On the Size and the Approximability of Minimum Temporally Connected Subgraphs*

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Abstract

We consider *temporal graphs* with discrete time labels and investigate the size and the approximability of minimum temporally connected spanning subgraphs. We present a family of minimally connected temporal graphs with n vertices and $\Omega(n^2)$ edges, thus resolving an open question of (Kempe, Kleinberg, Kumar, JCSS 64, 2002) about the existence of sparse temporal connectivity certificates. Next, we consider the problem of computing a minimum weight subset of temporal edges that preserve connectivity of a given temporal graph either from a given vertex r (r-MTC problem) or among all vertex pairs (MTC problem). We show that the approximability of r-MTC is closely related to the approximability of Directed Steiner Tree and that r-MTC can be solved in polynomial time if the underlying graph has bounded treewidth. We also show that the best approximation ratio for MTC is at least $O(2^{\log^{1-\varepsilon} n})$ and at most $O(\min\{n^{1+\varepsilon}, (\Delta M)^{2/3+\varepsilon}\})$, for any constant $\varepsilon > 0$, where M is the number of temporal edges and Δ is the maximum degree of the underlying graph. Furthermore, we prove that the unweighted version of MTC is APX-hard and that MTC is efficiently solvable in trees and 2-approximable in cycles.

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1 Introduction

Graphs and networks are ubiquitous in Computer Science, as they provide a natural and useful abstraction of many complex systems (e.g., transportation and communication networks) and processes (e.g., information spreading, epidemics, routing), and also of the interaction between individual entities or particles (e.g., social networks, chemical and biological networks). Traditional graph theoretic models assume that the structure of the network and the strength of interaction are time-invariant. However, as observed in e.g., [3, 19], in many applications of graph theoretic models, the availability and the weights of the edges are actually time-dependent. For instance, one may think of information spreading and distributed computation in dynamic networks (see e.g., [6, 12, 19, 20]), of mobile adhoc and sensor networks (see e.g.,

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[22]), of transportation networks and route planning (see e.g., [3, 15]), of epidemics, biological and ecological networks (see e.g., [17, 19]), and of influence systems and coevolutionary opinion formation (see e.g., [5, 9]).

Several variants of time-dependent graphs have been suggested as abstractions of such settings and computational problems (see e.g., [6] and the references therein). No matter the particular variant, the main research questions are usually related either to optimizing or exploiting temporal connectivity or to computing short time-respecting paths (see e.g., [1, 2, 3, 13, 15, 19, 21]). In this work, we adopt the simple and natural model of temporal graphs with discrete time labels [19] (and its extension with multiple labels per edge [21]), and study the existence of dense minimally connected temporal graphs and the approximability of temporally connected spanning subgraphs with minimum total weight.

Temporal Graphs and Temporal Connectivity. A *temporal graph* is defined on a time-invariant set of n vertices. Each (undirected) edge e is associated with a set of discrete time labels denoting when e is available. If every edge is associated with a single time label, as in [19], the temporal graph is *simple*. An edge e available at time t comprises a temporal edge (e, t) and there is a positive weight $w(e, t)$ associated with it. A (resp. strict) *temporal* (or *time-respecting*) path is a sequence of temporal edges with non-decreasing (resp. increasing) time labels. So, temporal paths respect the time availability constraints of the edges.

Given a source vertex r, a temporal graph is (temporally) *r-connected* if there is a temporal path from r to any other vertex. A temporal graph is (temporally) *connected* if there exists a temporal path from any vertex to any other vertex. We study the existence of dense minimally connected temporal graphs and the optimization problems of computing a minimum weight subset of temporal edges that preserve either r-connectivity or connectivity. We refer to these optimization problems as (Minimum) Single-Source Temporal Connectivity (or r-MTC, in short) and (Minimum) All-Pairs Temporal Connectivity (or MTC, in short). They arise as natural generalizations of Minimum Arborescence and Minimum Spanning Tree in temporal networks, and to the best of our knowledge, their approximability has not been determined so far (but see [1, 18] for some results on variants or special cases).

Previous Work and Motivation. The model of simple temporal graphs with discrete time labels was introduced in [19]. It is essentially equivalent to the model of scheduled networks [3], where each edge is available in a time interval. [3, 19] investigated how time availability restrictions on the edges affect certain graph properties. Berman [3] presented an algorithm for reachability by temporal paths and proved that an analogue of the max-flow-min-cut theorem holds for temporal graphs. Kempe et al. [19] focused on vertex-disjoint temporal paths and showed that Menger's theorem does not generalize to temporal graphs. They also identified a simple forbidden topological minor for Menger's theorem in temporal graphs. Mertzios at al. [21] introduced multiple labels per edge and studied the number of temporal edges required for a temporal design to guarantee certain graph properties. Interestingly, they proved that a variant of Menger's theorem, which also takes time into account, holds in all temporal graphs. A key technical tool in [3, 19, 21] is the time-expanded version of a temporal graph, which reduces reachability, edge-disjoint path and vertex-disjoint path questions in temporal graphs to similar questions in standard directed graphs.

Our motivation comes from a natural open question in [19, Section 6]. Attempting an analogy between spanning trees of (standard undirected) graphs and connectivity certificates of temporal graphs, Kempe et al. asked whether any simple temporal graph admits a sparse connectivity certificate. They observed that any r-connected temporal graph has a time-respecting arborescence with $n-1$ edges that serves as a sparse r-connectivity certificate. For all-pairs temporal connectivity, however, minimum temporal connectivity certificates may have different sizes. Kempe et al. observed that an allocation of time labels to the edges of the hypercube makes it minimally temporally connected. Hence, there are temporal graphs on n vertices with temporal connectivity certificates of $\Omega(n \log n)$ edges. An open question in [19, Section 6] was to determine the tightest function $c(n)$ for which any temporally connected graph on n vertices has a temporal connectivity certificate with at most $c(n)$ edges. A trivial upper bound on $c(n)$ is $O(n^2)$, since taking n time-respecting arborescences, each rooted at a different vertex, results in a temporally connected subgraph. Kempe et al. observed that if we consider strict temporal paths and allow for the same time label at different edges, $c(n) = \Omega(n^2)$ (e.g., consider K_n with the same time label on all edges). Nevertheless, for connectivity with strict temporal paths and distinct time labels, the best known lower bound on $c(n)$ is $\Omega(n \log n)$ ([1], again by a labeling of the hypercube).

