# Popular Half-Integral Matchings

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## Abstract

In an instance $G = (A \cup B, E)$ of the stable marriage problem with strict and possibly incomplete preference lists, a matching $M$ is popular if there is no matching $M'$ where the vertices that prefer $M'$ to $M$ outnumber those that prefer $M$ to $M'$. All stable matchings are popular and there is a simple linear time algorithm to compute a maximum-size popular matching. More generally, what we seek is a *min-cost* popular matching where we assume there is a cost function $c : E \to \mathbb{Q}$. However there is no polynomial time algorithm currently known for solving this problem. Here we consider the following generalization of a popular matching called a popular *half-integral* matching: this is a fractional matching $\vec{x} = (M_1 + M_2)/2$, where $M_1$ and $M_2$ are the 0-1 edge incidence vectors of matchings in $G$, such that $\vec{x}$ satisfies popularity constraints. We show that every popular half-integral matching is equivalent to a stable matching in a larger graph $G^*$. This allows us to solve the min-cost popular half-integral matching problem in polynomial time.

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## 1 Introduction

Let $G = (A \cup B, E)$ be an instance of the stable marriage problem on $n$ vertices and $m$ edges. Each vertex has a strict preference list ranking its neighbors. A matching $M$ is stable if $M$ admits no *blocking edge*, i.e., an edge $(a, b)$ such that both $a$ and $b$ prefer each other to their respective assignments in $M$. The existence of stable matchings in $G$ and the Gale-Shapley algorithm [7] to find one are classical results in graph algorithms.

Stability is a very strict condition and here we consider a relaxation of this called *popularity*. This notion was introduced by Gärdenfors [9] in 1975. We say a vertex $u \in A \cup B$ *prefers* matching $M$ to matching $M'$ if $u$ is matched in $M$ and unmatched in $M'$ or it is matched in both and $M(u)$ ranks better than $M'(u)$ in $u$'s preference list. For any two matchings $M$ and $M'$ in $G$, let $\phi(M, M')$ be the number of vertices that prefer $M$ to $M'$.

▶ **Definition 1.** A matching $M$ is *popular* if $\phi(M, M') \geq \phi(M', M)$ for every matching $M'$ in $G$, i.e., $\Delta(M, M') \geq 0$ where $\Delta(M, M') = \phi(M, M') - \phi(M', M)$.

Every stable matching is popular [9]. In fact, it is known that every stable matching is a minimum-size popular matching [10]. In applications such as matching students to projects or applicants to posts, it may be useful to consider a weaker notion (such as popularity) than the total absence of blocking edges for the sake of obtaining larger-sized matchings. Popularity provides "global stability" since a popular matching never loses an election to another matching; by relaxing stability to popularity, we have a larger pool of candidate matchings to choose from in such an application.

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When there is a cost function $c : E \to \mathbb{Q}$, what we seek is a min-cost popular matching. There are several polynomial time algorithms known [11, 5, 6, 16, 14, 15] for computing a min-cost stable matching in $G$. However, while a maximum-size popular matching can be computed in linear time [12], no polynomial time algorithm is currently known for computing a min-cost popular matching in an instance $G = (A \cup B, E)$ with strict preference lists, except when preference lists are complete [4].

A fractional matching $\vec{p}$ is a convex combination of matchings, i.e., $\vec{p} = \sum_i p_i \cdot I(M_i)$ where $\sum_i p_i = 1$, $p_i \geq 0$ for all $i$, $M_i$'s are matchings in $G$, and $I(M)$ is the 0-1 edge incidence vector of $M$. The fractional matching $\vec{p}$ is *popular* if $\Delta(\vec{p}, M) \geq 0$ for all matchings $M$ in $G$ where $\Delta(\vec{p}, M) = \sum_i p_i \cdot \Delta(M_i, M)$ (see Definition 1). It follows by linearity that if $\vec{p}$ is a popular fractional matching then $\Delta(\vec{p}, \vec{q}) \geq 0$ for all fractional matchings $\vec{q}$.

Let $\mathcal{P}$ be the polytope defined by the constraints that $\vec{p}$ belongs to the matching polytope of $G$ and $\Delta(\vec{p}, M) \geq 0$ for all matchings $M$ in $G$. A simple description of $\mathcal{P}$ was given in [13]. Thus a min-cost popular *fractional* matching can be computed in polynomial time.

**Our results and techniques.** Our main result is a polynomial time algorithm to compute a min-cost popular *half-integral* matching in $G$. A popular half-integral matching is a vector $\vec{x} \in \{0, \frac{1}{2}, 1\}^m \cap \mathcal{P}$. For any two popular matchings $M_1$ and $M_2$ in $G$, the half-integral matching $(I(M_1) + I(M_2))/2$ is popular. However not every popular half-integral matching is a convex combination of popular matchings – we show such an example in Section 2. Thus if $\mathcal{Q}$ is the convex hull of popular half-integral matchings in $G$, then $\mathcal{Q}$ need not be integral.

We show that every extreme point of $\mathcal{Q}$ is a stable matching in a new (larger) graph $G^*$ that we construct here. Thus the min-cost popular half-integral matching problem in $G$ becomes the min-cost stable matching problem in $G^*$ which can be solved in polynomial time. This also gives us a simple description of the polytope $\mathcal{Q}$ via the stable matching polytope of $G^*$ (i.e., the convex hull of stable matchings in $G^*$).

The main tool that we use here is the description of the polytope $\mathcal{P}$ from [13]. We first show that every stable matching $S$ in the new graph $G^*$ can be mapped to a half-integral matching in $G$ whose incidence vector belongs to $\mathcal{P}$. We then show that every extreme point $\vec{p}$ of the convex hull $\mathcal{Q}$ of popular half-integral matchings in $G$ can be realized as a stable matching in $G^*$. We use the fact that $\vec{p} \in \mathcal{P}$ along with the fact that $G$ is bipartite to show a "helpful witness" $(\alpha_u)_{u \in A \cup B} \in \{\pm 1, 0\}^n$. This witness will guide us in building a stable matching $S$ in $G^*$ that corresponds to $\vec{p}$.

A graph $G'$, similar to the graph $G^*$ used here, was recently used in [4] to show that any stable matching in $G'$ maps to a maximum-size popular matching $M$ in $G$. However every maximum-size popular matching in $G$ need not be obtained as a stable matching in $G'$. In the special case when preference lists are complete (i.e., $G$ is $K_{|A|,|B|}$), all popular matchings in $G$ can be realized as stable matchings in $G'$. The method used in [4] is similar to the method used in previous algorithms to compute maximum-size popular matchings [10, 12] – these show that there is no *popularity-improving* alternating path or cycle with respect to the matching returned. In contrast, our technique here is based on linear programming.

A min-cost popular half-integral popular matching has applications – consider the problem of assigning projects to students where each project can be split into two half-projects. Each half-project can be assigned to a distinct student and a student can be assigned two half-projects. A min-cost popular half-integral matching is a feasible assignment here that is popular and has the least cost. While fractional matchings, in general, may not be feasible in typical applications, half-integral matchings are more natural and suitable to applications.

