Hyperbolic Random Graphs: Separators and Treewidth

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Abstract

Hyperbolic random graphs share many common properties with complex real-world networks; e.g., small diameter and average distance, large clustering coefficient, and a power-law degree sequence with adjustable exponent β. Thus, when analyzing algorithms for large networks, potentially more realistic results can be achieved by assuming the input to be a hyperbolic random graph of size n. The worst-case run-time is then replaced by the expected run-time or by bounds that hold *with high probability (whp)*, i.e., with probability $1-O(1/n)$. Though many structural properties of hyperbolic random graphs have been studied, almost no algorithmic results are known.

Divide-and-conquer is an important algorithmic design principle that works particularly well if the instance admits small separators. We show that hyperbolic random graphs in fact have comparatively small separators. More precisely, we show that they can be expected to have balanced separator hierarchies with separators of size $O(\sqrt{n^{3-\beta}})$, $O(\log n)$, and $O(1)$ if $2<\beta<3$, $\beta=3$, and $3<\beta$, respectively. We infer that these graphs have whp a treewidth of $O(\sqrt{n^{3-\beta}})$, $O(\log^2 n)$, and $O(\log n)$, respectively. For $2<\beta<3$, this matches a known lower bound.

To demonstrate the usefulness of our results, we give several algorithmic applications.

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1 Introduction

A geometric random graph is obtained by randomly placing vertices into the plane and connecting two vertices if and only if they are close. When using the hyperbolic plane, one obtains a (threshold) hyperbolic random graph, as introduced by Krioukov et al. [21]. More precisely, vertices are placed in a disk D_R of radius R (which depends on n) and two vertices are connected if their distance is at most R. An important property of the hyperbolic plane is that the perimeter of a circle grows exponentially with the radius. Thus, when sampling uniformly, we obtain many vertices close to the boundary and few close to the center of D_R. As distances close to the boundary are larger (due to the exponentially growing perimeter), this leads to many vertices of low degree and few vertices of high degree. The resulting degree distribution actually follows a power law with exponent $\beta = 3$ [18, 28]. One can tweak this exponent using a parameter α. Choosing $1/2 \le \alpha < 1$ increases the probability

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of vertices with small radius (and thus of higher degree), while for $\alpha > 1$, the vertices are shifted towards the boundary of D_R. The resulting power-law exponent is $\beta = 2\alpha + 1$.

Besides a power-law degree-distribution, hyperbolic random graphs exhibit other properties of large real-world graphs. Due to the geometric notion of closeness, vertices with a common neighbor are likely also connected, leading to a constant clustering coefficient [18]. We note that this property distinguishes hyperbolic random graphs from other models that also generate scale-free graphs (i.e., graphs with a power-law degree-distribution) such as the Chung-Lu model [7] and the Barabási-Albert model [2]. These models produce graphs that have clustering coefficient $o(1)$ [5, 29]. Beyond a non-vanishing clustering coefficient, hyperbolic random graphs have polylogarithmic diameter [15, 20] and average distance $O(\log\log n)$ between vertex pairs [1, 6], i.e., they are ultra-small-world networks. Thus, hyperbolic random graphs seem to be well suited for representing large real-world networks. This is further supported by the work of Boguñá, Papadopoulos and Krioukov [4] who embedded the internet into the hyperbolic plane and demonstrated that the resulting coordinates lead to an almost optimal greedy routing.

Despite these promising properties, we note that hyperbolic random graphs are clearly not a perfect and domain-independent representation of the real world. The degree distribution of real-world data does for example not always follow a power-law [8]. Moreover, very large cliques [14] seem unrealistic as one would expect at least a few edges to be missing. Such missing edges can be achieved by considering the so-called binomial model, in which vertices are connected with a certain probability depending on their hyperbolic distance (see Section 6). Recently, hyperbolic random graphs have been generalized to geometric inhomogeneous random graphs (GIRGs) [6]. In a GIRG, the degree distribution depends on chosen weights and is thus not necessarily fixed to a power law. Moreover, GIRGs allow for more flexibility in the choice of the underlying (potentially higher dimensional) geometry.

Divide-and-conquer algorithms separate the given instance into smaller subinstances, solve these subinstances recursively, and then combine the results to a solution of the original instance. Such an approach works well if (i) both subinstances have roughly the same size (leading to a logarithmic recursion depth); and (ii) a small interface between the subinstances allows a combination of partial solutions with only few tweaks. For graphs, one is thus interested in finding small *balanced separators*, i.e., sets of vertices whose removal separates the graph into disconnected subgraphs of roughly the same size. A famous example is the planar separator theorem by Lipton and Tarjan [24] stating that every planar graph with n vertices has a separator of size $O(\sqrt{n})$ such that both resulting subgraphs have at least $1/3n$ vertices. This for example leads to a PTAS for Independent Set on planar graphs [25].

Closely related to separators is the concept of treewidth, which is a key concept in the field of parameterized complexity as many NP-hard graph problems are actually FPT with respect to the treewidth [9, 10]. The treewidth has been intensively studied on different random graph models (all results we mention in the following hold with probability tending to 1 for $n \to \infty$). Erdős-Rényi graphs [12] have linear treewidth if the edge-vertex ratio is above $1/2$ [16, 17, 22]. This bound is sharp as the treewidth is 2 if the edge-vertex ratio is below $1/2$ [22]. For random intersection graphs [19] and for the Barabási-Albert model (which produces scale-free graphs) [2], Gao [17] gave linear lower bounds for the treewidth.

Besides these negative results, there are positive results for geometric random graphs (in the euclidean geometry). Depending on the maximum distance for which two vertices are connected, the treewidth can be shown to be $\Theta(\log n/\log\log n)$ [27] or $\Theta(r\sqrt{n})$ [23, 27]. Recently, Bringmann et al. [6] showed that a GIRG has a balanced cut of sub-linear size, which implies the same for hyperbolic random graphs.

Contribution & Outline. We present a hierarchical decomposition (the *hyperdisk decomposition*) of a disk in the hyperbolic plane into equally sized regions that have large distance from each other while the separators have small area; see Section 3. This decomposition carries over to hyperbolic random graphs leading to a hierarchy of balanced separators, each of expected size $O(n^{1-\alpha})$, $O(\log n)$, and $O(1)$ if $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$, respectively.

In Section 4, we infer that hyperbolic random graphs have whp treewidth $O(n^{1-\alpha})$, $O(\log^2 n)$, and $O(\log n)$ if $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$, respectively. For $\alpha < 1$, this matches the lower bound implied by the clique number of hyperbolic random graphs [14]. For $\alpha = 1$ and $\alpha > 1$, this is above the lower bounds by factors of $\log n \log\log n$ and $\log\log n$, respectively.

