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New Interpretation and Generalization of the Kameda-Weiner Method*

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Abstract

We present a reinterpretation of the Kameda-Weiner method of finding a minimal nondeterministic finite automaton (NFA) of a language, in terms of atoms of the language. We introduce a method to generate NFAs from a set of languages, and show that the Kameda-Weiner method is a special case of it. Our method provides a unified view of the construction of several known NFAs, including the canonical residual finite state automaton and the atomaton of the language.

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1 Introduction

Nondeterministic finite automata (NFAs), introduced by Rabin and Scott [11] in 1959, have played a major role in the theory and applications of finite automata. In particular, the problem of finding NFAs with the minimal number of states has received much attention. Different approaches have been used over the years when trying to solve this problem, of which the work done by Kameda and Weiner [10] in 1970 seems to be among the most classical ones. Kameda and Weiner studied the problem of NFA minimization using a matrix based on the states of the minimal deterministic finite automata (DFAs) for a given language and its reverse. They suggested a method of finding a minimal NFA using grids of this matrix.

We present a reinterpretation of the Kameda-Weiner method, using the recently introduced atoms of regular languages [3, 5], and continuing the work started by Brzozowski and Tamm in [4], where the Kameda-Weiner method was formulated in terms of quotients and atoms of a language. We show that the matrix used by Kameda and Weiner can be viewed as the *quotient-atom* matrix of the language, and that any maximal grid of this matrix can be seen as the set of atoms that the grid involves. We also show that, instead of applying the rather complicated *intersection rule* of the Kameda-Weiner method, to construct an NFA corresponding to a cover of the matrix, consisting of maximal grids, one can use sets of atoms associated with grids, and form an NFA based on these sets. We note that essentially the same approach to the Kameda-Weiner method, which uses projections of grids (corresponding to sets of atoms), has been presented by Champarnaud and Coulon [6].

Furthermore, we generalize the idea of constructing an NFA using sets of atoms. Namely, we introduce a method to generate NFAs from a set of languages, and show that the

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Kameda-Weiner method of constructing a minimal NFA is a special case of this method. The introduced method provides a unified view of the construction of several known NFAs including, for example, the canonical residual finite state automaton and the átomaton of the language.

The structure of the rest of the paper is as follows. In Section 2, we provide definitions for automata, quotients, and atoms of a regular language, and recall some results related to atoms. Section 3 describes the Kameda-Weiner method of finding a minimal NFA of a language and shows how the Kameda-Weiner construction of an NFA can be expressed in terms of atoms. In Section 4, we introduce a method to generate NFAs from a set of languages and present a few examples of known NFAs that can be constructed using this method. In Section 5, we show that the NFA minimization method presented by Kameda and Weiner is a special case of generating an NFA by our method. Section 6 concludes the paper.

2 Automata, Quotients, and Atoms of Regular Languages

A *nondeterministic finite automaton (NFA)* is a quintuple $\mathcal{N} = (Q, \Sigma, \delta, I, F)$, where Q is a finite, non-empty set of *states*, Σ is a finite non-empty *alphabet*, $\delta : Q \times \Sigma \to 2^Q$ is the *transition function*, $I \subseteq Q$ is the set of *initial states*, and $F \subseteq Q$ is the set of *final states*. We extend the transition function to functions $\delta' : Q \times \Sigma^* \to 2^Q$ and $\delta'' : 2^Q \times \Sigma^* \to 2^Q$, using δ for all these functions. An NFA $\mathcal{N}' = (Q', \Sigma', \delta', I', F')$ is a *subautomaton* of $\mathcal{N}$ if $Q' \subseteq Q$, $\Sigma' \subseteq \Sigma$, $I' \subseteq I$, $F' \subseteq F$, and $q \in \delta'(p, a)$ implies $q \in \delta(p, a)$ for every $p, q \in Q'$ and $a \in \Sigma'$.