**Background.** Algorithms for computing popular matchings [1] were first considered in the one-sided preference lists model where it is only vertices in $A$ that have preferences and cast votes while vertices in $B$ have no preferences. Popular matchings need not always exist in this model, however it was shown in [13] that popular fractional matchings always exist and using the description of $\mathcal{P}$, such a fractional matching can be found in polynomial time (via linear programming).

In the two-sided preference lists model, when preference lists have ties, $G = (A \cup B, E)$ need not always admit a popular matching and it is known that determining if $G$ admits a popular matching or not is an NP-complete problem [2, 3]. When preference lists are strict, every stable matching is popular. The min-cost stable matching problem in an instance $G = (A \cup B, E)$ with strict preference lists is well-studied and descriptions of the stable matching polytope were given by Vande Vate [16], Rothblum [14], and Teo and Sethuraman [15].

We discuss preliminaries in Section 2. Section 3 describes the graph $G^*$ and shows that every stable matching in $G^*$ is a popular half-integral matching in $G$. Section 4 shows how every popular half-integral matching in $G$ that is an extreme point of $\mathcal{Q}$ (the popular half-integral matching polytope) can be obtained as a stable matching in $G^*$.

## 2 Preliminaries

For any vertex $u \in A \cup B$ and neighbors $v$ and $w$, we will use the following function to show $u$'s preference for $v$ vs $w$: $\mathsf{vote}_u(v, w) = 1$ if $u$ prefers $v$ to $w$, it is -1 if $u$ prefers $w$ to $v$, else (i.e., when $v = w$) it is 0. We will be using this function in the description of the popular fractional matching polytope $\mathcal{P}$.

Recall that a popular fractional matching is a point $\vec{x} = (x_e)_{e \in E}$ in the matching polytope of $G$ such that $\Delta(\vec{x}, M) \geq 0$ for all matchings $M$ in $G$. It will be convenient to assume that each vertex $u \in A \cup B$ is completely matched in every fractional matching $\vec{x}$ in $G$. So we will revise $\vec{x}$ so that each vertex $u$ gets matched to an artificial last-resort neighbor $\ell(u)$ (which is placed at the bottom of $u$'s preference list) with weight $(1 - \sum_{e \in E(u)} x_e)$, where the sum is over all the edges $e$ incident on $u$.

For convenience, we will continue to use $\vec{x}$ to denote the revised $\vec{x}$ in $[0,1]^{m+n}$. We use $\tilde{E}$ to denote the edge set $E \cup \{(u, \ell(u)) : u \in A \cup B\}$ and $\tilde{E}(u)$ is the set of edges in $\tilde{E}$ that are incident on $u$. The following simple description of $\mathcal{P}$ was given in [13]. In the constraints below, a variable $\alpha_u$ is associated with each $u \in A \cup B$ and not to last-resort neighbors.

$$\alpha_a + \alpha_b \;\geq\; \sum_{(a,b') \in \tilde{E}(a)} x_{(a,b')} \cdot \mathsf{vote}_a(b, b') \;+\; \sum_{(a',b) \in \tilde{E}(b)} x_{(a',b)} \cdot \mathsf{vote}_b(a, a') \quad \forall (a,b) \in \tilde{E}$$

$$\sum_{u \in A \cup B} \alpha_u \;=\; 0 \quad \text{and} \quad \sum_{e \in \tilde{E}(u)} x_e \;=\; 1 \quad \forall u \in A \cup B \quad \text{and} \quad x_e \geq 0 \quad \forall e \in \tilde{E}.$$

The constraints above arise as the dual to the maximum weight matching problem in the graph $\tilde{G}_x$ which is $G$ augmented with last-resort neighbors and with edge set $\tilde{E}$, where the weight of an edge $(a, b)$ is $\sum_{(a,b') \in \tilde{E}(a)} x_{(a,b')} \cdot \mathsf{vote}_a(b, b') + \sum_{(a',b) \in \tilde{E}(b)} x_{(a',b)} \cdot \mathsf{vote}_b(a, a')$. The constraint $\sum_{u \in A \cup B} \alpha_u = 0$ is equivalent to saying that the maximum weight of a matching in $\tilde{G}_x$ is 0, in other words, $\vec{x}$ is popular. We refer the reader to Section 3 of [13] for all the details.

For any fractional matching $\vec{x}$, if there exists $\vec{\alpha} = (\alpha_u)_{u \in A \cup B}$ such that $\vec{x}$ and $\vec{\alpha}$ satisfy the above constraints, then we say $\vec{x} \in \mathcal{P}$. The vector $\vec{\alpha}$ will be called a *witness* to $\vec{x}$'s popularity.

| | | | |
|---|---|---|---|
| $a_0$ | $v_1$ | | |
| $a_1$ | $b_1$ | $v_1$ | |
| $a_2$ | $b_1$ | $b_2$ | |
| $u_1$ | $v_1$ | $v_2$ | $b_0$ |
| $u_2$ | $v_2$ | $b_2$ | $v_1$ |

| | | | | |
|---|---|---|---|---|
| $b_0$ | $u_1$ | | | |
| $b_1$ | $a_2$ | $a_1$ | | |
| $b_2$ | $a_2$ | $u_2$ | | |
| $v_1$ | $u_2$ | $a_1$ | $u_1$ | $a_0$ |
| $v_2$ | $u_1$ | $u_2$ | | |

**Figure 1** The above table describes the preference lists of all the men and women in $G$. Here $a_0$ has a single neighbor $v_1$ while $a_1$'s top choice is $b_1$, second choice is $v_1$ and so on for each vertex.

**$\mathcal{P}$ is not integral.** We now show an example of a graph $G$ and a fractional matching $\vec{p} \in \mathcal{P}$, however $\vec{p}$ is not a convex combination of popular matchings. Let $A = \{a_0, a_1, a_2, u_1, u_2\}$, $B = \{b_0, b_1, b_2, v_1, v_2\}$, and the preference lists of vertices are described in Figure 1.

Consider the half-integral matching $\vec{p}$ which has $p_{(a_1,b_1)} = p_{(a_2,b_2)} = 1$ and $p_e = \frac{1}{2}$ for $e \in \{(u_1, v_1), (u_2, v_2), (u_1, v_2), (u_2, v_1)\}$. For any other edge $e$, we have $p_e = 0$. This fractional matching belongs to $\mathcal{P}$ by using the following $\alpha$ values: $\alpha_{a_0} = \alpha_{b_0} = 0$; $\alpha_{a_2} = \alpha_{b_1} = 1$; $\alpha_{a_1} = \alpha_{b_2} = -1$; and $\alpha_w = 0$ for $w \in \{u_1, u_2, v_1, v_2\}$.

There is only one way to express $\vec{p}$ as a convex combination of integral matchings, that is, $\vec{p} = (I(M_1) + I(M_2))/2$, where $M_1 = \{(a_1, b_1), (a_2, b_2), (u_1, v_1), (u_2, v_2)\}$ and $M_2 = \{(a_1, b_1), (a_2, b_2), (u_1, v_2), (u_2, v_1)\}$. We show below that neither $M_1$ nor $M_2$ is popular.