We demonstrate algorithmic applications in Section 5. For Independent Set, we give an approximation scheme whose algorithms have expected approximation ratio $1-\varepsilon$ ($\varepsilon > 0$) and expected polynomial run-time (in n). Choosing ε suitably, the polynomial run-time and the approximation ratio $1-O(1)/\log^\alpha n$ hold whp. Moreover, we show that maximum matchings in hyperbolic random graphs can whp be computed in time $O(n^{2-\alpha})$ ($O(n^{2-\alpha}\log n)$ with edge weights). As $\alpha \geq 1/2$, this improves upon the worst-case complexity of the fastest known algorithm for general graphs with run-time $O(\sqrt{n}m)$ [26]. For both results, we assume that the geometry of the graph is known, which lets us compute the hyperdisk decomposition. Otherwise, one can still apply the results by Fomin et al. [13] to obtain fast algorithms for various problems; see Section 5.3.

In Section 6, we consider the binomial model, where two vertices are connected with a certain probability depending on their distance. For $\alpha < 1$ we obtain the treewidth $O(n^{2-(\alpha+1)/(\alpha t+1)})$ (t is a constant and $t \to 0$ leads to the threshold model).

2 Preliminaries

Hyperbolic Random Graphs. We consider three parameters, the number of vertices n, the parameter $\alpha \geq 1/2$ controlling the power-law exponent, and C controlling the average degree. We obtain a hyperbolic random graph by sampling n points in the disk D_R with radius $R = 2\log n + C$. A point is described using radial coordinates (r, θ) with the center of D_R as origin. The angle θ is drawn uniformly from $[0, 2\pi]$ and the radius r is chosen according to the density $d(r) = \alpha\sinh(\alpha r)/(\cosh(\alpha R) - 1)$. Thus, the points are distributed according to the following density function (which depends on the radius r but not on the angle θ).

$$f(r,\theta) = f(r) = \frac{\alpha}{2\pi} \cdot \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1} \tag{1}$$

For $S \subseteq D_R$, the probability measure $\mu(S) = \int_S f(r)\,\mathrm{d}r$ gives the probability that a specific vertex lies in S. Note that $\mu(D_R) = 1$ and for $\alpha = 1$, $\mu(S)$ is the area of S divided by the area of D. Two vertices are connected if and only if their (hyperbolic) distance is at most R.

We are often interested in the number of vertices lying in a certain region S. To this end, let $X_i \in \{0, 1\}$ be the random variable with the interpretation that $X_i = 1$ if and only if the vertex i lies in S. Note that $\mathbb{E}[X_i] = \mu(S)$. Moreover, the random variable $X = \sum_{i=1}^n X_i$ describes the number of vertices in S. The following theorem directly follows from Chernoff bounds [11] and helps to give bounds that hold whp (i.e., with probability $1 - O(1/n)$).

▶ **Theorem 1.** *Let $X_1, \ldots, X_n$ be independent random variables with $X_i \in \{0, 1\}$ and let X be their sum. Let $f(n) = \Omega(\log n)$. If $f(n)$ is an upper (resp. lower) bound for $\mathbb{E}[X]$, then for each constant c there is a constant c' such that $X \leq c' f(n)$ (resp. $X \geq c' f(n)$) holds with probability $1 - O(n^{-c})$.*

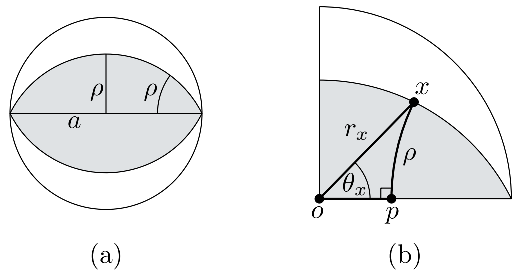

Figure 1 (a) The hyperdisk (gray) with radius ρ and horizontal axis a going through the origin. (b) A right triangle in the hyperbolic plane.

Separator Hierarchy. Let D be a metric space (e.g., a disk in hyperbolic space or a graph). A *separator hierarchy* of D is a rooted tree T with t nodes where each node i is associated with a subset $S_i \in D$ such that $\mathcal{S} = \{S_1, \dots, S_t\}$ partitions D, i.e., $D = \bigcup_{S_i \in \mathcal{S}} S_i$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. For every node i, let $U_i = S_i \cup \bigcup_{j \in \mathrm{desc}(i)} S_j$, where $\mathrm{desc}(i)$ are the descendants of i in T. For $(T, \mathcal{S})$ to be a separator hierarchy, we require that for every pair of sibling nodes i and j (i.e., nodes with the same parent) the set U_i is disconnected from U_j. For the parent k of i and j, we also say that S_k is the *separator* that separates U_i from U_j.

The *diameter* of the separator S_k (separating U_i and U_j) is the largest value d such that every element in U_i has distance at least d form every element in U_j. The diameter of the separator hierarchy $(T, \mathcal{S})$ is the minimum diameter of all its separators. For a given measure μ on D, the separator S_k is *balanced* if $\mu(U_i) \leq 1/2\mu(U_k)$ and $\mu(U_j) \leq 1/2\mu(U_k)$. If these inequalities hold in expectation, we say that S_k is *expected to be balanced*. The separator hierarchy is balanced (in expectation) if each separator is balanced (in expectation).

Treewidth. Let T be a tree with t nodes and let $\mathcal{X} = \{X_1, \dots, X_t\}$ be a family of sets. For each node i of T, the set X_i is called the *bag* of i. The pair $(T, \mathcal{X})$ is a *tree decomposition* of a graph $G = (V, E)$ if the bags are subsets of V satisfying the following two properties.

(i) For each vertex $v \in V$, the nodes of T whose bags contain v induce a subtree of T.
(ii) For each edge $uv \in E$, there exist a bag $X \in \mathcal{X}$ with $u \in X$ and $v \in X$.

The *width* of a tree decomposition is the size of the largest bag minus 1. The *treewidth* $\mathrm{tw}(G)$ of a graph G is the smallest k for which G has a tree decomposition of width k.

3 Hyperdisk Decomposition

In this section, we define a separator hierarchy for a disk D in the hyperbolic plane. We assume that D is centered at the origin. The separators are (parts of) hyperdisks, which are defined as follows. Let a be a line in the hyperbolic plane. The set of points with distance ρ from a form the *hypercircle* with *radius* ρ and *axis* a. The set of points with distance at most ρ from a is the corresponding *hyperdisk*; see[1] Fig. 1a. We usually consider lines through the origin as axes. By a_γ, we denote such a line whose points have angle γ or $\gamma + \pi$.

