Voronoi Choice Games*

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Abstract

We study novel variations of Voronoi games and associated random processes that we call *Voronoi choice games*. These games provide a rich framework for studying questions regarding the power of small numbers of choices in multi-player, competitive scenarios, and they further lead to many interesting, non-trivial random processes that appear worthy of study.

As an example of the type of problem we study, suppose a group of n miners (or players) are staking land claims through the following process: each miner has m associated points independently and uniformly distributed on an underlying space (such as the unit circle, the unit square, or the unit torus), so the kth miner will have associated points $p_{k1}, p_{k2}, \ldots, p_{km}$. We generally here think of m as being a small constant, such as 2. Each miner chooses one of these points as the base point for their claim. Each miner obtains mining rights for the area of the square that is closest to their chosen base; that is, they obtain the Voronoi cell corresponding to their chosen point in the Voronoi diagram of the n chosen points. Each player's goal is simply to maximize the amount of land under their control. What can we say about the players' strategy and the equilibria of such games?

In our main result, we derive bounds on the expected number of pure Nash equilibria for a variation of the 1-dimensional game on the circle where a player owns the arc starting from their point and moving clockwise to the next point. This result uses interesting properties of random arc lengths on circles, and demonstrates the challenges in analyzing these kinds of problems. We also provide several other related results. In particular, for the 1-dimensional game on the circle, we show that a pure Nash equilibrium always exists when each player owns the part of the circle nearest to their point, but it is NP-hard to determine whether a pure Nash equilibrium exists in the variant when each player owns the arc starting from their point clockwise to the next point. This last result, in part, motivates our examination of the random setting.

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1 Introduction

Consider the following prototypical problem: a group of miners are staking land claims. The kth miner – or player – has m associated points $p_{k1}, p_{k2}, \ldots, p_{km}$ in the unit torus (which is the unit square with wraparound at the boundaries, providing symmetry). Each miner via some process will choose exactly one of their m points as the base for their claim. The resulting n points yield a Voronoi diagram, and each miner obtains their corresponding Voronoi cell. Each player's goal is simply to maximize the amount of land under their control. We wish to study player behavior in this and similar games, focusing on equilibria.

As another application, political candidates can often be mapped according to their political views into a small-dimensional space; e.g., American candidates are often viewed as being points in a two-dimensional space, measuring how liberal/conservative they are on economic issues in one dimension and social issues on the other. Suppose parties must choose a candidate simultaneously, and their probability of winning is increasing in the area of the political space closest to their point. Again, the goal in this case is to maximize the corresponding area in a Voronoi diagram.

There are numerous variations one can construct from this setting. Most naturally, if the players are (lazy) security guards instead of miners, who have to patrol the area closest to their chosen base, their goal might be to minimize the area under their purview. Other alternatives stem from variations such as whether player choices are simultaneous or sequential, how the points for players are chosen, the underlying metric space, the type of equilibrium sought, and the utility function used to evaluate the final outcome.

However, the variations share the following fundamental features. There are n players, with the kth player having m_k associated points in some metric space. (We will focus on $m_k = m$ for a fixed m for all players.) Each player will have to choose to adopt one of their available points. A Voronoi diagram is then constructed, and each player is then associated with the corresponding area in the diagram. We refer to this general setting as *Voronoi choice games*. We discuss below how Voronoi choice games differ from similar recent work, but the key point is in the problems we study different players have different available choices; this asymmetry creates new problems and requires distinct methods.

We are particularly interested in the setting where each player's points are chosen uniformly at random from the underlying space. While uniform random points are not motivated by practice, the framework leads to an interesting and, from the standpoint of probabilistic analysis and geometry, very natural class of games. Our work suggests many potential connections, to work on Voronoi diagrams for random point sets, and to work on balanced allocations (or "the power of two choices"), where choice is used to improve load balancing. Moreover, looking at the setting of uniform random points gives us the opportunity to understand the nature of these games at a high level; specifically, do most instances have no pure Nash equilibrium, or could they have exponentially many possible pure Nash equilibria?

In general, however, we find that results for these types of problems seem very challenging. In our main result, we limit ourselves to the setting where each player has m associated points chosen uniformly at random from the unit circle, and each player owns the arc starting from their point clockwise to the next point – that is, the distance is unidirectional around the circle. We derive bounds on the expected number of pure Nash equilibria. Even in this simple setting, our result is quite technical, requiring a careful analysis based on interesting properties of distributions of random arcs on a circle. This appears, however, to be the "easiest" interesting version of the problem; currently, higher-dimensional Voronoi diagrams are beyond our reach. However, our work suggests that further results are likely to involve interesting mathematics.