The matching $M_1' = \{(u_1, b_0), (a_1, v_1), (a_2, b_1), (u_2, v_2)\}$ is more popular than $M_1$ and the matching $M_2' = \{(a_0, v_1), (u_2, b_2), (a_2, b_1), (u_1, v_2)\}$ is more popular than $M_2$. Thus $\vec{p}$ is not in the convex hull of popular matchings in $G$.

**The graph $G'$.** Our input is a graph $G = (A \cup B, E)$ on $n$ vertices and $m$ edges. Note that there are no last-resort neighbors here – they were added only for the formulation of the polytope $\mathcal{P}$. Vertices in $A$ and in $B$ are usually referred to as men and women, respectively, and we follow the same convention here.

The construction of the following graph $G' = (A' \cup B', E')$, based on $G$, was shown in [4]. The set $A'$ has two copies $a_0$ and $a_1$ of each man $a \in A$, the men in $\{a_0 : a \in A\}$ are called *level 0 men* of $G'$ and those in $\{a_1 : a \in A\}$ are called *level 1 men* of $G'$. The set $B'$ consists of all the women in $B$ along with *dummy vertices* $\cup_{a \in A}\{d(a)\}$, where there is one dummy vertex per man in $A$. The preference lists of the vertices are as follows:

- each level 0 man $a_0$ has the same preference list as the corresponding man $a$ in $G$ except that the dummy vertex $d(a)$ occurs as his *least* preferred neighbor at the bottom of his preference list
- each level 1 man $a_1$ has the same preference list as the corresponding man $a$ in $G$ except that the dummy vertex $d(a)$ occurs as his *most* preferred neighbor at the top of his preference list
- each dummy vertex $d(a)$ has $a_0$ and $a_1$ as its neighbors: top choice is $a_0$, followed by $a_1$
- every woman $b \in B$ has the following preference list in $G'$: all her level 1 neighbors (in the same order of preference as in $G$) followed by all her level 0 neighbors (in the same order of preference as in $G$).

We will be using this graph $G'$ here; in fact, we will have two such graphs $G'$ and $G''$ combining to form our new graph $G^*$. The graph $G''$ is analogous to the graph $G'$ except that the roles of men and women (and also that of levels 0 and 1) are swapped here.

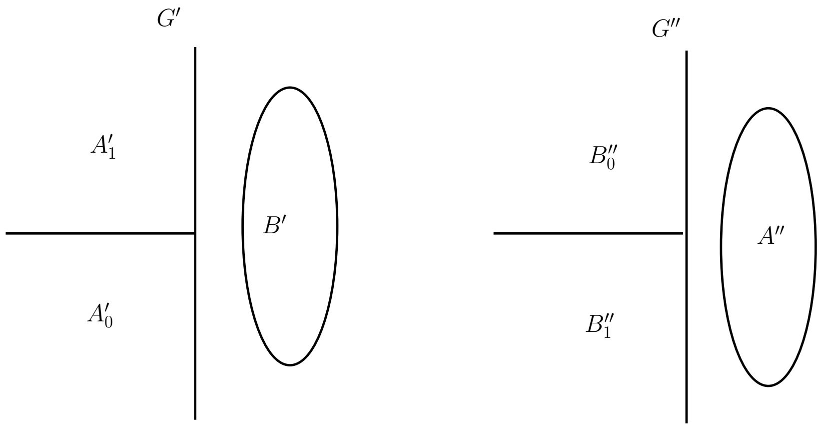


**Figure 2** The graph $G'$ on the left and the graph $G''$ on the right in $G^*$. For $i = 0, 1$, we use $A'_i$ to refer to level $i$ men in $G'$ and we use $B''_i$ to refer to level $i$ women in $G''$.

## 3 The graph $G^*$

We define the graph $G^*$ as follows: $G^*$ consists of two vertex-disjoint subgraphs $G'$ and $G''$ (see Figure 2). The graph $G'$ was described in Section 2.

In the graph $G'' = (B'' \cup A'', E'')$, women are on the left side of $G''$ and men are on the right side – the set $B''$ has two copies $b_0$ and $b_1$ of each woman $b \in B$, the women in $\{b_0 : b \in B\}$ are called *level 0 women* of $G''$ and those in $\{b_1 : b \in B\}$ are called *level 1 women* of $G''$.

The set $A''$ consists of all the men in $A$ along with new dummy vertices $\cup_{b \in B}\{d(b)\}$, where there is one dummy vertex per woman in $B$. The preference lists of the vertices are as follows:

- each level 0 woman $b_0$ has the same preference list as the corresponding woman $b$ in $G$ except that the dummy vertex $d(b)$ occurs as her *most* preferred neighbor at the top of her preference list
- each level 1 woman $b_1$ has the same preference list as the corresponding woman $b$ in $G$ except that the dummy vertex $d(b)$ occurs as her *least* preferred neighbor at the bottom of her preference list
- each dummy vertex $d(b)$ has only $b_0$ and $b_1$ as its neighbors: its top choice is $b_1$, followed by $b_0$
- every man $a \in A$ has the following preference list in $G''$: all his level 0 neighbors (in the same order of preference as in $G$) followed by all his level 1 neighbors (in the same order of preference as in $G$).

We want all stable matchings in $G^*$ to be perfect matchings – note that all level 0 men in $G'$ and all level 1 women in $G''$ will be matched in any stable matching in $G^*$ since they are top-choice neighbors for their respective dummy neighbors. However the same cannot be said about level 1 men in $G'$ and level 0 women in $G''$.

In order to take care of these vertices, we add the following "self-loop" edges to $G^*$: the edge $(a_1, a)$ for each man $a$ in $A$, where $a_1 \in A'_1$ and $a \in A''$, and the edge $(b_0, b)$ for each woman $b$ in $B$, where $b_0 \in B''_0$ and $b \in B'$. The vertex $a_1 \in A'_1$ regards $a \in A''$ as his worst ranked neighbor and similarly, $b_0 \in B''_0$ regards $b \in B'$ as her worst ranked neighbor.

For any man $a \in A''$, the vertex $a_1$ is in the middle of his preference list, sandwiched between all his level 0 neighbors and all his level 1 neighbors as shown in (1) below. More precisely, $a_1$ is sandwiched between $b''_0$ and $b'_1$, where $b' > \cdots > b''$ is $a$'s preference list in $G$. Thus $b''_0$ is $a$'s worst level 0 neighbor and $b'_1$ is $a$'s best level 1 neighbor.

$$a:\ b'_0 > \cdots > b''_0 > \underline{a_1} > b'_1 > \cdots > b''_1; \qquad b:\ a'_1 > \cdots > a''_1 > \underline{b_0} > a'_0 > \cdots > a''_0. \quad (1)$$

Similarly, for any woman $b \in B'$, the vertex $b_0$ is in the middle of her preference list, sandwiched between all her level 1 neighbors and all her level 0 neighbors as shown in (1). More precisely, $b_0$ is sandwiched between $a''_1$ and $a'_0$, where $a' > \cdots > a''$ is $b$'s preference list in $G$. Using the fact that all stable matchings in $G^*$ match the same set of vertices [8], it can be shown that every stable matching in $G^*$ is perfect.