Let $x = (r_x, \theta_x)$ be a point on the hypercircle with axis a_0 and radius ρ. Moreover, let o be the origin and let p be the point on a_γ such that the line through x and p is perpendicular to a_γ. Then o, p, and x form a triangle with right angle at p; see Fig. 1b. The angle at o

[1] We use the Poincaré disk model in illustrations. Thus, the disk shown in Fig. 1 (as well as the outer-most disk in every later illustration) is not D but the Poincaré disk representing the whole hyperbolic plane.

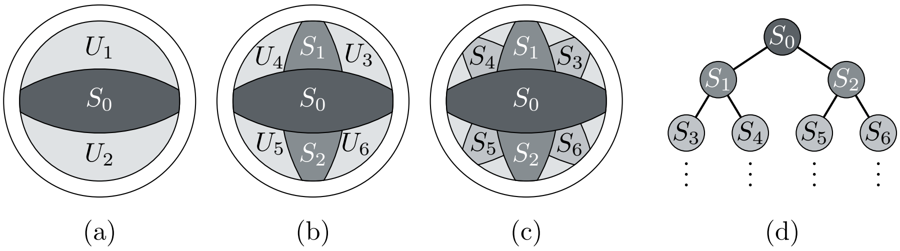

Figure 2 (a) The disk D (gray area) is separated by S_0 (dark gray) into the two regions U_1 and U_2 (light gray). (b, c) The separators on levels 1 and 2. (d) The corresponding separator hierarchy represented as tree in which each node corresponds to a separator.

is θ_x, the length of the opposite side xp is ρ, and the length of the hypotenuse is r_x. The trigonometry of hyperbolic right triangles yields the following equation, which we need later.

$$\sin(\theta_x) = \frac{\sinh(\rho)}{\sinh(r_x)} \tag{2}$$

The *hyperdisk decomposition* of D is the following separator hierarchy. The top-level separator (level 0) S_0 is the intersection of D with the hyperdisk with axis a_0 (all hyperdisks we consider have arbitrary but fixed radius ρ). This symmetrically separates D into regions U_1 and U_2 above and below S_0; see Fig. 2a. The region U_1 (and analogously U_2) is again symmetrically separated into two parts by its intersection with the hyperdisk with axis $a_{\pi/2}$. Denote the resulting separator by S_1 and the two separated regions by U_3 and U_4; see Fig. 2b. On the next level, U_3 (and analogously $U_4, \dots, U_6$) is separated by its intersection with the hyperdisk with axis $a_{\pi/4}$; see Fig. 2c. We continue this decomposition until $S_i = U_i$. Clearly, this leads to a separator hierarchy $(T, \mathcal{S})$ and T is a complete binary tree; see Fig. 2d.

3.1 Properties of the Hyperdisk Decomposition

In the following, we investigate different properties of the hyperdisk decomposition, depending on the radius R of the disk D, on the radius ρ of the hyperdisks, and on a measure μ on D. We start with two simple observations. The first observation follows from the fact that two points on different sides of a hyperdisk with radius ρ have distance at least 2ρ (they have distance ρ from the hyperdisk's axis and the line segment connecting them crosses the axis).

▶ **Observation 2.** *The hyperdisk decomposition has diameter at least* 2ρ*.*

We will later set $\rho = R/2$ (Section 3.2) or to something even larger (Section 6). However, the bounds we prove in this section hold for more general choices of ρ.

The next observation follows from the fact that for two nodes i and j on the same level, the regions U_i and U_j are symmetric with respect to rotation around the origin.

▶ **Observation 3.** *The hyperdisk decomposition is balanced for every measure that is invariant under rotation around the origin.*

The measure we are particularly interested in is μ given by $\mu(S) = \int_S f(r)\,\mathrm{d}r$ with the density function $f(r)$ as defined in Equation (1). Note that μ is clearly invariant under rotation around the origin as f does not depend on the angle of a given point. In the remainder of this section, we always assume μ to be this measure.

Our main goal in the following is to bound the measure of the separators in the hyperdisk decomposition. Clearly, the measure of the separators decreases for increasing level. To quantify this, we first show the following lemma.

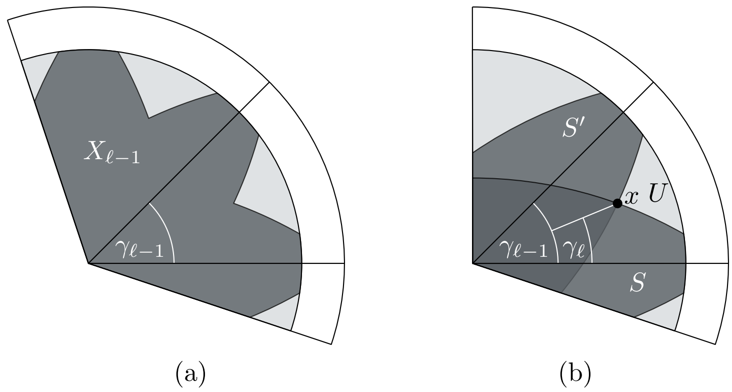

Figure 3 (a) The region $X_{\ell-1}$. (b) Two consecutive hyperdisks enclosing the region U.

▶ **Lemma 4.** *Let S be a separator on level $\ell \geq 1$ of the hyperdisk decomposition and let $x \in S$ be a point with radius r_x. Then the following holds:*

$$r_x \geq r_{\min} = \max\{\rho, \rho + \log(1 - e^{-2\rho}) - \log(2^{2-\ell} - 2^{2-2\ell})\}.$$

Proof. We show that the inequality holds for every point x that is not contained in a separator of level less than ℓ. Clearly, $r_x \geq \rho$ holds as every point with smaller radius is contained in the top-level separator. It remains to show $r_x \geq \rho + \log(1 - e^{-2\rho}) - \log(2^{2-\ell} - 2^{2-2\ell})$.

First assume $\ell = 1$. In this case $\log(2^{2-\ell} - 2^{2-2\ell}) = 0$. Moreover, $\log(1 - e^{-2\rho}) < 0$. Thus the claim is weaker than $r_x \geq \rho$, which we already proved above. We assume in the following that $\ell \geq 2$, which implies that there are at least two separators with level less than ℓ.