The random case of this specific version of the problem is also motivated by the following results. We show it is NP-hard to determine whether a pure Nash equilibrium exists when a player owns the arc starting from their point clockwise to the next point for $m \geq 4$, nearly resolving the worst case. Further, for the different setting when a player owns an arc of the circle corresponding to the standard Voronoi diagrams, that is a player own all points nearest to their point, we show a Nash equilibrium always exists (as long as all the possible choices for the players are distinct).

While other similar Voronoi game models have been introduced previously, our primary novelty is to introduce this natural type of asymmetric "choice" into these types of games. We believe this addition provides a rich framework with many interesting combinatorial, geometric, and game theoretic problems, as we describe throughout the paper. As such, we leave many natural open questions.

1.1 Related Work

The classical foundations for problems of this type can be found in the work by Hotelling [11], who studied the setting of two vendors who had to determine where to place their businesses along a line, corresponding to the main street in a town, with the assumption of uniformly distributed customers who would walk to the nearer vendor. Hotelling games have been considered for example in work on regret minimization and the price of anarchy, where the model studied players choosing points on a general graph instead of on the line as in the original model [4]. Recent work has also shown that for a Hotelling game on a given graph, once there are sufficiently many players a pure Nash equilibrium always exists [8]. A useful survey on economic location-based models is provided by Gabszewicz and Thisse [9].

Other variations of Voronoi games have appeared in the literature. More recent work refers to these generally as *competitive location games*; see for example [7, 14, 18, 13], which discuss Voronoi games on graphs, for additional references.

Our setting appears different from previous work, in that it focuses on players with *limited* sets of choices that *vary* among the players. Our starting point was aiming to build connections between Voronoi games and random processes based on "the power of two choices" [3, 15, 6, 1]. While in our games, each player has a limited (typically constant) set of distinct points to choose from, in previous work generally *all players* could choose from *any point* in the universe of possible choices. In economic terms, in relation to the Hotelling model, our work models that different businesses may have available a limited number of differing locations where they may establish their business. For example, businesses may have optioned the right to set up a franchise at specific locations in advance, and must then choose which location to actually build. While they could know the options available to other competing franchises, they may have to decide where to build without knowing the choices made by competitors. In other situations, it may be possible for franchises to move (at some cost) to an alternative location. We emphasize that our model is very different than previously studied symmetric versions of the game; we do not recover earlier results, and earlier results do not appear to apply once asymmetry is introduced.

1.2 Models

Before beginning, we explain the general class of games we are interested in. We refer to the following as the *k-D Simultaneous Voronoi Game*:

- Each of the n players has m associated points from the k-dimensional unit torus $[0,1]^k$. We assume that all players know about all of the possible points that can be chosen by every player (it is a game of complete information).
- The n players must simultaneously choose one of their m associated points.
- A Voronoi diagram is constructed for the n chosen points, and each player receives utility equal to the volume of its point's Voronoi cell in the maximization variation of the game. (In the minimization version, the utility could be the negation of the corresponding volume.)

The easiest version to think about is the 1-D version; each player chooses from m points on the unit circle, and after their choice they own an arc of the circle corresponding to all points closest to their chosen point. If each player tries to maximize their arc length, then the utility of a player is the length of their arc. (Or, if each player tries to minimize their arc length, the negation of the arc length is the utility.) On the unit circle, there is another variant that we refer to as the *One Way 1-D Simultaneous Voronoi Game*, in which a player owns the arc starting from their point and continuing in a clockwise direction until the next chosen point. Such a variation is quite natural in one dimension; it corresponds to assigning a "direction" to the unit circle. This variation is chiefly motivated by our connections to the power-of-two choices. In particular, it resembles the distributed hashing scheme of [6] in which peers correspond to points on a circle and keys are mapped to the closest peer in one direction along the circle.

Our contributions include highlighting differences between the 1-D problem and the One Way 1-D problem, showing that in this case a small difference in the model subtlety leads to large differences in the behavior with respect to equilibria. Indeed, as we explain, we believe the One Way 1-D problem potentially offers more insight into the behavior of the k-D Simultaneous Voronoi Game for $k \geq 2$ with respect to pure Nash equilibria.

We focus on analyzing the equilibria of these games. The most common equilibrium to study is the Nash equilibrium [16], in which each player has a random distribution on strategies such that no player can improve their expected utility by changing their distribution. While Nash's results imply the Voronoi games above all have Nash equilibria, we do not determine the complexity of finding Nash equilibria for these games; this is left as an open question. We here focus on pure Nash equilibrium. A pure Nash equilibrium is a Nash equilibrium in which each player's distribution has a support of size one. In other words, each player picks a single strategy to play and, given the other players' strategies, no player can improve their utility by choosing a different strategy. Unlike the Nash and correlated equilibria, a pure Nash equilibrium is not guaranteed to exist.