**The function $f$.** We now define a function $f$ : {stable matchings in $G^*$} $\to$ {half-integral matchings in $G$}. Observe that every stable matching in $G^*$ has to match all dummy vertices since each of these is a top-choice neighbor for someone. Thus out of $a_0$ and $a_1$ in $A'$, only one is matched to a non-dummy neighbor and similarly, out of $b_0$ and $b_1$ in $B''$, only one is matched to a non-dummy neighbor.

Let $S$ be any stable matching in $G^*$. By removing all self-loops that occur in $S$ and those edges in $S$ that contain a dummy vertex, the resulting matching is the union of two matchings $S'$ and $S''$ in $G$. We define $f(S)$ to be $(I(S') + I(S''))/2$, where $I(M) \in \{0,1\}^m$ is the 0-1 edge incidence vector of $M$. So $f(S)$ is a valid half-integral matching in $G$.

▶ **Theorem 2.** *For any stable matching $S$ in $G^*$, the half-integral matching $f(S)$ is popular in $G$.*

**Proof.** We are given a stable matching $S$ in $G^*$. Recall that we pruned all edges that contain a dummy vertex and all self-loops from $S$ to define $f(S)$. We now prune all dummy vertices, their partners in $S$, and self-loops from $G^*$ also – let $H^*$ denote the pruned graph $G^*$. Let $H'$ denote the pruned subgraph $G'$ and let $H''$ denote the pruned subgraph $G''$.

The men in the graph $H'$ consist of one copy of each $a \in A$ – some of these are in level 0 and the rest are in level 1. The women in $H'$ are exactly those in $B$. The women in $H''$ consist of one copy of each $b \in B$ – some of these are in level 0 and the rest are in level 1. The men in $H''$ are exactly those in $A$. Thus $H'$ and $H''$ are two copies of the graph $G$.

Let $S'$ be the pruned matching (resulting from $S$) restricted to $H'$ and let $S''$ be the pruned matching (resulting from $S$) restricted to $H''$. Let $\tilde{A}'_i$ denote the set of level $i$ men in $H'$, for $i = 0, 1$ (see Figure 3). Let $\tilde{B}'_i$ consist of women matched in $S'$ to men in $\tilde{A}'_i$, for $i = 0, 1$. Women unmatched in $S'$ are added to $\tilde{B}'_1$.

Similarly, $\tilde{B}''_i$ consists of level $i$ women in the $H''$ part of $H^*$ and $\tilde{A}''_i$ denotes the set of men matched in $S''$ to women in $\tilde{B}''_i$, for $i = 0, 1$. Men unmatched in $S''$ are added to $\tilde{A}''_0$.

For each edge $e = (a, b) \in H'$, define the function $w'(e)$ as follows: $w'(e) = \mathsf{vote}_a(b, S'(a)) + \mathsf{vote}_b(a, S'(b))$. If $S'(u)$ is undefined for any vertex $u$, then $\mathsf{vote}_u(v, S'(u)) = 1$ for any neighbor $v$ of $u$ since every vertex prefers being matched to being unmatched. Note that if $(a, b) \in S'$ then $w'(e) = 0$.

Similarly, for each edge $e = (a, b) \in H''$, define the function $w''(e)$ as follows: $w''(e) = \mathsf{vote}_a(b, S''(a)) + \mathsf{vote}_b(a, S''(b))$. For any vertex $u$ that is unmatched in $S''$, we take $\mathsf{vote}_u(v, S''(u)) = 1$, for any neighbor $v$ of $u$. Note that $w'(e)$ and $w''(e)$ always take values in $\{-2, 0, 2\}$. Due to the stability of the matching $S$ in $G^*$, the following observations hold:

- Every edge $e \in \tilde{A}'_1 \times \tilde{B}'_0$ has to satisfy $w'(e) = -2$. Similarly, every edge $e \in \tilde{A}''_1 \times \tilde{B}''_0$ has to satisfy $w''(e) = -2$.

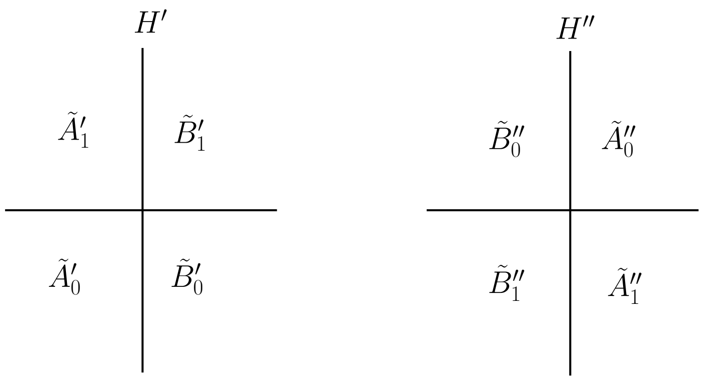


**Figure 3** The graph $H'$ on the left and the graph $H''$ on the right in the graph $H^*$.

Consider an edge $(a_1, b)$ in $\tilde{A}'_1 \times \tilde{B}'_0$. It follows from the definition of preference lists of women in $G'$ that the woman $b$ prefers $a_1$ (a level 1 man) to her partner $S'(b)$ (a level 0 man). Since $S$ is stable, it follows that $a_1$ prefers his partner $S'(a_1)$ to $b$. Moreover, $a_0$ prefers $b$ to $S'(a_0) = d(a)$, since $d(a)$ is $a_0$'s last choice. Thus $b$ prefers her partner $S'(b)$ to $a_0$. So $\mathsf{vote}_a(b, S'(a)) = \mathsf{vote}_b(a, S'(b)) = -1$. A similar proof holds for any edge $e \in \tilde{A}''_1 \times \tilde{B}''_0$.

- Every edge $e$ such that $w'(e) = 2$ has to be in $\tilde{A}'_0 \times \tilde{B}'_1$. Similarly, every edge $e$ such that $w''(e) = 2$ has to be in $\tilde{A}''_0 \times \tilde{B}''_1$.

If $e$ is an edge in $H'$ such that $w'(e) = 2$, then $e \notin \tilde{A}'_i \times \tilde{B}'_i$ (for $i = 0, 1$) as such an edge would block $S$. We have already seen that any edge $e \in \tilde{A}'_1 \times \tilde{B}'_0$ satisfies $w'(e) = -2$. Thus any edge $e$ such that $w'(e) = 2$ has to be in $\tilde{A}'_0 \times \tilde{B}'_1$. We can similarly show that any edge $e$ in $H''$ such that $w''(e) = 2$ has to be in $\tilde{A}''_0 \times \tilde{B}''_1$.