Let $\gamma_\ell = \pi/2^\ell$ and consider all hyperdisks that have an axis whose angle is a multiple of $\gamma_{\ell-1}$. Let $X_{\ell-1}$ be the union of these hyperdisks; see Fig. 3a. By the definition of the hyperdisk decomposition, the union of all separators of level less than ℓ equals $X_{\ell-1}$. We show that all points not in $X_{\ell-1}$ (and thus all points in a separator of level ℓ) satisfy the claimed inequality. To this end, first note that rotating the disc D by a multiple of $\gamma_{\ell-1}$ around the origin maps $X_{\ell-1}$ to itself. Thus, it suffices to prove the claim for points with angles between 0 and $\gamma_{\ell-1}$.

Let S and S' be the hyperdisks whose axes have angles 0 and $\gamma_{\ell-1}$, respectively, let U be the region between them, and let x be the point where S, S', and U touch; see Fig. 3b. In the following we first show that actually no point in U has radius smaller than x (which is intuitively true when looking at Fig. 3a). Afterwards it remains to show the claimed lower bound for the point x.

Let $y \in U$ be a point with coordinates (r_y, θ_y) and let ρ_y be the distance of y from the horizontal axis a_0. By Equation (2), we have $\sinh(\rho_y) = \sinh(r_y)\sin(\theta_y)$. Thus, for all relevant angles θ_y, the distance ρ_y is increasing with increasing radius and with increasing angle. Hence, in case $\theta_y \leq \theta_x$, the assumption $r_y < r_x$ implies that $\rho_y < \rho$ (recall that ρ is the distance of x from the horizontal axis). Thus, y is contained in the hyperdisk S and cannot be contained in U. Symmetrically, if $\theta_y \geq \theta_x$ and $r_y < r_x$, then y is contained in S'. Hence, no point in U has smaller radius than x.

It thus remains to show the claimed inequality for the point x. Note that x has the angle $\theta_x = \gamma_\ell$. Thus, by Equation (2), we have the following.

$$\sin(\gamma_\ell) = \frac{\sinh(\rho)}{\sinh(r_x)}$$

$$
\begin{aligned}
\Leftrightarrow \qquad\qquad & \sinh(r_x) = \frac{\sinh(\rho)}{\sin(\gamma_\ell)} \\
\Leftrightarrow \qquad\qquad & e^{r_x} - e^{-r_x} = \frac{e^{\rho} - e^{-\rho}}{\sin(\gamma_\ell)} \\
\Rightarrow \qquad\qquad & e^{r_x} \geq \frac{e^{\rho} - e^{-\rho}}{\sin(\gamma_\ell)} \\
\Leftrightarrow \qquad\qquad & r_x \geq \log(e^{\rho} - e^{-\rho}) - \log(\sin(\gamma_\ell)) \\
& \quad = \rho + \log(1 - e^{-2\rho}) - \log(\sin(\gamma_\ell))
\end{aligned}
$$

We use that $\sin(\gamma) \leq 1 - (1 - 2/\pi \cdot \gamma)^2$ for $\gamma \in [0, \pi/2]$ to obtain the following.

$$
\sin(\gamma_\ell) \leq 1 - \left(1 - \frac{2}{\pi}\gamma_\ell\right)^2 = 1 - \left(1 - 2^{1-\ell}\right)^2 = 2^{2-\ell} - 2^{2-2\ell}
$$

Together with the previous inequality, this yields the claimed bound. ◀

The above lemma together with a simple calculation shows the following. With the height of the hyperdisk decomposition (which is a separator hierarchy), we refer to the height of the corresponding tree, i.e., to the maximum distance from a node to the root.

▶ **Lemma 5.** *The hyperdisk decomposition has height $O(R)$ if $\rho \geq \varepsilon$ for a constant ε.*

Proof. We show that setting $\ell = \log_2(c \cdot e^R) + 2 = O(R)$ (for a suitable constant c) results in $r_{\min} \geq R$. Thus, the separators with smaller levels already cover the whole disk of radius R. The last part of the formula given for $r_{\min}$ in Lemma 4 can be rewritten as follows.

$$
-\log\left(2^{2-\ell} - 2^{2-2\ell}\right) \geq -\log\left(2^{2-\ell}\right) = \log\left(2^{\ell-2}\right) = \log(c) + R
$$

When choosing $c = (1 - e^{-2\varepsilon})^{-1}$, Lemma 4 yields $r_{\min} \geq R$. ◀

In the following, we upper bound the measure of a separator S on level ℓ of the hyperdisk decomposition. Let H be the hyperdisk corresponding to S. To simplify the calculations, we rotate the disk such that a_0 (i.e., the horizontal line through the origin) is the axis of H. First assume that $\ell \geq 1$ and let $r_{\min}$ be the lower bound for the radius shown in Lemma 4. Consider the set S' of all points in H with radius at least $r_{\min}$ that lie to the right of the origin (angle between $-\pi/2$ and $\pi/2$). Clearly $S \subseteq S'$ and thus $\mu(S) \leq \mu(S')$.

To compute $\mu(S')$, let $\theta_\rho(r)$ be the angle between 0 and $\pi/2$ such that the point $(r, \theta_\rho(r))$ lies on the hypercircle bounding H. By Equation (2), we have $\theta_\rho(r) = \arcsin(\sinh(\rho)/\sinh(r))$. We obtain the following (recall that $f(r)$ is the density function).

$$
\mu(S') = \int_{S'} f(r)\,\mathrm{d}r = \int_{r_{\min}}^{R} \int_{-\theta_\rho(r)}^{\theta_\rho(r)} f(r)\,\mathrm{d}\theta\,\mathrm{d}r = \int_{r_{\min}}^{R} 2\theta_\rho(r) f(r)\,\mathrm{d}r \tag{3}
$$

Note that $\theta_\rho(r)$ is only well-defined if $r \geq \rho$. For $\ell \geq 1$, this is not an issue as $r_{\min} \geq \rho$ by Lemma 4. However, the case that $S = H$ is the top-level separator needs special treatment. In this case, we partition S into the disk D_ρ of radius ρ and the subsets S' and S'' of H whose points have radius at least ρ and lie to the right and left of the origin, respectively. Due to symmetry, $\mu(S') = \mu(S'')$. Thus, $\mu(S) = \mu(D_\rho) + 2\mu(S')$. Gugelmann et al. [18] showed that $\mu(D_\rho) = e^{-\alpha(R-\rho)}(1 + o(1))$. Moreover, for $\mu(S')$ we again obtain Equation (3) with $r_{\min} = \rho$. We thus obtain the following lemma by bounding $\mu(S')$ from above. Note that we require ρ to be linear in R here.