Contribution. In this work, we resolve the open question of [19] and derive upper and lower bounds on the approximability of Single-Source and All-Pairs Temporal Connectivity.

In Section 3, we construct a family of simple temporal graphs with $3n$ vertices and roughly $n(n+9)/2$ edges which are almost minimally temporally connected, in the sense that the removal of any subset of $5n$ edges results in a disconnected temporal graph [1] (Theorem 1). Hence, we show that $c(n) = \Theta(n^2)$ (i.e., there are graphs with dense minimum temporal connectivity certificates), thus resolving the open question of [19]. Our construction is essentially best possible and can be easily extended to connectivity by strict temporal paths (with distinct time labels on the edges). An interesting feature of our construction (and an indication of its tightness) is that slightly increasing the time label of a single temporal edge results in a temporal connectivity certificate with $O(n)$ edges!

Given the huge gap on the size of temporal connectivity certificates, it is natural to ask about the complexity and the approximability of Single-Source and All-Pairs Temporal Connectivity. Previous work shows that we can decide if a temporal graph is connected in polynomial time (see e.g., [1, 3, 19]) and that Single-Source Temporal Connectivity can be solved in polynomial time in the unweighted case. Another interesting observation is that if we use the time-expanded version of a temporal graph for Minimum Temporal Connectivity, the resulting optimization problems are quite similar to Group Steiner Tree problems. In fact, this observation serves as the main intuition behind several of our results.

In Section 4, we show that the polynomial-time approximability of Single-Source Temporal Connectivity (r-MTC) is closely related to the approximability of the classical Directed Steiner Tree problem. Using a transformation from Directed Steiner Tree to r-MTC (Theorem 2) and [16, Theorem 1.2], we show that r-MTC cannot be approximated within a ratio of $O(\log^{2-\varepsilon} n)$, for any constant $\varepsilon > 0$, unless $\text{NP} \subseteq \text{ZTIME}(n^{\text{poly}\log n})$. Our transformation also implies that any $o(n^\varepsilon)$-approximation for r-MTC would improve the best known approximation ratio of Directed Steiner Tree. On the positive side, using a transformation from r-MTC to Directed Steiner Tree and the algorithm of [7], we obtain a polynomial-time $O(n^\varepsilon)$-approximation, for any constant $\varepsilon > 0$, and a quasipolynomial-time $O(\log^3 n)$-approximation for r-MTC (Theorem 3). We also show that r-MTC can be solved in polynomial time if the underlying graph has bounded treewidth (Theorem 4).

[1] Based on Theorem 1, we can easily obtain a family of minimally connected temporal graphs with $\Omega(n^2)$ edges (e.g., we remove temporal edges from the graph, as long as connectivity is preserved). For simplicity and clarity, we avoid presenting a tight (but more complicated) construction of dense minimally connected temporal graphs, and stick to almost minimal graphs in the proof of Theorem 1.

In Section 5, we consider the approximability of All-Pairs Temporal Connectivity (MTC). Theorem 3 implies an $O(n^{1+\varepsilon})$-approximation for MTC (Corollary 5). An approximation-preserving reduction to Directed Steiner Forest and [14, Theorem 1.1] imply a polynomial-time $O((\Delta M)^{2/3+\varepsilon})$-approximation for MTC, where M is the number of temporal edges and Δ is the maximum degree of the underlying graph (Theorem 6). If M is quasilinear and Δ is polylogarithmic, we obtain an $O(n^{2/3+\varepsilon})$-approximation. On the negative side, a reduction from Symmetric Label Cover implies that MTC cannot be approximated within a factor of $O(2^{\log^{1-\varepsilon} n})$ unless NP $\subseteq$ DTIME$(n^{\text{poly}\log n})$ (Theorem 7, see also [10, Section 4]). We also show that the unweighted version of MTC is APX-hard (Theorem 8).

In Section 6, we show that MTC can be solved optimally, in polynomial time, if the underlying graph is a tree (Theorem 9), and that MTC is 2-approximable if the underlying graph is a cycle (Theorem 10, but it is open whether MTC remains NP-hard for cycles).

For clarity, we focus on connectivity by (non-strict) temporal paths. However, all our results can be extended (with small changes in the proofs and with the same approximation guarantees and running times) to the case of connectivity by strict temporal paths.

Comparison to Previous Work. Akrida et al. [1] study connectivity by strict temporal paths. Allocating distinct time labels to the hypercube, they obtain a minimal temporally connected graph with $\Omega(n \log n)$ edges. They also show that any allocation of distinct labels to K_n results in a temporal graph that is not minimally connected. However, they do not give any lower bound on the size of temporal connectivity certificates for K_n. Our Theorem 1 improves on the lower bound of [1] from $\Omega(n \log n)$ to $\Omega(n^2)$. [1] also shows that computing the maximum number of edges that are redundant for temporal connectivity is APX-hard.

Huang et al. [18] consider the Single-Source (but not the All-Pairs) version of Minimum Temporal Connectivity in simple scheduled networks [3]. They show that the problem is APX-hard. Using a transformation to Directed Steiner Tree, they show that the approximation guarantees of [7] carry over to Single-Source Temporal Connectivity for scheduled networks. Although the approximation guarantees are the same, the reduction of [18] is slightly different and less general than ours in Theorem 3 (which we discovered independently). In addition to the approximability result, we present strong inapproximability bounds for r-MTC and show that it is polynomially solvable for graphs with bounded treewidth.

Erlebach et al. [13] study the problem of computing a shortest exploration schedule of a temporal graph, i.e., a shortest strict temporal walk that visits all vertices. They prove that it is NP-hard to approximate the shortest exploration schedule within a factor of $O(n^{1-\varepsilon})$, for any $\varepsilon > 0$, and construct temporal graphs whose exploration requires $\Theta(n^2)$ steps. Since the notion of exploration schedules is much stronger than (r-)connectivity, their results do not have any immediate implications for r-MTC and MTC (e.g., the $\Theta(n^2)$-explorable graphs of [13, Lemma 4] admit a temporally connected subgraph with $O(n)$ edges).