We will now show that $f(S) \in \mathcal{P}$ by assigning appropriate $\alpha_u$ values for all $u$ in $A \cup B$. We first define $\alpha'_u$ and $\alpha''_u$:

- let $\alpha'_u = -1$ if $u \in \tilde{A}'_1 \cup \tilde{B}'_0$ and let $\alpha'_u = 1$ if $u \in \tilde{A}'_0 \cup \tilde{B}'_1$.
- let $\alpha''_u = -1$ if $u \in \tilde{A}''_1 \cup \tilde{B}''_0$ and let $\alpha''_u = 1$ if $u \in \tilde{A}''_0 \cup \tilde{B}''_1$.

The following is an immediate corollary of the above observations and the definitions of $\alpha'_u$ and $\alpha''_u$: $\alpha'_a + \alpha'_b \ge w'(a,b)$ and $\alpha''_a + \alpha''_b \ge w''(a,b)$ for all edges $(a,b)$. Also for any vertex $u$ that is unmatched in $S'$ and $S''$, we have $\alpha'_u + \alpha''_u = 0$.

Define $\alpha_u = (\alpha'_u + \alpha''_u)/2$ for all $u \in A \cup B$. Observe that $\sum_{u:A\cup B} \alpha_u = 0$. The above constraints imply that $(\alpha_u)_{u \in A\cup B}$ and the incidence vector of $f(S)$ satisfy the constraints of the polytope $\mathcal{P}$. Thus $f(S)$ is a popular half-integral matching. ◀

## 4 Constructing a stable matching in $G^*$

We showed in the previous section that $f$ maps stable matchings in $G^*$ to popular half-integral matchings in $G$. In fact, $f(S)$ is what we will call a *full* half-integral matching, i.e., for every vertex $u \in A \cup B$, either $u$ is fully matched in $f(S)$ or it is fully unmatched in $f(S)$. Let $\vec{p} \in \{0, \frac{1}{2}, 1\}^m$ be a full half-integral matching that is popular. Since $\vec{p} \in \mathcal{P}$, there exists a witness $(\alpha_u)_{u \in A\cup B}$ to $\vec{p}$'s popularity. The following lemma will be useful to us.

▶ **Lemma 3.** *There exists a witness $(\alpha_u)_{u \in A\cup B}$ to $\vec{p}$'s popularity such that $\alpha_u \in \{\pm 1, 0\}$, for each $u \in A \cup B$.*

**Proof.** In order to show such a witness, we will consider the following linear program:

$$\text{minimize} \sum_{u \in A \cup B} \alpha_u \qquad \text{(LP1)}$$

subject to

$$\alpha_a + \alpha_b \;\geq\; \textstyle\sum_{(a,b') \in \tilde{E}(a)} p_{(a,b')} \cdot \mathsf{vote}_a(b, b') + \sum_{(a',b) \in \tilde{E}(b)} p_{(a',b)} \cdot \mathsf{vote}_b(a, a') \;\forall\, (a, b) \in \tilde{E}$$

Recall that $\tilde{E}$ is the set $E \cup \{(u, \ell(u)) : u \in A \cup B\}$, where $\ell(u)$ is the artificial last-resort neighbor of vertex $u$. In the above constraints, let us denote the right hand side quantity corresponding to edge $e$ by $\mathsf{value}_p(e)$. Since $\vec{p}$ is a full half-integral matching, it is easy to see that $\mathsf{value}_p(e)$ is integral for all edges $e$.

Consider the polyhedron defined by the above constraints $N \cdot \vec{\alpha} \geq \vec{c}$, where $N$ is the above $(m + n) \times n$ constraint matrix, $\vec{\alpha}$ is the column of unknowns $\alpha_u$, for $u \in A \cup B$, and $\vec{c}$ is the column vector of $\mathsf{value}_p(\cdot)$ values. The top $m \times n$ sub-matrix of $N$ is the edge-vertex incidence matrix $U$ of the graph $G$ and the bottom $n \times n$ matrix is the identity matrix $I$. Since the graph $G$ is bipartite, the matrix $U$ is totally unimodular and hence the matrix $N$ is totally unimodular. Since $\vec{c}$ is an integral vector, it follows that all the vertices of $N \cdot \vec{\alpha} \geq \vec{c}$ are integral.

Thus there is an integral optimal solution to (LP1), call it $\vec{\alpha}^*$. We need to now show that $\vec{\alpha}^* \in \{\pm 1, 0\}^n$. It follows from the constraints corresponding to the edges $(u, \ell(u))$ that $\alpha^*_u \geq -1$ if $u$ is matched in $\vec{p}$ and $\alpha^*_u \geq 0$ for $u$ unmatched in $\vec{p}$. We now show the following claim.

▶ **Claim 4.** *Let $e = (a, b)$ be any edge such that $p_e > 0$. Then the constraint in (LP1) corresponding to $e$ is tight, i.e., $\alpha^*_a + \alpha^*_b = \mathsf{value}_p(e)$.*

**Proof.** Consider the dual program of (LP1): it is the maximum weight matching problem in the graph $G$ augmented with last-resort neighbors and with edge set $\tilde{E}$, where the weight of edge $e$ is $\mathsf{value}_p(e)$. A maximum weight matching in this graph has weight 0 (because $\vec{p}$ is popular). Since $\Delta(\vec{p}, \vec{p}) = 0$, the fractional matching $\vec{p}$ is an optimal dual solution. It follows from complementary slackness conditions that if $p_{(a,b)} > 0$, then the constraint in (LP1) for edge $(a, b)$ is tight. ◀

Observe that for any vertex $u$, there has to be an edge $e$ incident on it with $p_e > 0$ and either $\mathsf{value}_p(e) = 0$ or $\mathsf{value}_p(e) = -1$ (the edge $e$ between $u$ and its worse partner $v$ in $\vec{p}$). Using Claim 4 and the fact that $\alpha^*_v \geq -1$, we can now conclude that $\alpha^*_u \leq 1$. ◀

We will use the above lemma to show the following theorem in this section.

▶ **Theorem 5.** *Let $\vec{p} \in \{0, \frac{1}{2}, 1\}^m$ be a full half-integral matching that is popular. Then $\vec{p} = f(S)$ for some stable matching $S$ in $G^*$.*

We will now build a stable matching $S$ in the graph $G^*$ such that $f(S) = \vec{p}$. For every edge $e = (a, b)$ such that $p_e > 0$, we need to decide which of the edges $(a_0, b), (a_1, b), (b_0, a), (b_1, a)$ will get included in $S$. In order to make this decision, we will build a graph $H^*$. The graph $H^*$ consists of two copies $H'$ and $H''$ of the input graph $G$.

Every vertex $u \in A \cup B$ gets assigned a level, denoted by $\mathsf{level}'(u)$, in $H'$. For $a \in A$, $\mathsf{level}'(a) = i$ fixes $a_i \in \{a_0, a_1\}$ to be the one that will be matched to a woman (i.e., a non-dummy vertex) in $S$. For $b \in B$, we say $\mathsf{level}'(b) = i$ to fix $b$ getting matched to some level $i$ man in $H'$. We will say $u$ is in level $i$ in $H'$ to mean $\mathsf{level}'(u) = i$.