▶ **Lemma 6.** *Let S be a separator on level ℓ of the hyperdisk decomposition of D_R with hyperdisks of radius $\rho \in \Omega(R)$. Then the following holds.*

$$\mu(S) = \begin{cases} O\left(e^{\alpha\rho-\alpha R}\right) \cdot \left(2^{1-\alpha}\right)^{-\ell} & \text{for } \alpha < 1 \\ O\left(e^{\rho-R} \cdot R\right) & \text{for } \alpha = 1 \\ O\left(e^{\rho-R}\right) & \text{for } \alpha > 1 \end{cases}$$

Proof. For the special case $\ell = 0$, $\mu(D_\rho) = O(e^{\alpha\rho-\alpha R})$ is dominated by the claimed bounds for all α. Thus, for all cases, it remains to prove the bounds for $\mu(S')$. Starting with Equation (3) and using that $\arcsin(x) \leq x \cdot \pi/2$ (for $x \geq 0$), we get the following.

$$\begin{aligned} \int_{r_{\min}}^{R} 2\theta_\rho(r) f(r)\, \mathrm{d}r &\leq \int_{r_{\min}}^{R} 2 \cdot \frac{\pi}{2} \cdot \frac{\sinh(\rho)}{\sinh(r)} \cdot \frac{\alpha}{2\pi} \cdot \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1}\, \mathrm{d}r \\ &= \frac{\alpha}{2} \cdot \frac{\sinh(\rho)}{\cosh(\alpha R) - 1} \cdot \int_{r_{\min}}^{R} \frac{\sinh(\alpha r)}{\sinh(r)}\, \mathrm{d}r \\ &= \frac{\alpha}{2} \cdot \frac{\sinh(\rho)}{\cosh(\alpha R) - 1} \cdot \int_{r_{\min}}^{R} \frac{e^{\alpha r} - e^{-\alpha r}}{e^r - e^{-r}}\, \mathrm{d}r \\ &= \frac{\alpha}{2} \cdot \frac{\sinh(\rho)}{\cosh(\alpha R) - 1} \cdot \int_{r_{\min}}^{R} \frac{e^r}{e^r - e^{-r}} \cdot \frac{e^{\alpha r} - e^{-\alpha r}}{e^r}\, \mathrm{d}r \\ &\leq \frac{\alpha}{2} \cdot \frac{\sinh(\rho)}{\cosh(\alpha R) - 1} \cdot \frac{e^\rho}{e^\rho - e^{-\rho}} \cdot \int_{r_{\min}}^{R} \frac{e^{\alpha r}}{e^r}\, \mathrm{d}r \\ &= O\left(e^{\rho - \alpha R}\right) \cdot \int_{r_{\min}}^{R} e^{(\alpha-1)r}\, \mathrm{d}r \end{aligned}$$

The last inequality follows from the facts that $\frac{e^r}{e^r - e^{-r}}$ is monotonically decreasing (and thus maximal for $r = r_{\min} \geq \rho$) and that $e^{-\alpha r}$ is positive. In case $\alpha = 1$, the integral equals to $R - r_{\min} \leq R$, yielding $\mu(S') = O(e^{\rho-R} \cdot R)$. Otherwise, we have the following.

$$\int_{r_{\min}}^{R} \frac{e^{\alpha r}}{e^r}\, \mathrm{d}r = \left[\frac{e^{(\alpha-1)r}}{\alpha - 1}\right]_{r_{\min}}^{R} = \frac{e^{(\alpha-1)R} - e^{(\alpha-1)r_{\min}}}{\alpha - 1}$$

If $\alpha > 1$, the integral is dominated by $e^{(\alpha-1)R}$, yielding $\mu(S') = O(e^{\rho-\alpha R} \cdot e^{\alpha R - R}) = O(e^{\rho-R})$. If $\alpha < 1$, the integral is dominated by $e^{(\alpha-1)r_{\min}}$. Using the bound for $r_{\min}$ given by Lemma 4, we obtain the following.

$$\begin{aligned} \mu(S') &= O\left(e^{\rho-\alpha R}\right) \cdot e^{(\alpha-1)r_{\min}} \\ &= O\left(e^{\rho-\alpha R}\right) \cdot e^{(\alpha-1)\rho} \cdot \left(1 - e^{-2\rho}\right)^{\alpha-1} \cdot \left(2^{2-\ell} - 2^{2-2\ell}\right)^{1-\alpha} \end{aligned}$$

In this product, the first two factors simplify to $O(e^{\alpha\rho-\alpha R})$. The third factor tends to 1 for increasing ρ (and $\rho \in \Omega(R)$). The fourth factor can be written as $2^{-\ell(1-\alpha)}(2^2 - 2^{2-\ell})^{1-\alpha}$. As $(2^2 - 2^{2-\ell})^{1-\alpha}$ is bounded by the constant $4^{1-\alpha}$, we obtain the claimed bound. ◀

3.2 Decomposing Hyperbolic Random Graphs

Recall that one obtains a hyperbolic random graph G by randomly placing n vertices in the disk D of radius $R = 2\log n + C$ according to the probability measure μ and connecting two vertices if and only if their distance is less than R. For a subset $S \subseteq D$, let $G[S]$ be the subgraph of G induced by vertices in S.

Let $(T, \mathcal{S})$ be the hyperdisk decomposition of D with hyperdisks of radius $\rho = R/2$. Let further k be a node of T with children i and j. By Observation 2, the diameter of the separator S_k (separating U_i from U_j) is at least $2\rho = R$. This implies that no vertex in U_i is connected to a vertex in U_j. Thus, the graph $G[U_k]$ is separated by the vertices in S_k into the subgraphs $G[U_i]$ and $G[U_j]$. Hence, the hyperdisk decomposition of D translates into a separator hierarchy of the graph G. We also call this separator hierarchy the *hyperdisk decomposition* of G. The properties of the separators in D directly translate to the separators in G, i.e., we can expect the separators to be balanced (Observation 3) and small (Lemma 6).

▶ **Theorem 7.** *The hyperdisk decomposition of a hyperbolic random graph is expected to be balanced and for a separator S on level ℓ, the following holds.*

$$\mathbb{E}\left[|S|\right] = \begin{cases} O\left(n^{1-\alpha}\right) \cdot \left(2^{1-\alpha}\right)^{-\ell} & \text{for } \alpha < 1 \\ O\left(\log n\right) & \text{for } \alpha = 1 \\ O\left(1\right) & \text{for } \alpha > 1 \end{cases}$$

4 The Treewidth of Hyperbolic Random Graphs

The treewidth of a graph G is closely related to the size of separators in G. If G has treewidth k, it is known to have a balanced separator of size $k + 1$. This follows from the fact that a tree (e.g., the decomposition tree of width k) has a (weighted) balanced separator of size 1. Conversely, if G can be recursively decomposed by small balanced separators, i.e., if it has a balanced separator hierarchy with small separators, its treewidth is also small.