Pure Nash equilibria can be viewed as a setting where each player can choose to switch to any of their adopted points at any time. The question is then what are the stable states, where no player individually has the incentive to switch their adopted point. These stable states correspond to pure Nash equilibria, and may not even exist. A natural question is whether simple local dynamics – such as myopic best response, where at each time step some subset of players decides whether or not to switch the point it has adopted – reach a stable state quickly. To motivate our study of the random case, we examine the computational complexity of determining the existence of stable states in the 1-D Simultaneous Voronoi Game and One Way 1-D Simultaneous Voronoi Game in Section 2. For the former, we show (making use of known techniques) that a pure Nash equilibrium always exists; for the latter, we show that determining whether a pure Nash equilibrium exists is NP-complete.

In Section 3 we consider the existence of a pure Nash equilibrium for the Randomized One Way 1-D Simultaneous Voronoi Game, where each players possible choices for points are selected uniformly at random from the unit circle. Here we bound the expected number of pure Nash equilibria, through a careful analysis based on properties of distributions of random arcs on a circle.

We also note that we have some results for another type of equilibrium, known as the correlated equilibrium. Whereas Nash equilibria have the players independently choosing their strategies, a correlated equilibrium allows the players' random distributions to be correlated (for example, by an external party). The stability requirement is then that given knowledge only of the overall distribution of outcomes and their own randomly chosen strategy, a player cannot improve their expected utility by deviating from their given strategy distribution [2]. Since a Nash equilibrium is a special case of a correlated equilibrium, a correlated equilibrium for the above games must exist. We discuss the computational complexity of finding a correlated equilibrium for the k-D Simultaneous Voronoi Game in Section 4.

We provide additional results in the full paper [5], including an empirical investigation of the probability that myopic best response will find a stable state in the Randomized One Way 1-D Simultaneous Voronoi Game and Randomized 2-D Simultaneous Voronoi Game, and several related conjectures related to the Randomized One Way 1-D Simultaneous Voronoi Game.

2 Pure Nash Equilibria

In this section, we show a fundamental difference between the One Way 1-D Simultaneous Voronoi Game and the 1-D Simultaneous Voronoi Game. Recall that for these problems each of the n players has a choice of m points on the unit circle; all players simultaneously choose one of their m points. The utility for the One Way variation of given player is equal to the distance to the nearest chosen point clockwise from its chosen point, while for the standard variation the utility is the size of the Voronoi cell (in this case, an arc).

We show the standard Voronoi variation always has at least one pure Nash equilibrium (for any number of choices per player), while it is NP-hard to determine if the maximization version of the One Way variation has a pure Nash equilibrium. We also suggest the implications of these results for the higher dimensional setting.

2.1 Existence of Pure Nash Equilibria in the 1-D Simultaneous Voronoi Game

In the argument that follows we assume the choices of points are distinct. The analysis can be easily modified for the case where multiple players can choose the same point if ownership of that point is determined by a fixed preference order (and other players have zero utility). However, if players choosing the same point share utility, then the theorem does not hold, as shown in [14].

▶ **Theorem 1.** *A pure Nash equilibrium for the maximization and minimization versions of the 1-D Simultaneous Voronoi Game exists for any set of points.*

Proof. We follow an approach utilized in [7, Lemma 4]. We define a natural total ordering on multi-sets of numbers A and B. For any two such multi-sets A and B, if $|A| < |B|$ we have $A \succ B$. When $|A| = |B|$, we have $A \succ B$ if $\max A > \max B$. If $\max A = \max B$, then let A' be A with one copy of the value $\max A$ removed, and similarly for B'; then $A \succ B$ also when $\max A = \max B$ and $A' \succ B'$.

Now consider a collection of choices in the 1-D Simultaneous Voronoi Game, and let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be the corresponding arc lengths, starting from some chosen point