2 The Model and Preliminaries

Throughout, we let $[k] \equiv \{1, \dots, k\}$, for any integer $k \geq 1$. An (edge weighted) *temporal graph* $\mathcal{G}(V, E, L)$ with vertex set V, edge set E and lifetime L is a sequence of (undirected edge-weighted) graphs $(G_t(V, E_t, w_t))_{t \in [L]}$, where $E_t \subseteq E$ is the set of edges available at time t and $w_t(e)$ (or $w(e, t)$) is the nonnegative weight of each edge $e \in E_t$. We often write $\mathcal{G}$ or $\mathcal{G}(V, E)$, for brevity. A temporal graph $\mathcal{G}$ is *unweighted* if $w(e, t) = 1$ for all $e \in E_t$ and all $t \in [L]$. For each edge $e \in E_t$, we say that (e, t) is a *temporal edge* of $\mathcal{G}$. For each edge $e \in E$, $L_e = \{t \in [L] : e \in E_t\}$ denotes the set of time units (or *time labels*) when e is available. A temporal graph is *simple* if $|L_e| = 1$ for all edges $e \in E$.

We let n be the number of vertices and $M = \sum_e |L_e|$ be the number of temporal edges of $\mathcal{G}$. For temporal connectivity problems, we can assume that at least one temporal edge is available in each time unit, and thus, $L \leq M$. The (static) graph $G(V, E)$ is the *underlying graph* of $\mathcal{G}$. We say that $\mathcal{G}$ has some (non-temporal) graph theoretic property (e.g., is a tree, a cycle, a clique, has bounded treewidth) if the underlying graph G has this property.

For a vertex set S, $G[S]$ (resp. $\mathcal{G}[S]$) is the underlying (resp. temporal) graph induced by S. A spanning subgraph $\mathcal{G}'$ of a temporal graph $\mathcal{G} = (G_t(V, E_t, w_t))_{t\in[L]}$ is a sequence of graphs $(G'_t(V, E'_t, w_t))_{t\in[L]}$ such that $E'_t \subseteq E_t$. The total weight of $\mathcal{G}'$ is $\sum_{t\in[L]} \sum_{e\in E'_t} w(e, t)$.

A *temporal* (or *time-respecting*) path is an alternating sequence of vertices and temporal edges $(v_1, (e_1, t_1), v_2, (e_2, t_2), \ldots, v_k, (e_k, t_k), v_{k+1})$, such that $e_i = \{v_i, v_{i+1}\} \in E_{t_i}$, for all $i \in [k]$, and $1 \leq t_1 \leq t_2 \leq \cdots \leq t_k$. A temporal path is *strict* if $t_1 < t_2 < \cdots < t_k$. Such a temporal path is from v_1 to v_{k+1} (or a temporal $v_1 - v_{k+1}$ path).

A temporal graph $\mathcal{G}$ is (temporally) *r-connected*, for a given source $r \in V$, if there is a temporal path from r to any vertex $u \in V$. A temporal graph $\mathcal{G}$ is (temporally) *connected*, if there is a temporal path from u to v for any ordered pair $(u, v) \in V \times V$. If all temporal paths are strict, $\mathcal{G}$ is strictly connected (or strictly r-connected). An *(r-)connectivity certificate* of $\mathcal{G}$ is any spanning subgraph of $\mathcal{G}$ that is also (r-)connected.

Given a temporal graph $\mathcal{G}$ and a source vertex r, the problem of (Minimum) *Single-Source Temporal Connectivity* (r-MTC) is to compute a temporally r-connected spanning subgraph of $\mathcal{G}$ with minimum total weight. The optimal solution to r-MTC is a simple temporal graph whose underlying graph is a tree (see e.g., [19, Section 6]). Given a temporal graph $\mathcal{G}$, the problem of (Minimum) *All-Pairs Temporal Connectivity* (MTC) is to compute a temporally connected spanning subgraph of $\mathcal{G}$ with minimum total weight.

An algorithm A has *approximation ratio* $\rho \geq 1$ for a minimization problem, such as Single-Source and All-Pairs Temporal Connectivity, if for any instance I, the cost of A on I is at most ρ times I's optimal cost.

Directed Steiner Tree and Forest. To understand the approximability of r-MTC and MTC, we use reductions from and to Directed Steiner Tree and the Directed Steiner Forest.

Given a directed edge-weighted graph $G(V, E)$ with n vertices, a source $r \in V$ and a set of k terminals $S \subseteq V$, the Directed Steiner Tree (DST) problem asks for a subgraph of G that includes a directed path from r to any vertex in S and has minimum total weight. The best known algorithm for DST is due to Charikar et al. [7] and achieves an approximation ratio of $O(k^\varepsilon)$, for any constant $\varepsilon > 0$, in polynomial time, and of $O(\log^3 k)$ in quasipolynomial time. On the negative side, [16, Theorem 1.2] shows that DST cannot be approximated within a factor $O(\log^{2-\varepsilon} n)$, for any constant $\varepsilon > 0$, unless NP $\subseteq$ ZTIME($n^{\text{poly}\log n}$).

Given a directed edge-weighted graph $G(V, E)$ with n vertices and m edges, and a collection $D \subseteq V \times V$ of k ordered vertex pairs, the Directed Steiner Forest (DSF) problem asks for a subgraph of G that contains an $s - t$ path for each $(s, t) \in D$ and has minimum total weight. [14] presents a polynomial-time $O(n^\varepsilon \min\{n^{4/5}, m^{2/3}\})$-approximation for DSF, for any constant $\varepsilon > 0$.

3 A Lower Bound on the Size of Temporal Connectivity Certificates

In this section, we construct an infinite family of simple temporal graphs with $\Theta(n)$ vertices and lifetime $\Theta(n)$ such that any temporal connectivity certificate has $\Omega(n^2)$ edges. Our construction is essentially best possible, since any temporal graph with n vertices and lifetime L admits a connectivity certificate with $O(\min\{n^2, nL\})$ edges.

▶ **Theorem 1.** *For any even $n \geq 2$, there is a simple connected temporal graph with $3n$ vertices, $n(n+9)/2 - 3$ edges and lifetime at most $7n/2$, so that the removal of any subset of $5n$ edges results in a disconnected temporal graph.*

Proof sketch. For any even n, we construct a simple connected temporal graph $\mathcal{G}$ with $\Theta(n)$ vertices and $\Theta(n^2)$ edges so that virtually any edge is essential for temporal connectivity.