The following statement is easy to prove and for example used to show that planar graphs have treewidth $\sqrt{n}$ based on the planar separator theorem.

▶ **Lemma 8.** *Let $(T, \mathcal{S})$ be a separator hierarchy of G. For each node i of T, let X_i be the union of S_i and all separators S_j for which j is an ancestor of i in T. Then $(T, \mathcal{X} = \{X_1, \dots, X_t\})$ is a tree decomposition of G.*

Proof. We have to show that for each vertex $v \in V$ the bags containing v form a subtree of T and that for each edge $uv \in E$, there exists a bag containing u and v. For the former, let v be a vertex and let i be the unique node of T such that $v \in S_i$. Then $v \in X_j$ if and only if $j = i$ or j is an descendant of i. The node i together with its descendants clearly is a subtree of T. For the second condition, let $uv \in E$, let $u \in S_i$ and $v \in S_j$. Due to their edge, u and v cannot be separated, which implies that i is an ancestor of j or vice versa. Without loss of generality, we assume i is an ancestor of j. Then X_j includes S_j and S_i, which proves the claim. ◀

Using the properties of the hyperdisk decomposition as stated in Lemma 5 and Theorem 7 together with Lemma 8, we obtain a tree decomposition with upper bounds on the expected size of each bag. Applying a Chernoff bound (Theorem 1) leads to the following theorem.

▶ **Theorem 9.** *For the treewidth* $\mathrm{tw}(G)$ *of a hyperbolic random graph* G*, the following holds with high probability.*

$$\mathrm{tw}(G) = \begin{cases} \Theta\left(n^{1-\alpha}\right) & \textit{for } \alpha < 1 \\ O\left(\log^2 n\right) & \textit{for } \alpha = 1 \\ O\left(\log n\right) & \textit{for } \alpha > 1 \end{cases}$$

Note that the theorem states a matching lower bound if $\alpha < 1$. It follows from the fact that hyperbolic random graphs have clique number $\Theta(n^{1-\alpha})$ if $\alpha < 1$ [14]. For $\alpha \geq 1$, the clique number is $\Theta(\log n/ \log\log n)$ [14]. Thus, our upper bounds for $\alpha = 1$ and $\alpha > 1$ differ from this lower bound by factors $\log n \log\log n$ and $\log\log n$, respectively.

5 Applications

Our results from the previous sections have several algorithmic implications. In particular, the logarithmic treewidth for $\alpha > 1$ leads to efficient algorithms for numerous NP-hard problems, e.g., Vertex Cover, Independent Set, Dominating Set, Odd Cycle Traversal, and Max Cut [9]. We note that the size of the largest connected component in a hyperbolic random graph is whp polynomial even for $\alpha > 1$ as the maximum degree [18] is a lower bound for the size of the largest component. Thus, for $\alpha > 1$, the treewidth is not only logarithmic in the size of the whole (potentially disconnected) hyperbolic random graph but also in the size of the largest component.

For $\alpha < 1$, the separators are larger and thus algorithmic applications are less obvious. Moreover, there exists a *giant component* [3], i.e., a connected component of linear size. In the following sections, we present several algorithmic applications for the case that G is the giant component of a hyperbolic random graph with $\alpha < 1$. We note that all results still hold when considering the whole hyperbolic random graph instead of only the giant component (in fact, some arguments actually get simpler).

We give an approximation scheme for Independent Set (Section 5.1) and a fast algorithm for computing maximum matchings (Section 5.2). Both results assume that the geometry of the hyperbolic random graph is known (which is a strong but not completely unreasonable assumption [4]). In Section 5.3 we give applications that do not rely on knowing the geometry.

5.1 An Approximation Scheme for Independent Set

As an independent set forms a clique in the complement graph and vice versa, it is NP-hard to approximate Independent Set with approximation ratio better than $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$ [30]. However, based on the planar separator theorem (stating that planar graphs have balanced separators of size $O(\sqrt{n})$), Lipton and Tarjan [25] showed that Independent Set on planar graphs has a *PTAS (polynomial-time approximation scheme)*, i.e., for every constant $\varepsilon > 0$, it admits an efficient approximation algorithm with approximation ratio $1 - \varepsilon$. We adapt their approach to show that there is an approximation scheme for Independent Set if the input is a hyperbolic random graph given in its geometric representation.

The PTAS for planar graphs is based on two facts. First, planar graphs have a balanced separator hierarchy with separators of size $O(\sqrt{n})$. Second, planar graphs have independent sets of linear size. The latter follows directly from the fact that planar graphs have bounded chromatic number. For hyperbolic random graphs, this is not true, as they include comparatively large cliques [14]. However, as the degree sequence follows a power law, we find a subgraph of linear size whose vertices have bounded degree. This subgraph can then be colored with a constant number of colors which implies a large independent set. To obtain the following lemma, we have to apply this argument to the giant component of a hyperbolic random graph.

▶ **Lemma 10.** *The giant component of a hyperbolic random graph has whp independent sets of linear size.*

Following the approach of Lipton and Tarjan [25], we prove the following lemma. The rough idea is to choose V' to be the union of all separators of the hyperdisk decomposition with level at most $\lfloor \log_2(n/k) \rfloor$.

▶ **Lemma 11.** *Let $G = (V, E)$ be a hyperbolic random graph with $\alpha < 1$ and let $k \in \mathbb{N}$. Then there is a vertex set $V' \subseteq V$ of expected size $O(n)/k^\alpha$ such that the connected components of $G - V'$ have expected size at most k.*

Proof. Consider the hyperdisk decomposition $(T, \mathcal{S})$ of G and let i be a node on level ℓ. Recall that U_i denotes the set of vertices such that $G[U_i]$ is the subgraph whose separators are represented by the descendants of i in T. Due to the fact that the hyperdisk decomposition is balanced (Theorem 7), the expected size of U_i is at most $n2^{-\ell}$. We choose V' to be the union of all separators with level at most $\ell_{\max} = \lfloor \log_2(n/k) \rfloor$. Each connected component of $G - V'$ is then a subgraph of $G[U_i]$ for a vertex i with level $\ell_{\max} + 1$. Thus, the expected size of these components is at most k.