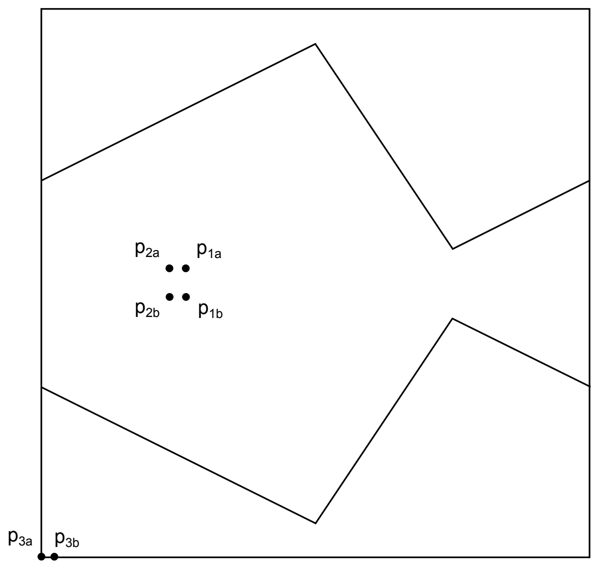

Figure 1 This is an example of a 2-D Simultaneous Voronoi Game in which no pure Nash equilibria exists. Player 1 has points $(p_{1a}, p_{1b}) = ((1/4 + \varepsilon^2, 1/2 + \varepsilon), (1/4 + \varepsilon^2, 1/2 - \varepsilon))$, player 2 has points $(p_{2a}, p_{2b}) = ((1/4 - \varepsilon^2, 1/2 + \varepsilon), (1/4 - \varepsilon^2, 1/2 - \varepsilon))$, and player 3 has points $(p_{3a}, p_{3b}) = ((0, 0), (\varepsilon, 0))$. The Voronoi diagram shown is constructed from the points $(1/4, 1/2)$ and $(0, 0)$.

and then in clockwise order around the circle, induced by the choice of points. Each player's payoff is given by a value of the form $(a_i + a_{i+1})/2$ for a suitable value of i. (One player's payoff is $(a_{n-1} + a_0)/2$.) Consider the maximization version of the game. If some player has a move that improves their utility, let them make that move. Without loss of generality, suppose this player's payoff was given by $(a_i + a_{i+1})/2$, and it moves somewhere on the arc with length a_j. Note this means $a_j > a_i + a_{i+1}$. We see that the arc lengths A' are those of A but with a_i, a_{i+1}, and a_j replaced by $a_i + a_{i+1}, x$, and y where $x + y = a_j$. Hence $A \succ A'$, so after a finite number of moves, this version of myopic best response converges to a pure Nash equilibrium.

The argument for the minimization variation is analogous. ◀

We note that we leave as an open question to determine a bound on the number of steps a myopic best response approach would take to reach a pure Nash equilibrium; in particular, we do not yet know if the approach of Theorem 1 yields a pure Nash equilibrium in a polynomial number of steps.

We might have hoped that the above technique could allow us to show that for the 2-D Simultaneous Voronoi Game (and higher dimensions) that a pure Nash equilibrium exists. Unfortunately, that is not the case. One can readily find choices of 2 points for each of 3 players where no pure Nash equilibrium exists for both the maximization and minimization version of the problem. We have generated many such examples randomly, computing the Voronoi diagrams for the eight resulting configurations. One example in two dimensions is depicted in Figure 1. In this example, player 1 has choices $((1/4+\varepsilon^2, 1/2+\varepsilon), (1/4+\varepsilon^2, 1/2-\varepsilon))$, player 2 has choices $((1/4 - \varepsilon^2, 1/2 + \varepsilon), (1/4 - \varepsilon^2, 1/2 - \varepsilon))$, and player 3 has choices $((0, 0), (\varepsilon, 0))$ for sufficiently small $\varepsilon > 0$. The idea here is that player 3's choice is irrelevant, and players 1 and 2 are dividing up the Voronoi cell owned by point $(1/4, 1/2)$ in the Voronoi diagram of the points $(1/4, 1/2)$ and $(0, 0)$. If both player 1 and 2 choose their first point, or both choose their second point, then they divide the cell with a vertical line, which due to the geometry of the cell favors player 1. However, if one of them chooses their first point and the other chooses their second point then they divide the cell with a roughly horizontal line, which gives each of them roughly half. Thus, there is no choice of points for which neither wants to deviate. This example applies equally in higher dimensions (by making higher dimensional coordinates all zero).

This example can be extended to show that for any number of players n, there are settings of points for the players so no pure Nash equilibrium exists, showing that this setting differs from previous work on symmetric Hotelling games on graphs, where it has been shown that when there are sufficiently many players a pure Nash equilibrium always exists [8]. Specifically, use the same example but for players $4, \ldots, n$ both of their choices will be in an epsilon-small neighborhood of $(0, 0)$. We note that such examples do not disprove the possibility that a pure Nash equilibrium exists with high probability if the points are chosen randomly.

Given that the 1-D Simultaneous Voronoi Game appears to have a very special structure (in terms of the existence of pure Nash equilibrium) that differs from k-D Simultaneous Voronoi Game for $k \geq 2$, it is natural to seek a 1-D variant that might shed more insight into the behavior in higher dimensions. This motivates us to look at the One Way 1-D Simultaneous Voronoi Game.

2.2 NP-Hardness of the One Way 1-D Simultaneous Voronoi Game

In contrast to the result of the previous subsection, we prove the NP-hardness of the maximization version of the One Way 1-D Simultaneous Voronoi Game whenever each player has $m \geq 4$ choices. We leave as an open question to find a reduction for $m = 2$ or 3, as well as for the minimization version.