We start with describing the construction. For any even n, $\mathcal{G}$ consists of $3n$ vertices partitioned into three sets $A = \{a_1, \ldots, a_n\}$, $H = \{h_1, \ldots, h_n\}$ and $C = \{c_1, \ldots, c_n\}$, with n vertices each. The underlying graph $G[A]$ is the complete graph K_n and comprises the *dense* part of the construction with $\Theta(n^2)$ edges. The edges of $G[A]$ are partitioned into $n/2$ edge-disjoint paths $p_1, \ldots, p_{n/2}$. Each path p_i has length $n-1$ and spans all vertices in A (see Figure 1.a). All edges of each path p_i have time label i.

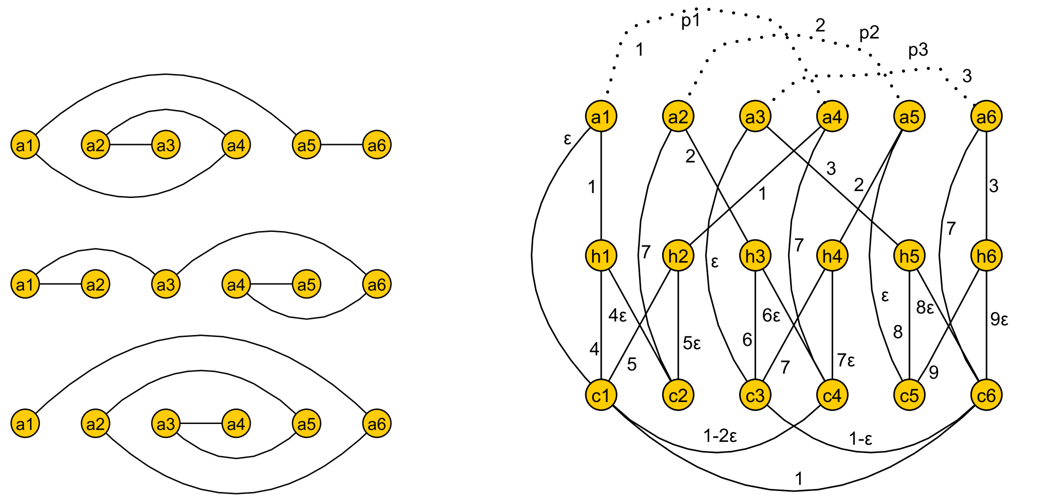
The vertices of H comprise the *intermediate* part of the construction. There are no edges with both endpoints in H. For every $i \in [n/2]$, one endpoint of the path p_i is connected to h_{2i-1} and the other endpoint is connected to h_{2i}. Both edges have time label i.

The vertices of C form the *interconnecting* part of the construction. For each $i \in [n/2]$, we refer to c_{2i-1} (resp. c_{2i}) as the *entry vertex* (resp. the *exit vertex*) for the vertices h_{2i-1} and h_{2i}. There are two edges connecting c_{2i-1} to h_{2i-1} and h_{2i} with labels $n/2 + 2i - 1$ and $n/2+2i$, respectively, and two edges connecting c_{2i} to h_{2i-1} and h_{2i} with labels $(n/2+2i-1)\epsilon$ and $(n/2 + 2i)\epsilon$, respectively, for some fixed $\epsilon \in (0, 1/(4n))$. We also connect the vertices of C to each other. For every $i \in [n/2 - 2]$, there are edges connecting c_{2i-1} to c_{2i+2} and to c_n, and a single edge connecting c_{n-3} to c_n. To allocate time labels to these edges, we order them in decreasing order of their endpoint with higher index, breaking ties by ordering them in increasing order of their endpoint with lower index, i.e., the order is $\{c_1, c_n\}$, $\{c_3, c_n\}, \ldots, \{c_{n-3}, c_n\}$, $\{c_{n-5}, c_{n-2}\}$, $\{c_{n-7}, c_{n-4}\}, \ldots, \{c_1, c_4\}$. The time label of the k-th edge in this order is $1 - (k-1)\epsilon$. Finally, for every $i \in [n/2]$, there are an edge with time label ϵ connecting the vertex c_{2i-1} to the vertex a_{2i-1} in A and an edge with time label $n+1$ connecting the vertex c_{2i} to the vertex a_{2i} in A (see also Figure 1.b).

The total number of edges is $n(n + 9)/2 - 3$, the number of different time labels is at most $7n/2$, and each edge has a single label.

Next, we present the intuition and discuss the main technical claims. The construction is based on the collection $p_1, \ldots, p_{n/2}$ of $n/2$ edge-disjoint paths, where all edges in each path p_i have label i. Extending each path p_i to vertices h_{2i-1} and h_{2i}, we get a path that connects h_{2i} to h_{2i-1} (and vice versa) and to all vertices in A at time i. Moreover, different time labels make these paths essentially independent of each other, in the sense that if a temporal walk begins and ends at time i, it can use only edges with label i (i.e., only edges of this path) to connect h_{2i} to h_{2i-1}. Formalizing this intuition, we can show that the unique temporal path from h_{2i} to h_{2i-1} is through path p_i. Therefore, all edges of $G[A]$ must be present in any temporally connected spanning subgraph of $\mathcal{G}$. To achieve a dense underlying graph $G[A]$, we observe that the collection of $n/2$ edge-disjoint paths can be defined so that they go through the same n vertices, in a different order each (see Figure 1.a). This describes the main intuition behind our construction and explains how the dense and the intermediate parts work. The only problem now is that H-vertices with high indices, e.g., h_n, cannot reach H-vertices with low indices, e.g., h_1. The vertices in the interconnecting part C serve to carefully connect each h_j to each h_i, with $j > i + 1$, without destroying the property that the only temporal path from h_{2i} to h_{2i-1} is through path p_i.

For every vertex pair $h_{2i-1}, h_{2i} \in H$, we introduce a vertex pair $c_{2i-1}, c_{2i} \in C$. As an entry vertex, c_{2i-1} is connected to h_{2i-1} and h_{2i} with "large" labels (larger than $n/2$). Hence, starting from the rest of $\mathcal{G}$, we can reach h_{2i-1} and h_{2i} through c_{2i-1}, but we cannot continue



(a) Partition into 3 Hamiltonian paths. (b) Putting the 3 parts together.