It remains to bound $\mathbb{E}[|V'|]$. Recall from Theorem 7 that a separator on level ℓ has expected size $O(n^{1-\alpha}) \cdot 2^{-\ell(1-\alpha)}$. Moreover, as T is a complete binary tree, there are 2^ℓ separators on level ℓ. Thus, the total size of separators on level ℓ is $O(n^{1-\alpha}) \cdot 2^{-\ell(1-\alpha)} 2^\ell = O(n^{1-\alpha}) \cdot 2^{\alpha\ell}$. We thus obtain the following for the expected size of V'.

$$\begin{aligned} \mathbb{E}\left[|V'|\right] &= \sum_{\ell=0}^{\ell_{\max}} O\left(n^{1-\alpha}\right) \cdot 2^{\alpha\ell} \\ &= O\left(n^{1-\alpha}\right) \cdot \sum_{\ell=0}^{\ell_{\max}} 2^{\alpha\ell} \end{aligned}$$

To conclude the proof, it remains to proof that the sum equals $(n/k)^\alpha \cdot O(1)$, which follows from the following calculation and from the fact that the geometric series converges.

$$\begin{aligned} \sum_{\ell=0}^{\ell_{\max}} 2^{\alpha\ell} &= \sum_{i=0}^{\ell_{\max}} 2^{\alpha(\ell_{\max}-i)} \\ &= 2^{\alpha\ell_{\max}} \sum_{i=0}^{\ell_{\max}} 2^{-\alpha i} \\ &= \left(n/k\right)^\alpha \cdot O(1) \end{aligned}$$ ◀

To approximate Independent Set, one can apply Lemma 11 to the given hyperbolic random graph G and then compute for each of the resulting connected components an optimal independent set in expected time $O(k2^k)$. For all $O(n/k)$ connected components, this takes expected $O(n2^k)$ time. The union of these independent sets is an independent set of G. Let I be this independent set, restricted to the giant component H of G. Comparing this to the size of an optimal independent set $I^\star$ of H, we miss at most the vertices in the separator V', thus $\mathbb{E}[|I^\star| - |I|] = O(n)/k^\alpha$. As $|I^\star|$ is linear with high probability (Lemma 10), dividing by $|I^\star|$ yields $\mathbb{E}[1 - |I|/|I^\star|] = k^{-\alpha} \cdot O(1)$ and thus $\mathbb{E}[|I|/|I^\star|] = 1 - k^{-\alpha} \cdot O(1)$.

This directly implies the claimed approximation scheme: for every given $\varepsilon > 0$, one can choose the k such that the expected approximation ratio is $1 - \varepsilon$. As the k we have to chose does not depend on n, the resulting running time is polynomial in n (but exponential in $1/\varepsilon$). Additionally applying concentration bounds if $k = \log n$ yields the following theorem.

▶ **Theorem 12.** *For the giant components of hyperbolic random graphs given in their geometric representation,* Independent Set *can be approximated in expected $O(n2^k)$ time with expected approximation ratio $1 - k^{-\alpha} \cdot O(1)$. If $k = \log n$, the algorithm runs whp in polynomial time and the bound on the approximation ratio holds whp.*

Proof. It remains to consider the case $k = \log n$. We apply Lemma 11 with $k = \log n$ leading to connected components of expected size $\log n$. Using a Chernoff bound (Theorem 1) shows that the size of the connected components is whp at most $c \log n$ for a constant c. Thus, the algorithm described above has whp polynomial run-time.

Concerning the approximation ratio, note that the separator V' has expected size $O(n)/\log^\alpha n$. Thus, there are constants c_1 and $n_{\min}$ such that the $\mathbb{E}[|V'|] \le c_1 n/\log^\alpha n$ if $n > n_{\min}$. As we can brute-force smaller instances, we can assume the latter assumption to be true. Applying a Chernoff bound (Theorem 1) we get that V' includes whp at most $c_2 n/\log^\alpha n$ vertices for another constant c_2. As before let I be the independent set we computed and let $I^\star$ be an optimal independent set of the giant component. Then, $|I^\star| - |I| \le c_2 n/\log^\alpha n$ holds whp. As whp $|I^\star| \ge c_3 n$ (at least for sufficiently large instances; see Lemma 10), the calculations from above show that $|I|/|I^\star| \ge 1 - c/\log^\alpha n$ holds whp for $c = c_2/c_3$. ◀

5.2 Computing Matchings in $O(n^{2-\alpha})$ Time

Lipton and Tarjan [25] also gave $O(n^{3/2})$ and $O(n^{3/2} \log n)$ algorithms that compute matchings of maximum cardinality and matchings of maximum weight, respectively, in a planar graph. Their algorithm uses the following divide-and-conquer strategy. Find a separator, recursively compute maximum matchings for both subgraphs, and finally combine these solutions by iteratively adding the vertices of the separator while maintaining a maximum matching. The latter can be done by finding a single augmenting path, which can be done in $O(m)$ and $O(m \log n)$ time for unweighted and weighted graphs, respectively (m is the number of edges).

To obtain the following theorem, we apply this divide-and-conquer strategy to hyperbolic random graphs, show that m is actually linear in n (even for the subgraphs for which we compute the augmenting paths), and apply concentration bounds.

▶ **Theorem 13.** *Let G be the giant component of a hyperbolic random graph given in its geometric representation. Whp, a maximum matching in G can be computed in $O(n^{2-\alpha})$ and $O(n^{2-\alpha} \log n)$ time if G is unweighted and weighted, respectively.*

5.3 In Case the Geometry is Unknown

The two previous applications relied on the fact that we know the geometric representation of the given hyperbolic random graph. However, even if the geometry is unknown, we can still benefit from the knowledge that hyperbolic random graphs have low treewidth by using an algorithm of Fomin et al. [13]. It takes a graph G and an integer k as input and either decides that $\mathrm{tw}(G) > k$ or returns a tree decomposition of width k^2. It runs in $O(k^7 n \log n)$

time. By spending an additional factor of $\log(\mathrm{tw}(G))$ we can actually compute the smallest k for which the algorithm succeeds to compute a tree decomposition.

Given a tree decomposition of width k, Fomin et al. [13] show (among other algorithms) how to compute a matching with maximum weight and a maximum vertex flow in a directed graph in $O(k^4 n \log^2 n)$ and $O(k^2 n \log n)$ time, respectively. For sufficiently large α, this leads to algorithms solving these problems faster than the best known algorithms for general graphs. E.g., for $\alpha = 31/32$, the treewidth of G is $O(n^{1/32})$ with high probability (Theorem 9). Thus, we get a tree decomposition of width $O(n^{1/16})$ in $O(n^{7/32} \cdot n \log^2 n)$ time. Computing a matching of maximum weight then takes $O(n^{1/4} \cdot n \log^2 n)$. Thus, the overall running time is $O(n^{5/4} \log^2 n)$. Also note that this approach leads to almost linear (i.e., linear up to a polylogarithmic factor) run-times if $\alpha \geq 1$.