▶ **Theorem 2** (For proof see [5]). *The problem of determining if a pure Nash equilibrium (PNE) exists in the maximization version of a One Way 1-D Simultaneous Voronoi Game is NP-hard for $m \geq 4$.*

We conjecture that determining if a pure Nash equilibrium exists in the k-D Simultaneous Voronoi Game is NP-hard for $k \geq 2$; however, we suspect that building the corresponding gadgets will prove technically challenging.

3 Random Voronoi Games

We now consider random variations of the Voronoi games we have considered, where each player's available choices are chosen uniformly at random from the underlying universe. While it is not clear such a model corresponds to any specific real-world scenario, such random problems are intrinsically interesting combinatorially and in relation to other similar studied problems. For example, in the context of load-balancing in distributed peer-to-peer systems, the authors of [6] study a model where one begins with a Voronoi diagram on N points (chosen uniformly at random from the universe, say the unit torus) corresponding to N servers. Then M agents sequentially enter; each is assigned k random points from the universe; and each agent chooses the one of its k points that lies in the Voronoi cell with the smallest number of agents, or *load* for that server. Other "power-of-choice" problems, such as the Achlioptas process [1], have spurred new understanding of phenomena such as explosive percolation.

Given our hardness results, a natural question is whether the One Way 1-D Simultaneous Voronoi Game has (or does not have) a pure Nash equilibrium with high probability in the random variation. While we have not proven this result, we have proven bounds on the expected number of pure Nash equilibria in the random setting that are interesting in their own right and nearly answer this question. In particular, our careful analysis builds on the interesting relationship between random arcs on a circle and weighted sums of exponentially distributed random variables.

The following is our main result.

▶ **Theorem 3.** *The expected number of PNE for the maximization version of the One Way 1-D Simultaneous Voronoi Game is at most m, and at least $0.19^{m-1}m$.*

Interestingly, our bounds depend on the number of choices m, not the number of players. Unfortunately, this means that we cannot use these expectation bounds directly to show that, for example, the probability of a PNE existing is exponentially small in n. But the bounds provide insight by showing that pure Nash equilibria are typically few in number in the random case.

In proving Theorem 3, we need the following well-known property of exponential random variables. Let $X_1, \ldots, X_n$ be i.i.d. exponential random variables. Let $S_n = \sum_{j=1}^{n} X_j$. Recall that S_n is said to have a *gamma distribution*, and we use facts about the gamma distribution later in our analysis. Similarly, S_{n-1}/S_n is said to have a *beta distribution*, and we use facts about beta distributions as well. For example,

▶ **Lemma 4.** $S_n, \frac{S_1}{S_2}, \frac{S_2}{S_3}, \ldots, \frac{S_{n-1}}{S_n}$ *are all mutually independent.*

See e.g. [10] for its discussion of exponential random variables. In particular, the X_i can be viewed as the gaps in the arrivals of a Poisson process; the $n-1$ arrivals before the last are uniformly distributed on the interval $[0, S_n]$, from which one can derive the lemma above.

Another important fact we use is the following:

▶ **Fact 5.** *We have* $\mathrm{E}\left[(S_j/S_{j+1})^t\right] = \prod_{i=0}^{t-1} \frac{j+i}{j+1+i} = \frac{j}{j+t}$, *since* $S_j/S_{j+1} \sim$ *Beta* $(j, 1)$.

We note that in some of our arguments, the ordering of the variables becomes reversed, and we consider sums of the form $\sum_{j=n-i+1}^{n} X_j$. Of course this has the same distribution as $\sum_{j=1}^{i} X_i$, and the corresponding version of Lemma 4 holds. Where convenient, we therefore refer to $\sum_{j=n-i+1}^{n} X_j = S_i$ where there is no ambiguity as to the desired meaning.

Proof of Theorem 3. We begin by using linearity of expectations to write the expected number of PNE in terms of the probability that each player choosing their first choice will yield a stable configuration.

$$\begin{aligned}\mathrm{E}[\#\ \text{PNE}] &= m^n \cdot \Pr[\text{first choices are stable}]\\ &= m^n \cdot \mathrm{E}[\Pr[\text{first choices are stable} \mid \text{position of first choices}]].\end{aligned}$$