The *language accepted* by an NFA $\mathcal{N}$ is $L(\mathcal{N}) = \{w \in \Sigma^* \mid \delta(I, w) \cap F \neq \emptyset\}$. The *right language* of a state q of $\mathcal{N}$ is $L_{q,F}(\mathcal{N}) = \{w \in \Sigma^* \mid \delta(q, w) \cap F \neq \emptyset\}$. A state is *empty* if its right language is empty. The *left language* of a state q of $\mathcal{N}$ is $L_{I,q} = \{w \in \Sigma^* \mid q \in \delta(I, w)\}$. A state is *unreachable* if its left language is empty. An NFA is *trim* if it has no empty or unreachable states. If $\mathcal{N}_1 = (Q_1, \Sigma, \delta_1, I_1, F_1)$ and $\mathcal{N}_2 = (Q_2, \Sigma, \delta_2, I_2, F_2)$ are NFAs, then a map φ from Q_1 into Q_2 is a *morphism* from $\mathcal{N}_1$ into $\mathcal{N}_2$ if and only if $\varphi(I_1) \subseteq I_2$, $\varphi(F_1) \subseteq F_2$, and $q \in \delta_1(p, a)$ implies $\varphi(q) \in \delta_2(\varphi(p), a)$.

A *deterministic finite automaton (DFA)* is a quintuple $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$, where Q, Σ, and F are as in an NFA, $\delta : Q \times \Sigma \to Q$ is the transition function, and q_0 is the initial state.

The following three operations on automata are commonly used: the *determinization* operation D applied to an NFA $\mathcal{N}$, yielding a DFA $\mathcal{N}^D$, obtained by the well-known subset construction, the *reversal* operation R which, when applied to an NFA $\mathcal{N}$, yields an NFA $\mathcal{N}^R$, where the sets of the initial and the final states of $\mathcal{N}$ are interchanged and all transitions are reversed, and the *trimming* operation T which, when applied to an NFA $\mathcal{N}$, results in an NFA $\mathcal{N}^T$ where all unreachable and empty states are removed.

The *left quotient*, or simply *quotient*, of a language L by a word $w \in \Sigma^*$ is the language $w^{-1}L = \{x \in \Sigma^* \mid wx \in L\}$. There is one *initial* quotient, $\varepsilon^{-1}L = L$. A quotient is *final* if it contains ε. It is well known that there is a one-to-one correspondence between the set of states $Q = \{q_0, \dots, q_{n-1}\}$ of the minimal DFA $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$ accepting L and the set of quotients $\{K_0, \dots, K_{n-1}\}$ of L, such that $L_{q_i,F}(\mathcal{D}) = K_i$ for $i = 0, \dots, n-1$.

An *atom* of a regular language L with quotients $K_0, \dots, K_{n-1}$ is any non-empty language of the form $\widetilde{K_0} \cap \dots \cap \widetilde{K_{n-1}}$, where $\widetilde{K_i}$ is either K_i or $\overline{K_i}$, and $\overline{K_i}$ is the complement of K_i with respect to Σ^*. Thus atoms of L are regular languages uniquely determined by L and they define a partition of Σ^*. They are pairwise disjoint and every quotient of L (including L itself) is a union of atoms. Also, every quotient of an atom of L is a union of atoms. It has been noticed that atoms are exactly the classes of the *left congruence* of L [9] defined as follows: for $x, y \in \Sigma^*$, x is equivalent to y if for every $u \in \Sigma^*$, $ux \in L$ if and only if $uy \in L$. This idea was used in [2], where this equivalence is called the *atom congruence*.

A regular language L with n quotients has at most 2^n atoms. An atom is *initial* if it has L (rather than $\overline{L}$) as a term; it is *final* if it contains ε. There is exactly one final atom, the atom $\widehat{K_0} \cap \cdots \cap \widehat{K_{n-1}}$, where $\widehat{K_i} = K_i$ if $\varepsilon \in K_i$, and $\widehat{K_i} = \overline{K_i}$ otherwise. Let $A = \{A_0, \dots, A_{m-1}\}$ be the set of atoms of L, let I_A be the set of initial atoms, and let A_{m-1} be the final atom. If $\overline{K_0} \cap \cdots \cap \overline{K_{n-1}}$ is an atom, then it is called the *negative* atom, all the other atoms are *positive*.