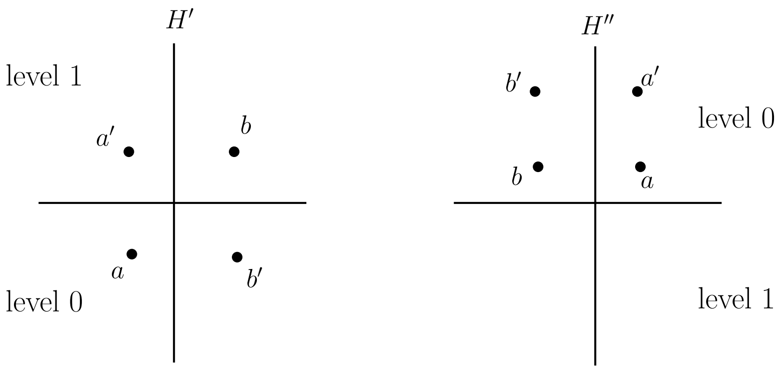


**Figure 4** Since $\alpha^*_a = 1$ and $\alpha^*_{b'} = -1$, we have $\mathsf{level}'(a) = \mathsf{level}'(b') = 0$ and similarly, $\mathsf{level}''(a) = \mathsf{level}''(b') = 0$. Since $\alpha^*_b = \alpha^*_{a'} = 0$, we have $\mathsf{level}'(b) = \mathsf{level}'(a') = 1$ and similarly, $\mathsf{level}''(b) = \mathsf{level}''(a') = 0$. So we place $a$ and $b'$ in level 0 in both $H$ and $H'$ and we place $a'$ and $b$ in level 1 in $H'$ and in level 0 in $H''$.

Similarly, every vertex $u \in A \cup B$ gets assigned a level, denoted by $\mathsf{level}''(u)$, in $H''$. For $b \in B$, $\mathsf{level}''(b) = j$ fixes $b_j \in \{b_0, b_1\}$ to be the one that will be matched to a man (i.e., a non-dummy vertex) in $S$. For $a \in A$, we say $\mathsf{level}''(a) = j$ to fix $a$ getting matched to some level $j$ woman in $H''$. We will say $u$ is in level $j$ in $H''$ to mean $\mathsf{level}''(u) = j$.

Since $\vec{p}$ is a full half-integral matching that is popular, we know from Lemma 3 that there exists a witness $\vec{\alpha}^* = (\alpha^*_u)_{u \in A \cup B}$ in $\{-1, 0, 1\}^n$ to the popularity of $\vec{p}$. We will use $\vec{\alpha}^*$ to fix $\mathsf{level}'(u)$ and $\mathsf{level}''(u)$ for each vertex $u$ as follows.

- $\alpha^*_u = -1$: If $u \in A$ then $\mathsf{level}'(u) = \mathsf{level}''(u) = 1$. If $u \in B$ then $\mathsf{level}'(u) = \mathsf{level}''(u) = 0$.
- $\alpha^*_u = 1$: If $u \in A$ then $\mathsf{level}'(u) = \mathsf{level}''(u) = 0$. If $u \in B$ then $\mathsf{level}'(u) = \mathsf{level}''(u) = 1$.
- $\alpha^*_u = 0$: For all $u \in A \cup B$, $\mathsf{level}'(u) = 1$ and $\mathsf{level}''(u) = 0$.

As an example, consider the 4-cycle $G$ on 2 men $a, a'$ and 2 women $b, b'$ where both $a$ and $a'$ prefer $b$ to $b'$ and both $b$ and $b'$ prefer $a$ to $a'$. Let $\vec{p}$ be the half-integral matching with $p_e = 1/2$ for each edge $e$. This is popular and $\alpha^*_a = 1$, $\alpha^*_b = \alpha^*_{a'} = 0$, and $\alpha^*_{b'} = -1$ is a witness to $\vec{p}$'s popularity. Figure 4 shows how these vertices get placed in $H'$ and in $H''$.

For any vertex $u$, let $v$ and $v'$ be its neighbors in $G$ such that $\vec{p}$ has positive support on $(u, v)$ and $(u, v')$. We will refer to $v$ and $v'$ as *partners* of $u$ in $\vec{p}$. We need to show that either (i) $\mathsf{level}'(u) = \mathsf{level}'(v)$ and $\mathsf{level}''(u) = \mathsf{level}''(v')$, or (ii) $\mathsf{level}'(u) = \mathsf{level}'(v')$ and $\mathsf{level}''(u) = \mathsf{level}''(v)$. In other words, we need to show that $u, v$ are level-compatible in one of $H', H''$ and $u, v'$ are level-compatible in the other graph in $H', H''$.

We will now show that our allocation of levels to men and women based on their $\alpha^*$-values ensures this. If $v = v'$ then $p_{(u,v)} = 1$ and the (tight) constraint for edge $(u, v)$ in the description of $\mathcal{P}$ is $\alpha^*_u + \alpha^*_v = 0$. Thus $(\alpha^*_u, \alpha^*_v)$ has to be one of $(1, -1), (0, 0), (-1, 1)$: in all three cases we have level-compatibility in both $H'$ and $H''$. The following lemma shows that even when $u$ has two distinct partners $v$ and $v'$ in $\vec{p}$, there is level-compatibility.

▶ **Lemma 6.** *Every vertex that has two distinct partners in $\vec{p}$ is level-compatible in $H'$ with one partner and is level-compatible in $H''$ with another partner.*

**Proof.** We will show this lemma for any vertex $b \in B$. An analogous proof holds for any vertex in $A$. Let $a \neq a'$ be the partners of $b$ in $\vec{p}$ and let $b$ prefer $a$ to $a'$. We know that $p_{(a,b)} = p_{(a',b)} = 1/2$. Since $\vec{p}$ is a full half-integral matching, $a$ (similarly, $a'$) has another neighbor $r(a)$ (resp., $r(a')$) with positive support in $\vec{p}$. We have four cases depending on how $a$ and $a'$ rank $b$ versus $r(a)$ and $r(a')$, respectively.