Partition the circle into arcs according to the players' first choice points. Let A_i be the length of the ith smallest arc. As shown in [10], the A_i are distributed jointly as

$$A_i \sim \frac{1}{S_n} \sum_{j=n-i+1}^{n} \frac{X_j}{j}$$

where again the X_j are i.i.d. exponential random variables of mean 1 and $S_n = \sum_{j=1}^{n} X_j$. We say that an arc is stable if the player whose point starts the arc (going clockwise) does not wish to deviate to any of their other points. Given the position of the first choices, the probability each arc is stable depends only on the other choices available to the player that owns the arc, and hence the stability of the arcs are independent. Therefore

$$\begin{aligned}&\Pr[\text{first choices are stable} \mid \text{position of first choices}]\\&\quad= \prod_{i=1}^{n} \Pr[i\text{th smallest arc is stable} \mid \text{position of first choices}].\end{aligned}$$

If the arc is the ith smallest, then it will be stable if the other choices fall in the same arc, one of the $i-1$ smaller arcs, or the front A_i-length portion of the $(n-i)$ larger arcs – except in the latter two cases, we must take into account that if a choice falls immediately backward into the arc directly counterclockwise of the current arc, then the arc is not stable. We therefore have the following calculation:

$$\begin{aligned}&\Pr[i\text{th smallest arc is stable} \mid \text{position of first choices}]\\
&\quad= \left((n-i)A_i + \sum_{j=1}^{i} A_j - \min(A_i, \text{length of arc before i})\right)^{m-1}\\
&\quad\le \left((n-i)A_i + \sum_{j=1}^{i} A_j\right)^{m-1}\\
&\quad= S_n^{-(m-1)}\left((n-i)\sum_{j=n-i+1}^{n}\frac{X_j}{j} + \sum_{j=1}^{i}\sum_{k=n-j+1}^{n}\frac{X_k}{k}\right)^{m-1}\\
&\quad= S_n^{-(m-1)}\left((n-i)\sum_{j=n-i+1}^{n}\frac{X_j}{j} + \sum_{k=n-i+1}^{n}(k-n+i)\frac{X_k}{k}\right)^{m-1}\\
&\quad= S_n^{-(m-1)}\left(\sum_{j=n-i+1}^{n}(n-i+j-n+i)\frac{X_j}{j}\right)^{m-1}\\
&\quad= S_n^{-(m-1)}\left(\sum_{j=n-i+1}^{n}X_j\right)^{m-1}\\
&\quad= \left(\frac{S_i}{S_n}\right)^{m-1}.\end{aligned}$$

Note we have used $\sum_{j=n-i+1}^{n} X_j = S_i$ for convenience. Our resulting bound has a surprisingly clean form in terms of the S_i.

Thus, by Lemma 4 and Fact 5:

$$\begin{aligned}\Pr[\text{first choices are stable}] &\le \mathrm{E}\left[\prod_{i=1}^{n}\left(\frac{S_i}{S_n}\right)^{m-1}\right]\\
&= \mathrm{E}\left[\prod_{i=1}^{n-1}\left(\frac{S_i}{S_{i+1}}\cdot\frac{S_{i+1}}{S_{i+2}}\cdot\dots\cdot\frac{S_{n-1}}{S_n}\right)^{m-1}\right]\\
&= \mathrm{E}\left[\prod_{i=1}^{n-1}\left(\frac{S_i}{S_{i+1}}\right)^{i(m-1)}\right] = \prod_{i=1}^{n-1}\mathrm{E}\left[\left(\frac{S_i}{S_{i+1}}\right)^{i(m-1)}\right]\\
&= \prod_{i=1}^{n-1}\frac{i}{i+i\,(m-1)} = \frac{1}{m^{n-1}}.\end{aligned}$$

It then follows that

$$\mathrm{E}\left[\#\mathrm{PNE}\right] \leq m^n \cdot \frac{1}{m^{n-1}} = m.$$

We can similarly find a lower bound, although some additional technical work is required.

$$\begin{aligned}
&\Pr[i\text{th smallest arc is stable} \mid \text{position of first choices}] \\
&\quad = \left((n-i)A_i + \sum_{j=1}^{i} A_j - \min(A_i, \text{length of arc before i}) \right)^{m-1} \\
&\quad \geq \left((n-i)A_i + \sum_{j=1}^{i} A_j - A_i \right)^{m-1} \\
&\quad = \left((n-i-1)A_i + \sum_{j=1}^{i} A_j \right)^{m-1} \\
&\quad = S_n^{-(m-1)} \left(\sum_{j=n-i+1}^{n} (n-i-1+j-n+i) \frac{X_j}{j} \right)^{m-1} \\
&\quad = S_n^{-(m-1)} \left(\sum_{j=n-i+1}^{n} \frac{(j-1)X_j}{j} \right)^{m-1}
\end{aligned}$$

Hence

$$\Pr[\text{first choices are stable}] \geq \mathrm{E}\left[S_n^{-(m-1)n} \prod_{i=1}^{n} \left(\sum_{j=n-i+1}^{n} \frac{(j-1)X_j}{j} \right)^{m-1} \right].$$