6 Binomial Hyperbolic Random Graphs

So far, we considered the so-called *threshold* model of hyperbolic random graphs where vertices are connected if and only if they have distance at most R. A more realistic (but technically more difficult) model is the *binomial* model, in which longer edges and shorter non-edges are allowed with a certain probability. More precisely, two vertices with distance d are connected with the following probability $p(d)$ depending on the constant t (usually $0 < t < 1$). Note that we obtain the threshold model for $t \to 0$.

$$p(d) = \left(1 + e^{\frac{1}{2t}(d-R)}\right)^{-1}$$

As before, we start with a hyperdisk decomposition $(T, \mathcal{S})$ of the disk D and then transfer it to a separator hierarchy of a hyperbolic random graph. In the threshold model, separators in $(T, \mathcal{S})$ translated to separators in the graph if $\rho \geq R/2$. This is not true in the binomial model as vertices with distance greater than R are still connected with a certain probability.

However, we obtain separators as follows. Let k be a node of T with children i and j, i.e., S_k separates U_k into U_i and U_j. An edge of the graph G is *critical* with respect to S_k if it connects a vertex located in U_i with a vertex in U_j. As before, let $G[U_k]$ be the graph induced by vertices in U_k. Then the vertices located in S_k together with the endvertices of critical edges separate $G[U_k]$ into (subgraphs of) $G[U_i]$ and $G[U_j]$. In this way, we again obtain a separator hierarchy for G, which we call the *extended hyperdisk decomposition*.

To bound the size of the resulting separators, we have to bound the number of vertices in S_k and the number of critical edges. For the former, we can use the previous results (in particular Lemma 6). For the latter, note that the expected number of vertices in U_i is at most $n2^{-\ell}$ if i has level ℓ (as the hyperdisk decomposition is balanced). As the same holds for U_j, there are only $(n2^{-\ell})^2$ vertex pairs that potentially form critical edges, each with a probability of at most $p(2\rho)$ (as their distance is at least 2ρ). Thus, the expected number of critical edges is at most $n^2 p(2\rho) 2^{-2\ell}$. Plugging a carefully chosen value for ρ into this formula as well as into the formula given by Lemma 6 leads to the following theorem.

▶ **Theorem 14.** *Let G be a binomial hyperbolic random graph with $\alpha < 1$ and consider its extended hyperdisk decomposition with hyperdisks of radius $\rho = \frac{2\alpha t + t + 1}{2\alpha t + 2} R$. For a separator S on level ℓ, the following holds.*

$$\mathbb{E}(|S|) = O\left(n^{2 - \frac{\alpha+1}{\alpha t + 1}}\right) \cdot \left(2^{1-\alpha}\right)^{-\ell}$$

Proof. As mentioned above, the number of critical edges is bounded by the following term.

$$n^2 p(2\rho) 2^{-2\ell} = n^2 p\left(\frac{2\alpha t + t + 1}{\alpha t + 1} R\right) 2^{-2\ell}$$

$$
\begin{aligned}
&= n^2 \left(1 + e^{\frac{1}{2t}\left(\frac{2\alpha t+t+1}{\alpha t+1}R - R\right)}\right)^{-1} 2^{-2\ell} \\
&= n^2 \left(1 + e^{\frac{1}{2}\frac{\alpha+1}{\alpha t+1}R}\right)^{-1} 2^{-2\ell} \\
&= n^2 \left(1 + e^{\frac{1}{2}\frac{\alpha+1}{\alpha t+1}(2\log n + C)}\right)^{-1} 2^{-2\ell} \\
&\le n^2 \left(1 + e^{\frac{\alpha+1}{\alpha t+1}(\log n + C/2)}\right)^{-1} 2^{-(1-\alpha)\ell} \\
&= O\left(n^{2-\frac{\alpha+1}{\alpha t+1}}\right) \cdot \left(2^{1-\alpha}\right)^{-\ell}
\end{aligned}
$$

For the second part, we go one step back and assume that S is a separator in the hyperdisk decomposition of the disk D_R in the hyperbolic plane (instead of a separator in graph). By Lemma 6, we get the following bound on the measure of S.

$$
\begin{aligned}
\mu(S) &= O\left(e^{\alpha\rho - \alpha R}\right)\left(2^{1-\alpha}\right)^{-\ell} \\
&= O\left(e^{\alpha\frac{2\alpha t+t+1}{2\alpha t+2}R - \alpha R}\right)\left(2^{1-\alpha}\right)^{-\ell} \\
&= O\left(e^{\alpha\frac{t-1}{2\alpha t+2}R}\right)\left(2^{1-\alpha}\right)^{-\ell} \\
&= O\left(e^{\alpha\frac{t-1}{2\alpha t+2}(2\log n + C)}\right)\left(2^{1-\alpha}\right)^{-\ell} \\
&= O\left(n^{\frac{\alpha t-\alpha}{\alpha t+1}}\right)\left(2^{1-\alpha}\right)^{-\ell} \\
&= O\left(n^{\frac{\alpha t+1-\alpha-1}{\alpha t+1}}\right)\left(2^{1-\alpha}\right)^{-\ell} \\
&= O\left(n^{1-\frac{\alpha+1}{\alpha t+1}}\right)\left(2^{1-\alpha}\right)^{-\ell}
\end{aligned}
$$

Multiplying with n (as we have n vertices) leads to the claimed bound. ◀

Note that this bound coincides with our result for the threshold model when $t \to 0$. Moreover, for $t \in (0, 1)$, we obtain separators of sublinear size. As for the threshold model, we can use Lemma 8 to obtain bounds for the treewidth (compare Section 4).

7 Conclusion

We have shown that hyperbolic random graphs have small separators, as well as a small treewidth (with a phase transition from polynomial to logarithmic at $\beta = 3$). This stands in stark contrast to other popular random graph models like Erdős-Rényi [12] or Barabási-Albert [2] that have linear separators [17]. Beyond providing new insights on the structural properties of hyperbolic random graphs, our results give rise to several algorithmic applications.

To judge the practical merit of these algorithms, an interesting next step is therefore to compare separators on real graphs with predictions made by the various models. It is, however, a challenge to compute small separators in a given massive graph. Depending on the precise problem formulation, this is likely to be an NP-complete problem.

A more theoretical open question is whether our results are tight or can be improved to achieve even smaller separators. Of interest is especially the binomial model, since it allows for long range edges in the graph; and there exists no lower bound on the treewidth in the current literature.

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