1. If both $a$ and $a'$ prefer $r(a)$ and $r(a')$ respectively to $b$, then $\mathsf{value}_p(a,b) = -\frac{1}{2}+\frac{1}{2} = 0$ and $\mathsf{value}_p(a',b) = -\frac{1}{2}-\frac{1}{2} = -1$. By Claim 4, we know that $\alpha^*_a + \alpha^*_b = 0$ and $\alpha^*_{a'} + \alpha^*_b = -1$. So $(\alpha^*_a, \alpha^*_b, \alpha^*_{a'})$ is either $(1,-1,0)$ or $(0,0,-1)$.
   - In the former case $\mathsf{level}'(a) = \mathsf{level}'(b) = 0$ and $\mathsf{level}''(a') = \mathsf{level}''(b) = 0$.
   - In the latter case $\mathsf{level}'(a') = \mathsf{level}'(b) = 1$ and $\mathsf{level}''(a) = \mathsf{level}''(b) = 0$.
2. If both $a$ and $a'$ prefer $b$ to $r(a)$ and $r(a')$ respectively, then $\mathsf{value}_p(a,b) = \frac{1}{2}+\frac{1}{2} = 1$ and $\mathsf{value}_p(a',b) = \frac{1}{2}-\frac{1}{2} = 0$. By Claim 4, we know that $\alpha^*_a + \alpha^*_b = 1$ and $\alpha^*_{a'} + \alpha^*_b = 0$. So $(\alpha^*_a, \alpha^*_b, \alpha^*_{a'})$ is either $(0,1,-1)$ or $(1,0,0)$.
   - In the former case $\mathsf{level}'(a) = \mathsf{level}'(b) = 1$ and $\mathsf{level}''(a') = \mathsf{level}''(b) = 1$.
   - In the latter case $\mathsf{level}'(a') = \mathsf{level}'(b) = 1$ and $\mathsf{level}''(a) = \mathsf{level}''(b) = 0$.
3. If $a$ prefers $b$ to $r(a)$ while $a'$ prefers $r(a')$ to $b$, then $\mathsf{value}_p(a,b) = \frac{1}{2}+\frac{1}{2} = 1$ and $\mathsf{value}_p(a',b) = -\frac{1}{2}-\frac{1}{2} = -1$. By Claim 4, we know that $\alpha^*_a + \alpha^*_b = 1$ and $\alpha^*_{a'} + \alpha^*_b = -1$. So $(\alpha^*_a, \alpha^*_b, \alpha^*_{a'})$ is $(1,0,-1)$.
   - Here $\mathsf{level}'(a') = \mathsf{level}'(b) = 1$ and $\mathsf{level}''(a) = \mathsf{level}''(b) = 0$.
4. If $a$ prefers $r(a)$ to $b$ while $a'$ prefers $b$ to $r(a')$, then $\mathsf{value}_p(a,b) = -\frac{1}{2}+\frac{1}{2} = 0$ and $\mathsf{value}_p(a',b) = \frac{1}{2}-\frac{1}{2} = 0$. By Claim 4, we know that $\alpha^*_a + \alpha^*_b = 0$ and $\alpha^*_{a'} + \alpha^*_b = 0$. So $(\alpha^*_a, \alpha^*_b, \alpha^*_{a'})$ is $(1,-1,1)$ or $(0,0,0)$ or $(-1,1,-1)$.
   - In the first case, all three vertices $a, b$, and $a'$ are in level 0 in both $H'$ and $H''$.
   - In the second case, all three vertices are in level 1 in $H'$ and in level 0 in $H''$.
   - In the third case, all three vertices are in level 1 in both $H'$ and $H''$. ◀

For any vertex $u$ with partners $v$ and $v'$ in $\vec{p}$, where $u$ prefers $v$ to $v'$, we call $v$ the *better partner* of $u$ and $v'$ the *worse partner* of $u$. If $p_{(a,b)} = 1$ for some edge $(a,b)$, then we regard $a$ as both the better partner and the worse partner of $b$.

We are now ready to describe the construction of our matching $S$. We give the following two pairing rules for any $b \in B$ (let $a$ be $b$'s better partner and $a'$ be $b$'s worse partner):

1. if $\alpha^*_b \in \{\pm 1\}$ then pair $b$ with $a$ in $H'$ and with $a'$ in $H''$.
2. if $\alpha^*_b = 0$ then pair $b$ with $a'$ in $H'$ and with $a$ in $H''$.

More precisely, if $\alpha^*_b = -1$ then we include $(a_0, b)$ and $(b_0, a')$ in $S$; if $\alpha^*_b = 1$ then we include $(a_1, b)$ and $(b_1, a')$ in $S$; and if $\alpha^*_b = 0$ then we include $(a'_1, b)$ and $(b_0, a)$ in $S$.

Note that the above rules for pairing vertices follow from the proof of Lemma 6. A woman $b$ with $\alpha^*_b = -1$ (similarly, $\alpha^*_b = 1$) is level-compatible with her better partner in level 0 (resp., level 1) in $H'$ and with her worse partner in level 0 (resp., level 1) in $H''$. Similarly, if $\alpha^*_b = 0$ then $b$ is level-compatible with her worse partner in level 1 in $H'$ and with her better partner in level 0 in $H''$.

Thus level-compatibility unambiguously fixes for us in which of $H, H'$ a vertex gets paired with which partner till we are left with a set $T$ of vertices forming a cycle: each vertex in $T$ has both its partners in $T$, and all these vertices are in the same level in both $H'$ and $H''$. We again know from the proof of Lemma 6 that this happens only when $(\alpha^*_a, \alpha^*_b) \in \{(1,-1),(0,0),(-1,1)\}$ for each edge $(a,b)$ in this cycle. The cycle can be resolved as per the two rules above (which is what our algorithm for constructing $S$ does). Thus rule 1 and rule 2 given above always work.

As the last step, we add the dummy vertices to $H'$ and $H''$. We also add the *inactive* men and women (the ones who will get matched to dummy vertices in $S$). We now add to $S$ the edges $(a_j, d(a))$ for all inactive men $a_j$ and similarly, the edges $(b_j, d(b))$ for all inactive women $b_j$. We also add self-loops to match each unmatched vertex with its copy on the other side, i.e., we add the edges $(a_1, a)$ for each $a \in A$ that is unmatched in $\vec{p}$ and the edges $(b_0, b)$

for each $b \in B$ that is unmatched in $\vec{p}$. Thus the final matching $S$ is a perfect matching in the graph $G^*$ and it follows from the construction of $S$ that $f(S) = \vec{p}$.

In order to prove that the matching $S$ is stable in $G^*$, we show in Lemmas 7 and 8 that $S$ has no blocking edge in $G'$. We can similarly show that $S$ admits no blocking edge in $G''$. Regarding the other edges in $G^*$, no self-loop $(a_1, a)$ or $(b_0, b)$ can be a blocking edge since $a$ is the least preferred neighbor of $a_1$ and similarly, $b$ is the least preferred neighbor of $b_0$. Similarly, since the dummy vertex $d(a)$ is the least preferred neighbor of $a_0$ and since $a_1$ is the least preferred neighbor of $d(a)$, no edge $(a_i, d(a))$ can block $S$. It is the same with edges $(b_i, d(b))$, for $i = 0, 1$. Hence $S$ is a stable matching in $G^*$ and Theorem 5 follows.

▶ **Lemma 7.** *Let $a \in A$ be in level 0 in $H'$ and $b$ be any neighbor of $a$ in $G$. Neither edge $(a_0, b)$ nor edge $(a_1, b)$ in $G'$ can block $S$.*

**Proof.** The following are the three cases that we need to consider here and show that none is a blocking edge to $S$:
1. the edge $(a_1, b)$,
2. the edge $(a_0, b)$ where $b$ is in level 1 in $H'$,
3. the edge $(a_0, b)$ where $b$ is in level 0 in $H'$.