A simple stochastic domination argument (provided in full version of the paper [5]) shows that the expectation on the right side decreases if, in each term in the product, we equalize the coefficient, so that instead of terms of the form $\frac{(j-1)X_j}{j}$, the coefficient for all terms of the sum in the ith term of the product is the average $c_i = \frac{1}{i}\sum_{j=n-i+1}^{n} \frac{j-1}{j}$. This gives

$$\begin{aligned}
\Pr[\text{first choices are stable}] &\geq \mathrm{E}\left[S_n^{-(m-1)n} \prod_{i=1}^{n} \left(\sum_{j=n-i+1}^{n} c_i X_j \right)^{m-1} \right] \\
&= \mathrm{E}\left[S_n^{-(m-1)n} \prod_{i=1}^{n} (c_i S_i)^{m-1} \right] \\
&= \left(\prod_{i=1}^{n} c_i^{m-1} \right) \mathrm{E}\left[\prod_{i=1}^{n} \left(\frac{S_i}{S_n} \right)^{m-1} \right].
\end{aligned}$$

Observe that this expectation is the same as the one we computed in the upper bound. Therefore

$$\mathrm{E}[\#\ \mathrm{PNE}] \geq \left(\prod_{i=1}^{n} c_i^{m-1} \right) m \geq 0.19^{m-1} m.$$

The proof of the final inequality is presented in the full version of the paper [5]. ◀

We note that similar calculations can be done for the minimization version, although we have not found a clean form for the upper bound. We can, however, state the following lower bound, showing the expected number of pure Nash equilibria is at least inverse polynomial in n for a fixed number of choices m.

▶ **Theorem 6** (For proof see [5]). *The expected number of PNE for the minimization version of the One Way 1-D Simultaneous Voronoi Game is at least* $\frac{m(m-1)}{(mn-1)n^{m-1}}$.

We have done several experiments regarding Random Voronoi games, which prove consistent with our theoretical results and suggest some interesting conjectures, particularly for the 2-D Simultaneous Voronoi Game. Chief among these conjectures is that the Random k-D Simultaneous Voronoi game has a pure Nash Equilibrium with probability approaching 1 as n grows. The complete discussion of these results can be found in the full version of the paper

4 Correlated Equilibria

Our goal in this section is to show that, for the k-D Simultaneous Voronoi Game, correlated equilibria can be found in polynomial time. We present the results for $k = 1, 2$, and 3. The results appear to extend to higher dimensions but the geometric details are technical; we note the time required to determine the correlated equilibrium appears to grow as $n^{O(k)}$. The results also apply to the One Way 1-D Simultaneous Voronoi Game.

▶ **Theorem 7.** *For $k = 1, 2$, and 3, and for a fixed m, there is a polynomial time algorithm for finding a correlated equilibrium in the k-D Simultaneous Voronoi Game.*

We appeal to [17] and [12], who present polynomial time algorithms for finding a correlated equilibrium of games *polynomial type*. (The running times for these algorithms are not specifically presented in the papers and appear rather large, but are still polynomial.) A game of polynomial type is one that can be represented in polynomial space such that given each player's strategy, their utilities can be computed in polynomial time. The k-D Simultaneous Voronoi Game is of polynomial type because it can be represented in $O(nmk)$ space by a list of each players point choices and, given the players' strategies, the utilities can be found by computing the Voronoi diagram of the chosen points.

▶ **Theorem 8** (Theorem 4.5, [12]). *Given a game of polynomial type and a polynomial time algorithm for computing the expected utility of a player under any product distribution on strategies, there exists a polynomial time algorithm for finding a correlated equilibrium in that game.*

Jiang et al. proved Theorem 8 by constructing a linear program with a variable for each of the 2^n possible strategy profiles. The LP's constraints are non-negativity, and the constraints requiring that the variables form a correlated equilibrium. They do not, however, enforce that the variables sum to one, or even at most one, and rather use the sum of these variables as the objective. Thus, since a correlated equilibrium is guaranteed to exist by Nash's Theorem, this LP is unbounded and its dual is infeasible. They then run the ellipsoid algorithm for a polynomial number of steps on the dual LP (this takes polynomial time, since the dual LP has only polynomially many variables). They argue that the intermediate steps of the ellipsoid algorithm can be used to construct product distributions of which there is a convex combination that is a valid correlated equilibrium, and which can be found with a second linear program.

