mathfrak{T}, \mathbb{Z}_2) = 0$ and n tetrahedra, $1 \leq n \leq 6$. Number "#sharp" of such triangulations satisfying equality in all bounds from Theorem 8 (third column), and the average number "$\overline{\mathrm{Adm}(\mathfrak{T}, r)}$" of admissible colourings in $\mathrm{Adm}(\mathfrak{T}, r)$, $5 \leq r \leq 7$ (columns 6, 9, and 12), compared to the naive upper bound "$(r-1)^{v+n}$" (columns 4, 7, and 10) and the new upper bounds given by Theorem 8 (columns 5, 8, and 11).

n	#trig.	#sharp	$(5-1)^{n+v}$	2^n+1	$\overline{\lvert\mathrm{Adm}(\mathfrak{T},5)\rvert}$	$(6-1)^{n+v}$	3^n+1	$\overline{\lvert\mathrm{Adm}(\mathfrak{T},6)\rvert}$	$(7-1)^{n+v}$	3^n+1	$\overline{\lvert\mathrm{Adm}(\mathfrak{T},7)\rvert}$
1	2	1	16	3	2.50	25	4	3.00	36	4	4.00
2	7	3	64	5	4.00	125	10	6.86	216	10	8.86
3	36	5	256	9	5.61	625	28	12.22	1,296	28	17.28
4	255	14	1,024	17	8.31	3,125	82	23.46	7,776	82	35.30
5	2305	30	4,096	33	12.02	15,625	244	43.00	46,656	244	70.44
6	24597	89	16,384	65	17.71	78,125	730	80.15	279,936	730	142.23

Table 3 Summary of the ability of $\mathrm{TV}_{r,1}$, $3 \leq r \leq 9$, to distinguish 3-manifolds with trivial (integral) homology up to complexity 11 from the 3-sphere. X/Y denotes the success rate, i.e., there are Y manifolds, X of which can be distinguished from the 3-sphere by the respective invariant.

n	$\mathrm{TV}_{3,1}$	$\mathrm{TV}_{4,1}$	$\mathrm{TV}_{5,1}$	$\mathrm{TV}_{6,1}$	$\mathrm{TV}_{7,1}$	$\mathrm{TV}_{8,1}$	$\mathrm{TV}_{9,1}$	$\mathrm{TV}_{5,1}$ and $\mathrm{TV}_{7,1}$
5	0/1	0/1	**1/1**	0/1	**1/1**	1/1	1/1	**1/1**
7	0/1	0/1	**1/1**	0/1	**1/1**	0/1	1/1	**1/1**
8	0/3	0/3	**1/3**	0/3	**3/3**	3/3	3/3	**3/3**
9	0/4	0/4	**3/4**	0/4	**3/4**	1/4	3/4	**4/4**
10	0/8	0/8	**5/8**	0/8	**7/8**	3/8	6/8	**8/8**
11	0/19	0/19	**11/19**	0/19	**16/19**	13/19	16/19	**18/19**

from the 3-sphere. Since homology can be computed in polynomial time, we only consider 3-manifolds M which cannot be distinguished from the 3-sphere using integral homology (i.e., $\beta_1(M, \mathbb{F}) = 0$, for any choice of field $\mathbb{F}$). There are 36 distinct such 3-manifolds of *complexity* at most 11, meaning, they can be triangulated with 11 tetrahedra or less. Due to Proposition 7(i) we already know that none of them can be distinguished from the 3-sphere by $\mathrm{TV}_{3,1}$. $\mathrm{TV}_{5,1}$, $\mathrm{TV}_{7,1}$, and $\mathrm{TV}_{9,1}$ distinguish 22, 31, and 30, a combination of $\mathrm{TV}_{5,1}$ and $\mathrm{TV}_{7,1}$ only fails once, and a combination of all three invariants never fails to distinguish them from the 3-sphere. See Table 3 for details.

Together with the favourable timings presented in Section 4, this indicates that our new approach to compute the Turaev-Viro invariants for odd values of r gives a practical fast way to distinguish manifolds with $\beta_1 = 0$ from the 3-sphere in many cases.

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