Figure 1 The temporal graph constructed in the proof of Theorem 1 for $n = 6$.

to the edges of p_i (with label $i \leq n/2$). As an exit vertex, c_{2i} is connected to h_{2i-1} and h_{2i} with "very small" labels (at most 1/4). Thus, starting from c_{2i}, we can reach first h_{2i-1} and h_{2i}, and then all vertices in A and any vertex h_j with index $j > 2i$. Moreover, to avoid creating a temporal path from h_{2i} to h_{2i-1}, the label of the edge $\{h_{2i}, c_{2i-1}\}$ (resp. $\{h_{2i}, c_{2i}\}$) is larger than the label of the edge $\{h_{2i-1}, c_{2i-1}\}$ (resp. $\{h_{2i-1}, c_{2i}\}$).

It remains now to connect the C-vertices to each other, without creating any alternative temporal paths from h_{2i} to h_{2i-1}, for any $i \in [n/2]$. For each $i \in [n/2]$, the edges between C-vertices should create temporal paths from c_{2i-1} and c_{2i} to any vertex c_j with index $j < 2i - 1$. On the other hand, they should not create any temporal $c_{2i} - c_{2i-1}$ paths, since then we would have a new temporal $h_{2i} - h_{2i-1}$ path. We introduce roughly n edges between C-vertices and carefully select their "small" labels in $[3/4, 1]$. Furthermore, to achieve temporal connectivity between all vertex pairs, we introduce an edge $\{c_{2i-1}, a_{2i-1}\}$ with the minimum time label ϵ and an edge $\{c_{2i}, a_{2i}\}$ with label $n + 1$, for each $i \in [n/2]$.

To complete the proof, we need to consider all possible types of ordered vertex pairs and to show that the temporal graph $\mathcal{G}$ is indeed connected. Moreover, since any subset of at least $5n$ edges includes some edges of $G[A]$, we can show that the removal of any edge from $G[A]$ with label i destroys the unique temporal path from h_{2i} to h_{2i-1}. ◀

We should highlight that increasing the label of edge $\{a_1, c_1\}$ from ϵ to 1, in the graph of Theorem 1, results in a temporal graph that admits a connectivity certificate of size $\Theta(n)$. Moreover, it is not hard to modify the construction of Theorem 1 so that all time labels of the edges are distinct, the temporal graph $\mathcal{G}$ is connected by strict temporal paths, and the removal of any subset of $5n$ edges results in a disconnected temporal graph. Therefore, the quadratic lower bound of Theorem 1 also applies to connectivity by strict temporal paths and improves on the lower bound of $\Omega(n \log n)$ in [1, Theorem 3].

4 The Approximability of Single-Source Temporal Connectivity

We proceed to study the approximability of Minimum Single-Source Temporal Connectivity. We show that the polynomial-time approximability of r-MTC is closely related to the approximability of the classical Directed Steiner Tree (DST) problem and that r-MTC can be solved in polynomial-time for graphs of bounded treewidth.

4.1 A Lower Bound on the Approximability of r-MTC

We start with an approximation-preserving transformation from DST to r-MTC. The intuition is that we can use strict temporal paths to “simulate” the directed edges of DST.

▶ **Theorem 2.** *Any polynomial-time $\rho(n)$-approximation algorithm for r-MTC on simple temporal graphs implies a polynomial-time $\rho(n^2)$-approximation algorithm for DST.*

Proof sketch. We present an approximation-preserving transformation from the DST to r-MTC. Given an instance $I = (G(V, E, w), S, r)$ of DST with $|V| = n$, we construct a temporal graph $\mathcal{G}'$ with n^2 vertices so that (i) any Steiner tree connecting r to S in G can be mapped to an r-connected subgraph of $\mathcal{G}'$ with no larger cost; and (ii) given any r-connected subgraph of $\mathcal{G}'$, we can efficiently compute a feasible Steiner tree for I with no larger cost.

Each vertex $u \in V$ corresponds to a vertex u in the temporal graph $\mathcal{G}'$. For every directed edge $e = (u, v)$ of G, we create $n - 1$ strict temporal $u - v$ paths of length 2. Specifically, for every $u \in V$, $\mathcal{G}'$ contains auxiliary vertices z_i^u, for all $i \in [n - 1]$, and temporal edges $\{u, z_i^u\}$ with time label i and weight 0. For every edge $e = (u, v) \in E$, $\mathcal{G}'$ contains temporal edges $\{z_i^u, v\}$ with time label $i + 1$ and weight $w(e)$, for all $i \in [n - 1]$. Let $Z = \{z_i^u\}_{u \in V, i \in [n-1]}$ be the set of all auxiliary vertices. For every vertex $x \in Z \cup (V \setminus S)$, $x \neq r$, $\mathcal{G}'$ contains a temporal edge $\{r, x\}$ with time label $n + 1$ and weight 0. These edges ensure that r is connected to all non-terminal and auxiliary vertices at no additional cost. ◀

Directed Steiner Tree cannot be approximated within a ratio of $O(\log^{2-\varepsilon} n)$, for any constant $\varepsilon > 0$, unless NP $\subseteq$ ZTIME$(n^{\text{poly} \log n})$ [16, Theorem 1.2]. Theorem 2 implies that this inapproximability result carries over to r-MTC. Moreover, any polynomial-time $o(n^\varepsilon)$-approximation algorithm for r-MTC would immediately improve the best known approximation ratio of the notoriously difficult DST problem.

4.2 An Approximation Algorithm for r-MTC

The following shows an approximation-preserving reduction from r-MTC to DST (see also the more general proof of Theorem 6). Then, we can use the algorithm of [7] and approximate r-MTC within a ratio of $O(n^\varepsilon)$, for any constant $\varepsilon > 0$, in polynomial time, and within a ratio of $O(\log^3 n)$ in quasipolynomial time.

▶ **Theorem 3.** *Any polynomial-time $\rho(k)$-approximation algorithm for DST implies a polynomial-time $\rho(n)$-approximation algorithm for r-MTC on general temporal graphs.*

4.3 A Polynomial-Time Algorithm for Graphs with Bounded Treewidth

The following shows that r-MTC can be solved in polynomial time, by dynamic programming, if the underlying graph has bounded treewidth (see e.g., [11] about nice tree decompositions and dynamic programming algorithms for graphs of bounded treewidth).