The second linear program's separation oracle requires as a subroutine a polynomial time algorithm for computing the expected utility of a player given a product distribution over the strategies. (This requirement is referred to in [17] and [12] as the *polynomial expectation property.*) Our work is to demonstrate polynomial time algorithms for this subroutine. We note that it is not immediate that such an algorithm should exist, even when each player has only $m = 2$ choices, as the number of possible configurations is m^n. Hence, we cannot simply sum over all configurations when calculating the expectation. In [17] it is noted that for certain congestion games, these expectations can be computed using dynamic programming, essentially adding one player in at a time and updating accordingly. Our approach is similar in spirit, but requires taking advantage of the underlying geometry.

▶ **Lemma 9** (For proof see [5]). *Computing a player's expected utility under a product distribution on strategies in the k-D Simultaneous Voronoi Game takes $O(n^{2k-1}\log n)$ time for $k \leq 3$.*

The proof is trivial for one dimension. For two and three dimensions, the key idea is to partition the space into regions where the possible Voronoi cell boundaries do not cross.

5 Conclusion

We have introduced a new but we believe important set of variants on Voronoi games, where each player has a disjoint set of points to choose from. We believe these variations are motivated both by natural economic settings, and because of the possible connections to other "power-of-choice" processes in which participants choose from a limited set of random options.

In particular, we note that the Voronoi choice games we propose offer the chance to consider randomized versions of the problem, where the set of possible choices for each player is chosen uniformly over of the space. We have conjectured that the Random k-D Simultaneous Voronoi Game has a pure Nash equilibrium with high probability, based on a simulation study. While this is perhaps the most natural open question in this setting, there remain several other questions for both the simultaneous and sequential versions of Voronoi choice problems, in the worst case and with random point sets.

References

1. Dimitris Achlioptas, Raissa M D'Souza, and Joel Spencer. Explosive percolation in random networks. *Science*, 323(5920):1453–1455, 2009.
2. Robert J Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1(1):67–96, 1974.
3. Yossi Azar, Andrei Z Broder, Anna R Karlin, and Eli Upfal. Balanced allocations. *SIAM journal on computing*, 29(1):180–200, 1999.
4. Avrim Blum, MohammadTaghi Hajiaghayi, Katrina Ligett, and Aaron Roth. Regret minimization and the price of total anarchy. In *Proceedings of the Fortieth Annual ACM Symposium on Theory of Computing*, pages 373–382. ACM, 2008.
5. Meena Boppana, Rani Hod, Michael Mitzenmacher, and Tom Morgan. Voronoi choice games, 2016. `arXiv:1604.07084`.
6. John Byers, Jeffrey Considine, and Michael Mitzenmacher. Simple load balancing for distributed hash tables. In *Peer-to-peer Systems II*, pages 80–87. Springer, 2003.
7. Christoph Dürr and Nguyen Kim Thang. Nash equilibria in voronoi games on graphs. In *Proceedings of the 5th Annual European Symposium on Algorithms*, pages 17–28. Springer, 2007.

8 Gaëtan Fournier and Marco Scarsini. Hotelling games on networks: efficiency of equilibria. *Available at SSRN 2423345*, 2014.

9 J. J. Gabszewicz and J.-F. Thisse. Location. In *Handbook of Game Theory with Economic Applications*. Volume 1, Chapter 9. R. Aumann and S. Hart, editors. Elsevier Science Publishers, 1992.

10 Lars Holst. On the lengths of the pieces of a stick broken at random. *Journal of Applied Probability*, pages 623–634, 1980.

11 Harold Hotelling. Stability in competition. *The Economic Journal*, 39(153):41–57, 1929.

12 Albert Xin Jiang and Kevin Leyton-Brown. Polynomial-time computation of exact correlated equilibrium in compact games. *Games and Economic Behavior*, 21(1-2):183–202, 2013.

13 Masashi Kiyomi, Toshiki Saitoh, and Ryuhei Uehara. Voronoi game on a path. *IEICE TRANSACTIONS on Information and Systems*, 94(6):1185–1189, 2011.

14 Marios Mavronicolas, Burkhard Monien, Vicky G Papadopoulou, and Florian Schoppmann. Voronoi games on cycle graphs. In *Proceedings of Mathematical Foundations of Computer Science*, pages 503–514. Springer, 2008.

15 M Mitzenmacher, Andrea W Richa, and R Sitaraman. The power of two random choices: A survey of techniques and results. In *Handbook of Randomized Computing*, pages 255–312, 2000.

16 John Nash. Non-cooperative games. *The Annals of Mathematics*, 54(2):286–295, 1951.

17 Christos H Papadimitriou and Tim Roughgarden. Computing correlated equilibria in multi-player games. *Journal of the ACM*, 55(3):14, 2008.

18 Sachio Teramoto, Erik D Demaine, and Ryuhei Uehara. Voronoi game on graphs and its complexity. In *IEEE Symposium on Computational Intelligence and Games*, pages 265–271. IEEE, 2006.