We use a one-to-one correspondence $A_i \leftrightarrow \mathbf{A}_i$ between atoms A_i of a language L and the states $\mathbf{A}_i$ of the NFA $\mathcal{A}$ defined as follows [5]:

▶ **Definition 1.** The *átomaton* of L is the NFA $\mathcal{A} = (\mathbf{A}, \Sigma, \alpha, \mathbf{I}_A, \{\mathbf{A}_{m-1}\})$, where $\mathbf{A} = \{\mathbf{A}_i \mid A_i \in A\}$, $\mathbf{I}_A = \{\mathbf{A}_i \mid A_i \in I_A\}$, and $\mathbf{A}_j \in \alpha(\mathbf{A}_i, a)$ if and only if $A_j \subseteq a^{-1}A_i$, for all $\mathbf{A}_i, \mathbf{A}_j \in \mathbf{A}$ and $a \in \Sigma$.

The right language of any state $\mathbf{A}_i$ of the átomaton is the atom A_i [5].

The next theorem is a slightly modified version of the result by Brzozowski [1]:

▶ **Theorem 2.** *If an NFA $\mathcal{N}$ has no empty states and $\mathcal{N}^R$ is deterministic, then $\mathcal{N}^D$ is minimal.*

Since it was shown in [5] that $\mathcal{A}^R$ is a minimal DFA for the reverse language of L, we know by Theorem 2 that $\mathcal{A}^R$ is isomorphic to $\mathcal{D}^{RD}$, where $\mathcal{D}$ is the minimal DFA of L. Thus, $\mathcal{A}$ is isomorphic to $\mathcal{D}^{RDR}$.

A new class of NFA's was defined in [5] as follows: an NFA $\mathcal{N} = (Q, \Sigma, \delta, I, F)$ is *atomic* if for every $q \in Q$, the right language $L_{q,F}(\mathcal{N})$ of q is a union of atoms of $L(\mathcal{N})$. Also, it was shown that for any NFA $\mathcal{N}$, $\mathcal{N}^D$ is a minimal DFA if and only if $\mathcal{N}^R$ is atomic.

3 NFA Minimization by Kameda and Weiner

Kameda and Weiner [10] have developed a theory of NFA minimization. They used minimal DFAs for a language L and its reverse L^R to form a matrix, and based on the grids in this matrix, a minimal NFA was found. We note that the biclique edge cover technique presented by Gruber and Holzer [8] as a lower bound method for the size of a minimal NFA, uses another representation of the same matrix.

We present main principles of the Kameda-Weiner method, using mostly our terminology and notation. Kameda and Weiner [10] consider a trim minimal DFA $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$ with Q of cardinality p, and its reversed determinized and trim version $\mathcal{D}^{RDT}$; the set of states of $\mathcal{D}^{RDT}$ is a subset S of cardinality r of $2^Q \setminus \emptyset$. They then form an $p \times r$ matrix T where the rows correspond to non-empty states $q_i \in Q$ of $\mathcal{D}$, which is the trim minimal DFA of a language L, and columns, to states $S_j \in S$ of $\mathcal{D}^{RDT}$, which is the trim minimal DFA of the language L^R by Theorem 2. The entry $t_{i,j}$ of the matrix T is 1 if $q_i \in S_j$, and 0 otherwise.

We use $\mathcal{D}^{RDRT}$, the trim átomaton, instead of $\mathcal{D}^{RDT}$, since the state sets of these two automata are identical. Interpret the rows of the matrix as non-empty quotients of L and columns, as positive atoms of L. Then $t_{i,j} = 1$ if and only if quotient K_i contains atom A_j as a subset, and it is clear that every regular language defines a unique such matrix, which we will call the *quotient-atom matrix*.