▶ **Theorem 4.** *Let $\mathcal{G}$ be a temporal graph on n vertices with lifetime L, source vertex r and treewidth at most k. Then, there is a dynamic programming algorithm which given a nice tree decomposition of G, computes an optimal solution to r-MTC in time $O(nk^2 3^k (L + k)^{k+1})$.*

5 The Approximability of All-Pairs Minimum Temporal Connectivity

In this section, we study the approximability of the all-pairs version of Minimum Temporal Connectivity in general temporal graphs. Reducing MTC to r-MTC and to Directed Steiner Forest, we obtain polynomial-time approximation algorithms for MTC, albeit with not so strong guarantees (Corollary 5 and Theorem 6). To justify the poor approximation ratios, we reduce Symmetric Label Cover (SLC) to MTC and show that any $\rho(n)$-approximation for MTC implies a $\rho(n^2)$-approximation for SLC (Theorem 7). Moreover, using an approximation-preserving reduction from the Steiner Tree problem, we show that the unweighted version of MTC is APX-hard (Theorem 8).

5.1 Approximation Algorithms for MTC

Using every vertex of the temporal graph as a source vertex and taking the union of the solutions obtained by the algorithm of Theorem 3 for r-MTC, we obtain the following.

▶ **Corollary 5.** *For any constant $\varepsilon > 0$, there is a polynomial-time $O(n^{1+\varepsilon})$-approximation algorithm for* MTC *on temporal graphs with n vertices.*

Next, we present a reduction from MTC to Directed Steiner Forest (DSF) that leads to a different algorithm. Although the approximation ratio may be worse than $O(n^{1+\varepsilon})$ in general, this algorithm gives significantly better guarantees if the total number of temporal edges is quasilinear (and if the maximum degree of the underlying graph is polylogarithmic).

▶ **Theorem 6.** *Let $\mathcal{G}$ be a temporal graph with n vertices and M temporal edges such that the underlying graph has maximum degree Δ. Then, for any constant $\varepsilon > 0$, there is a polynomial-time $O(M^\varepsilon \min\{M^{4/5}, (\Delta M)^{2/3}\})$-approximation algorithm for* MTC *on $\mathcal{G}$. If $M = O(n \operatorname{poly}\log n)$, we obtain an approximation ratio of $O(n^{4/5+\varepsilon})$. If both $M = O(n \operatorname{poly}\log n)$ and $\Delta = O(\operatorname{poly}\log n)$, we obtain an approximation ratio of $O(n^{2/3+\varepsilon})$.*

Proof. The reduction from DSF to MTC is a generalized version of the reduction used in the proof Theorem 3. Let I be an instance of MTC consisting of an underlying graph $G(V, E)$, a finite set of time labels L_e for each edge e, and a weight $w(e, t)$ for any temporal edge (e, t). We show how to transform I into an instance I' of DSF so that (i) any feasible solution of I can be mapped to a feasible solution of I' with no larger total weight; and (ii) given a feasible solution of I', we can compute a feasible solution of I with no larger total weight.

For convenience, we denote H the edge-weighted directed graph of the DSF instance I'. For every temporal edge (e, t) of $\mathcal{G}$, H contains two vertices $h^1_{(e,t)}$ and $h^2_{(e,t)}$. Intuitively, $h^1_{(e,t)}$ indicates that we may use (e, t) and $h^2_{(e,t)}$ indicates that we actually use (e, t). For each edge $e \in E$, let $l_1(e) < l_2(e) < ... < l_k(e)$ be the time labels in L_e. For every $i \in [k-1]$, H contains a directed edge $(h^1_{(e,l_i(e))}, h^1_{(e,l_{i+1}(e))})$ with weight 0. Intuitively, these edges indicate that we can wait and use e at some later time up to $l_k(e)$. Moreover, for every $i \in [k]$, H contains a directed edge $(h^1_{(e,l_i(e))}, h^2_{(e,l_i(e))})$ with weight $w(e, l_i(e))$. This edge indicates that we actually use the temporal edge $(e, l_i(e))$ and incur the corresponding cost.

For every ordered pair of temporal edges (e_1, t_1), (e_2, t_2) of $\mathcal{G}$, such that $e_1 \neq e_2$, t_2 is the smallest time label in L_{e_2} such that $t_2 \geq t_1$ ($t_2 > t_1$, for strict connectivity), and e_1 and e_2 share a common endpoint, H contains a directed edge $(h^2_{(e_1,t_1)}, h^1_{(e_2,t_2)})$ with weight 0.

For every vertex $v_i \in V$, $i \in [n]$, H contains a pair of terminal vertices s_i and t_i. For every temporal edge (e, t) incident to v_i, H contains a directed edge $(s_i, h^1_{(e,t)})$ with weight 0 and a directed edge $(h^2_{(e,t)}, t_i)$ with weight 0. The set of connection requirements of the DSF instance I' consists of all pairs (s_i, t_j) for all $i, j \in [n]$ with $i \neq j$.

By construction, any temporal $v_i - v_j$ path, which consists of a temporal edge sequence $((e_1, t_1), (e_2, t_2), \dots, (e_k, t_k))$, corresponds to a directed $s_i - t_j$ path in H of the form

$$s_i, h^1_{(e_1,t_1)}, h^2_{(e_1,t_1)}, h^1_{(e_2,t''_2)}, h^1_{(e_2,t'_2)}, h^1_{(e_2,t_2)}, h^2_{(e_2,t_2)}, \dots, h^1_{(e_k,t_k)}, h^2_{(e_k,t_k)}, t_j$$

with the same weight and vice versa. Using this observation, we can now establish claims (i) and (ii). Specifically, to show (i), we construct a feasible solution to I' that includes all directed edges of weight 0 and the directed edges $(h^1_{(e,t)}, h^2_{(e,t)})$ corresponding to the temporal edges (e, t) used in the feasible solution to I. Clearly, the two solutions have the same total weight and any temporal $v_i - v_j$ path in the solution to I corresponds to an $s_i - t_j$ path in the solution to I'. To show (ii), we first observe that any directed path from some s_i to some t_j should include some directed edges of the form $(h^1_{(e,t)}, h^2_{(e,t)})$ with weight $w(e, t)$. So, we construct a feasible solution to I that includes the temporal edges (e, t) corresponding to the positive-weight directed edges $(h^1_{(e,t)}, h^2_{(e,t)})$ included in the feasible solution to I'.