Consider Case 1. Since $a$ is in level 0 in $H'$, the vertex $a_1$ is matched to $d(a)$ in $S$. Since $d(a)$ is $a_1$'s most preferred neighbor, it follows that the edge $(a_1, b)$ cannot block $S$ for any neighbor $b$.

Consider Case 2. The woman $b$ is in level 1 and this implies that $S(b)$ is a level 1 vertex in $H'$. Since $b$ prefers any level 1 neighbor to a level 0 neighbor in $G'$, it follows that $b$ prefers $S(b)$ to $a_0$, thus $(a_0, b)$ cannot block $S$.

Consider Case 3. Since both $a$ and $b$ are in level 0 in $H'$, we have $\alpha^*_a = 1$ and $\alpha^*_b = -1$. These $\alpha^*$-values and $p_{(a,b)}$ satisfy the constraint corresponding to edge $(a, b)$ in the description of the popular matching polytope $\mathcal{P}$. Thus we have $0 \geq \mathsf{value}_p(a, b)$, where $\mathsf{value}_p(a, b)$ is the right hand side of the constraint for $(a, b)$ in $\mathcal{P}$. The following sub-cases can occur here (since $\mathsf{value}_p(a, b) \leq 0$):
- **(i)** both the partners of $a$ are better than $b$ or both the partners of $b$ are better than $a$
- **(ii)** $p_{(a,b)} = 1/2$ and either $a$ regards its other partner better than $b$ or vice-versa
- **(iii)** $a$ has one partner better than $b$ and the other worse than $b$ and similarly, $b$ has one partner better than $a$ and the other worse than $a$

Sub-case (i) is straightforward and it is easy to see that $(a_0, b)$ does not block $S$ here. In sub-cases (ii) and (iii), we know that a woman $b$ with $\alpha^*_b = -1$ gets matched to her better partner in $H'$. Thus in sub-case (ii) either $b$ is matched to $a$ (if $a$ is $b$'s better partner) or to a partner that $b$ prefers to $a$. Similarly, in sub-case (iii), $b$ gets matched to a neighbor that she prefers to $a$, thus $(a_0, b)$ does not block $S$ in any of these cases. This completes the proof of Lemma 7. ◀

▶ **Lemma 8.** *Let $a \in A$ be in level 1 in $H'$ and $b$ be any neighbor of $a$ in $G$. Neither edge $(a_0, b)$ nor edge $(a_1, b)$ in $G'$ can block $S$.*

**Proof.** The following are the three cases that we need to consider here and show that none is a blocking edge to $S$:
1. the edge $(a_0, b)$,
2. the edge $(a_1, b)$ where $b$ is in level 0 in $H'$,
3. the edge $(a_1, b)$ where $b$ is in level 1 in $H'$.

Consider Case 1. When $b$ is in level 1 in $H'$, she is matched to a level 1 man; since $b$ prefers any level 1 neighbor to a level 0 neighbor in $G'$, it follows that $b$ prefers $S(b)$ to $a_0$, thus $(a_0, b)$ cannot block $S$.

Let us consider the case when $b$ is in level 0 in $H'$. So $\alpha^*_b = -1$. Since $a$ is in level 1 in $H'$, we have either $\alpha^*_a = -1$ or $\alpha^*_a = 0$. So $\mathsf{value}_p(a, b) \le -1$. Hence $b$ prefers her better partner to $a$ and since $b$ satisfies $\alpha^*_b = -1$, she gets matched to her better partner in $H'$. Thus $(a_0, b)$ does not block $S$.

Consider Case 2. We will show that $a_1$ prefers his partner $S(a_1)$ to $b$. Either (i) $\alpha^*_a = -1$ in which case $\mathsf{value}_p(a, b) \le -2$ or (ii) $\alpha^*_a = 0$ in which case $\mathsf{value}_p(a, b) \le -1$.

In case (i), $\mathsf{vote}_a(b, S(a_1)) = -1$ and so $a$ prefers $S(a_1)$ to $b$. In case (ii), $\mathsf{vote}_a(b, S(a_1)) \le 0$ and so $a$ prefers his better partner in $\vec{p}$ to $b$. It follows from the proof of Lemma 6 that if $\alpha^*_a = 0$, then the man $a$ is matched to his better partner in $H'$. Thus $(a_1, b)$ does not block $S$ in either case.

Consider Case 3. There are four sub-cases here based on possible values of $(\alpha^*_a, \alpha^*_b)$: (i) $(\alpha^*_a, \alpha^*_b) = (-1, 1)$, (ii) $(\alpha^*_a, \alpha^*_b) = (-1, 0)$, (iii) $(\alpha^*_a, \alpha^*_b) = (0, 1)$, and (iv) $(\alpha^*_a, \alpha^*_b) = (0, 0)$.

- Cases (i) and (iv) are analogous to case 3 in the proof of Lemma 7 since $\mathsf{value}_p(a, b)$ is at most 0 in both these cases and a similar proof holds here for both these cases.
- In case (ii) above, we have $\mathsf{value}_p(a, b) \le -1$. So either (I) $a$ prefers both his partners in $\vec{p}$ to $b$ or vice-versa, in which case $(a_1, b)$ does not block $S$ or (II) $p_{(a,b)} = 1/2$ and both $a$ and $b$ prefer their other partners in $\vec{p}$ to each other, in which case $(a_1, b) \in S$.
- In case (iii) above, we know that both $a$ and $b$ get paired to their respective better partners in $H'$ (since $\alpha^*_a = 0$ and $\alpha^*_b = 1$). We have $\mathsf{value}_p(a, b) \le 1$ here. So either (I) $a$ prefers its better partner in $\vec{p}$ to $b$ or vice-versa (in which case $(a_1, b)$ does not block $S$) or (II) $p_{(a,b)} = 1/2$ and both $a$ and $b$ prefer each other to their other partners in $\vec{p}$, in which case $(a_1, b) \in S$. Thus $(a_1, b)$ does not block $S$ in any of these cases. ◀

Thus we have shown that $f$ is a surjective map from the set of stable matchings in $G^*$ to the set of full half-integral matchings in $G$ that are popular. It can be shown that if $\vec{p}$ is a popular half-integral matching that is *not* full, then the edge incidence vector of $\vec{p}$ is a convex combination of the edge incidence vectors of popular half-integral matchings that are full. Hence the extreme points of the convex hull $\mathcal{Q}$ of popular half-integral matchings are the full ones. Thus the description of $\mathcal{Q}$ can be obtained in a straightforward manner from the description of the stable matching polytope of $G^*$.

We have shown the following theorem.

▶ **Theorem 9.** *A min-cost popular half-integral matching in $G = (A \cup B, E)$ with strict preference lists and cost function $c : E \to \mathbb{Q}$ can be computed in polynomial time.*

**Conclusions.** We gave a simple description of the convex hull of popular half-integral matchings in a stable marriage instance $G = (A \cup B, E)$ with strict preference lists. This allowed us to solve the min-cost popular half-integral matching problem in $G$ in polynomial time. The main open problem here is to settle the complexity of the min-cost popular matching in $G$.

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