In the resulting DSF instance I', the total number of vertices is $O(n + M) = O(M)$ and the number of connection requirements is $O(n^2)$. If the maximum degree of the underlying graph is Δ, the total number of edges is dominated by the edges of the form $(h^2_{(e_1,t_1)}, h^1_{(e_2,t_2)})$, which are $O(\Delta M)$. Applying the approximation algorithm of [14, Theorem 1.1] to the DSF instance I', we obtain a polynomial-time $O(M^\varepsilon \min\{M^{4/5}, (\Delta M)^{2/3}\})$-approximation algorithm, for any constant $\varepsilon > 0$. In the special case where the number of temporal edges is $M = O(n\text{poly}\log n)$, we obtain an $O(n^{4/5+\varepsilon})$-approximation, for any constant $\varepsilon > 0$. If both $M = O(n\text{poly}\log n)$ and the maximum degree of the underlying graph $\Delta = O(\text{poly}\log n)$, we obtain a polynomial-time $O(n^{2/3+\varepsilon})$-approximation algorithm for any constant $\varepsilon > 0$. ◀

5.2 A Lower Bound on the Approximability of MTC

In this section, we present an approximation-preserving reduction from Symmetric Label Cover to MTC. Our reduction along with standard inapproximability results for Symmetric Label Cover indicate that MTC in general temporal networks is hard to approximate.

▶ **Theorem 7.** MTC *on temporal graphs with n vertices cannot be approximated within a factor of $O(2^{\log^{1-\varepsilon} n})$, for any constant $\varepsilon > 0$, unless* NP $\subseteq$ DTIME$(n^{\text{poly}\log n})$.

Proof. We present a polynomial-time approximation-preserving reduction from the Symmetric Label Cover (SLC) problem to MTC. In SLC (see e.g., [10, Definition 4.1]), we are given a complete bipartite graph $H(U, W)$, with $|U| = |W|$, a finite set of colors C and a binary relation $R(u, w) \subseteq C \times C$ for every vertex pair $(u, w) \in U \times W$. We seek to assign a color subset $\sigma(u) \subseteq C$ to each vertex $u \in U \cup W$ so that for every vertex pair $(u, w) \in U \times U$, there are colors $a \in \sigma(u)$ and $b \in \sigma(w)$ with $(a, b) \in R(u, w)$ and $\sum_{u \in U \cup W} |\sigma(u)|$, i.e., the total number of colors used, is minimized.

Given an instance of SLC, we create a temporal graph $\mathcal{G}$ whose vertex set V is partitioned into six sets V_U, $V_{C(U)}$, V_W, $V_{C(W)}$, V_X and $\{p, q\}$. There is a correspondence between the vertices of the bipartite graph H and the vertices of $\mathcal{G}$ in the sets V_U and V_W. The vertex sets $V_{C(U)} = V_U \times C$ and $V_{C(W)} = V_W \times C$ serve to encode the color assignment to the vertices of U and W in the SLC instance. Moreover, V_X contains a vertex (u, w, a, b) for every vertex pair $(u, w) \in U \times W$ and every allowable color pair $(a, b) \in R(u, w)$. Intuitively, the vertices of V_X serve to ensure that the color assignment is consistent. Finally, the vertices p and q ensure that the temporal graph $\mathcal{G}$ is connected.

For every $u \in V_U$ and $(u, a) \in V_{C(U)}$, $\mathcal{G}$ contains a temporal edge $\{u, (u, a)\}$ with label 1 and weight 1. Similarly, for every $w \in V_W$ and $(w, b) \in V_{C(W)}$, $\mathcal{G}$ contains a temporal edge $\{w,(w,b)\}$ with label 4 and weight 1. For every vertex $(u,w,a,b) \in V_X$, $\mathcal{G}$ contains the temporal edges $\{(u,a),(u,w,a,b)\}$ with label 2 and weight 0 and $\{(u,w,a,b),(w,b)\}$ with label 3 and weight 0. $\mathcal{G}$ contains temporal edges with label 5 and weight 0 between p and every vertex in $V_U \cup V_{C(U)} \cup V_{C(W)} \cup V_X$ and between q and every vertex in V_U. Moreover, $\mathcal{G}$ contains temporal edges with label 0 and weight 0 between p and every vertex in V_W and between q and every vertex in $V_W \cup V_{C(U)} \cup V_{C(W)} \cup V_X$. We note that $\mathcal{G}$ is temporally connected and has $O(k^2c^2)$ vertices, where $k = |U| = |W|$ and $c = |C|$ (in fact, the number of vertices of $\mathcal{G}$ is of the same order as the total size of all binary relations $R(u,w)$).

We next show that this reduction is approximation-preserving. We first show that any feasible solution to the SLC instance can be mapped to a temporally connected subgraph $\mathcal{G}'$ of $\mathcal{G}$ with at most the same weight. Let us fix any assignment σ of a color set to each vertex of H that is feasible for the SLC instance. We first include in $\mathcal{G}'$ all temporal edges of weight 0. For every vertex $u \in U$ with assigned colors $\sigma(u)$, we include in $\mathcal{G}'$ the temporal edges $\{u,(u,a)\}$, for all $a \in \sigma(u)$. The total weight of these edges is $|\sigma(u)|$. Similarly, for every vertex $w \in W$, we include in $\mathcal{G}'$ the temporal edges $\{w,(w,b)\}$, for all $b \in \sigma(w)$. The total weight of these edges is $|\sigma(w)|$. Therefore, the total weight of the temporal subgraph $\mathcal{G}'$ is equal to the cost of the solution σ for the SLC problem.

It remains to show that $\mathcal{G}'$ is temporally connected. All vertices in $V_U \cup V_{C(U)} \cup V_{C(W)} \cup V_X \cup \{p\}$ are connected with each other (through p) by temporal edges with time label 5. There are also temporal $p-q$ and $q-p$ paths consisting of edges with time label 5 through the vertices of V_U. Similarly, all vertices in $V_W \cup V_{C(U)} \cup V_{C(W)} \cup V_X \cup \{q\}$ are connected with each other (through q) by temporal edges with time label 0. Moreover, there is a temporal path (using edges with time labels 0 and 5) from every vertex in $V_W \cup V_{C(U)} \cup V_{C(W)} \cup V_X \cup \{q\}$ to every vertex in V_U. Also, p is connected to every vertex in V_W with temporal edges of time label 0 and vice versa. All these vertex pairs are connected through temporal paths entirely consisting of 0-weight edges. The really interesting case concerns vertex pairs $(u,w) \in V_U \times V_W$. By the feasibility of the solution σ, for every vertex pair $(u,w) \in U \times W$, there are colors $a \in \sigma(u)$ and $b \in \sigma(w)$ such that $(a,b) \in R(u,w)$. Therefore, the temporal $u-w$ path $(u,(u,a),(u,w,a,b),(w,b),b)$ is included in $\mathcal{G}'$. Hence, $\mathcal{G}'$ is temporally connected.

We also need to show that given a temporally connected subgraph $\mathcal{G}'$ of $\mathcal{G}$, we can efficiently compute an assignment σ of a color set to each vertex in $U \cup W$ that is feasible for the SLC instance and has total cost no larger than the total weight of $\mathcal{G}'$. For every $u \in V_U$ and every temporal edge of the form $(\{u,(u,a)\},1)$ included in $\mathcal{G}'$, we include the color a in $\sigma(u)$. Similarly, for every $w \in V_W$ and every temporal edge of the form $(\{w,(w,b)\},4)$ included in $\mathcal{G}'$, we include the color b in $\sigma(w)$. Since these are the only edges of $\mathcal{G}$ (and $\mathcal{G}'$) with positive weight, the total cost of σ is equal to the total weight of $\mathcal{G}'$.

It remains to show that σ is a feasible solution to the SLC instance. Let $(u,w) \in U \times W$ in the SLC instance. The crucial observation is that the only way to connect $u \in V_U$ to $w \in V_W$ in $\mathcal{G}'$ is through some temporal path $(\{u,(u,a)\},1)$, $(\{(u,a),(u,w,a,b)\},2)$, $(\{(w,b),(u,w,a,b)\},3)$, $(\{w,(w,b)\},4)$, for $(a,b) \in R(u,w)$. This claim immediately implies the feasibility of the assignment σ. To prove this claim, we observe that a temporal $u-w$ path cannot use any temporal edge incident to p or q, since all edges between V_U and $\{p,q\}$ have time label 5 and all edges between V_W and $\{p,q\}$ have time label 0. So, any temporal $u-w$ path in $\mathcal{G}'$ has to move from u to some vertex $(u,a) \in V_{C(U)}$. Such a vertex $(u,a) \in V_{C(U)}$ does not have any neighbors in V_U other than u. Hence, the next vertex of any temporal $u-w$ path in $\mathcal{G}'$ must be to visit some $(u,w,a,b) \in V_X$, where $w \in W$ and $(a,b) \in R(u,w)$. Similarly, since such a vertex $(u,w,a,b) \in V_X$ does not have any neighbors in $V_{C(U)}$ other than (u,a) and any neighbors in $V_{C(W)}$ other than (w,b), we conclude that the next step of any temporal $u-w$ path in $\mathcal{G}'$ must be to the vertex $(w,b) \in V_{C(W)}$. But now, the last temporal edge used has time label 3, which implies that the $u-w$ path cannot use any edges with time labels 1 and 2 anymore. Thus, the path cannot return to $V_U \cup V_{C(U)}$. The only choice now is that the path moves to w through the temporal edge $(\{w,(w,b)\},4)$, which establishes the claim about the structure of any temporal $u-v$ path in $\mathcal{G}'$.

The discussion above establishes the correctness of the reduction from SLC to MTC. Using the fact that the number of vertices of $\mathcal{G}$ is quadratic in the number of vertices of H and standard inapproximability results for SLC (e.g., [10]), we conclude the proof. ◀

Adjusting the proof of Theorem 7, we can get a reduction from the MinRep problem, which is considered in [8], to MTC. Thus, any polynomial-time $\rho(n)$-approximation for MTC on simple temporal graphs implies a polynomial-time $\rho(n^2)$-approximation for MinRep. Since the best known approximation ratio for MinRep is $O(n^{1/3}\log^{2/3} n)$ [8, Section 2], any $O(n^{1/6})$-approximation to MTC would imply an improved approximation ratio for MinRep.

5.3 Inapproximability of Unweighted MTC

The following shows an approximation-preserving reduction from the Steiner Tree problem on undirected graphs with edge weights either 1 or 2 to MTC on unweighted temporal graphs, where all temporal edges have weight equal to 1. Since this version of the Steiner Tree problem is known to be APX-hard [4], we obtain the following.

▶ **Theorem 8.** MTC *on unweighted temporal graphs is* APX-*hard, and thus it does not admit a PTAS, unless* P = NP.

6 All-Pairs Temporal Connectivity on Trees and Cycles

We can do better if the underlying graph is either a tree or a cycle. We can show that if the underlying graph is a tree, there is an optimal solution to the MTC problem that uses each edge with at most two time labels. Using this structural property, we can show that MTC can be solved efficiently by dynamic programming if the underlying graph is a tree.

▶ **Theorem 9.** *Let $\mathcal{G}$ be a temporal tree on n vertices with lifetime L. There is a dynamic programming algorithm that computes an optimal solution to* MTC *on $\mathcal{G}$ in time $O(nL^4)$.*

We also observe that if the underlying graph is a cycle $C_n = (v_0, v_1, \ldots, v_{n-1}, v_0)$, any temporally connected subgraph $\mathcal{G}'$ can be partitioned into *sectors*. A sector is a connected part $(v_i, v_{i+1}, \ldots, v_k)$ of the cycle for which there is a vertex $v_j \notin \{v_i, \ldots, v_{k-1}\}$ such that the temporal paths $p_{\text{incr}} = (v_i, v_{i+1}, \ldots, v_j)$ and $p_{\text{decr}} = (v_k, v_{k-1}, \ldots, v_{j+1})$ are present in $\mathcal{G}'$ (the vertex indices along C_n are taken modulo n). Intuitively, any vertex in the sector $(v_i, v_{i+1}, \ldots, v_k)$ can reach every vertex in C_n through the paths p_{incr} and p_{decr}. Then, we can show that there is an optimal solution to the MTC problem on C_n where each edge is shared by at most two different sectors. Then, ignoring edges shared by different sectors and using dynamic programming to determine a near optimal partitioning of C_n into sectors, we obtain the following.

▶ **Theorem 10.** *There is a polynomial-time 2-approximation algorithm for the* MTC *problem on any temporal cycle C_n.*

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