The ordered pair (K_i, A_j) is a *point* of T if $t_{i,j} = 1$. A *grid* g of T is the direct product $g = P \times R$ of a set P of quotients with a set R of atoms, such that every atom in R is a subset of every quotient in P. If $g = P \times R$ and $g' = P' \times R'$ are two grids of T, then $g \subseteq g'$ if and only if $P \subseteq P'$ and $R \subseteq R'$. Thus $\subseteq$ is a partial order on the set of all grids of T, and a grid is *maximal* if it is not contained in any other grid. We say that a grid $g = P \times R$ is *horizontally maximal* if for any grid $g' = P \times R'$, $R' \subseteq R$. Similarly, a grid $g = P \times R$ is *vertically maximal* if for any grid $g' = P' \times R$, $P' \subseteq P$. Clearly, any maximal grid is both horizontally and vertically maximal.

A *cover* G of T is a set $G = \{g_0, \dots, g_{k-1}\}$ of grids, such that every point (K_i, A_j) belongs to some grid g_i in G. A *minimal cover* has the minimal number of grids.

Let f_G be the function that assigns to every non-empty quotient K_i the subset of a cover G, consisting of grids $g = P \times R$ such that $K_i \in P$. The NFA constructed by the Kameda-Weiner method is $\mathcal{N}_G = (G, \Sigma, \eta_G, I_G, F_G)$, where G is a cover consisting of maximal grids, $I_G = f_G(K_0)$ is the set of grids corresponding to the initial quotient K_0, and F_G is defined by $g \in F_G$ if and only if $g \in f_G(K_i)$ implies that K_i is a final quotient. For every grid $g = P \times R$ and $a \in \Sigma$, we can compute $\eta_G(g, a)$ by the formula $\eta_G(g, a) = \bigcap_{K_i \in P} f_G(a^{-1}K_i)$. It is said that the NFA $\mathcal{N}_G$ is obtained from $\mathcal{D}$ by the *intersection rule*, using the (grid) cover G.

It may be the case that $\mathcal{N}_G$ does not accept the language L. A cover G is called *legal* if $L(\mathcal{N}_G) = L$. To find a minimal NFA of a language L, the method in [10] tests the covers of the quotient-atom matrix of L in the order of increasing size to see if they are legal. The first legal NFA is a minimal one.

Next, we will interpret the Kameda-Weiner method in terms of atoms. For this, we first show the relationship between maximal grids and certain sets of atoms. Let us start with the following definition:

▶ **Definition 3.** Let R be a set of atoms and let $U(R) = \bigcup_{A_j \in R} A_j$ be the union of these atoms. We define the *maximized* version of $U(R)$ to be the language $max(U(R)) = \bigcap_{U(R) \subseteq K_i} K_i$. We say that the set R is *maximal* if $max(U(R)) = U(R)$.

The following proposition is an easy observation:

▶ **Proposition 4.** *Let R be a set of atoms. Then*
1. $U(R) \subseteq max(U(R))$,
2. $max(max(U(R))) = max(U(R))$.

▶ **Proposition 5.** *Let R_i and R_j be sets of atoms. The following properties hold:*
1. *If $U(R_i) \subseteq U(R_j)$, then $max(U(R_i)) \subseteq max(U(R_j))$.*
2. *For every $a \in \Sigma$, $max(a^{-1}U(R_i)) \subseteq a^{-1}max(U(R_i))$.*

Proof. To prove the first claim, let $U(R_i) \subseteq U(R_j)$. Then it is easy to see that the inclusion $\{K_h \mid U(R_i) \subseteq K_h\} \supseteq \{K_k \mid U(R_j) \subseteq K_k\}$ holds, implying that also the inclusion $\bigcap_{U(R_i) \subseteq K_h} K_h \subseteq \bigcap_{U(R_j) \subseteq K_k} K_k$ holds. Thus, $max(U(R_i)) \subseteq max(U(R_j))$.

To prove the second property, consider the set of quotients K_h such that $U(R_i) \subseteq K_h$. Since $U(R_i) \subseteq K_h$ implies that $a^{-1}U(R_i) \subseteq a^{-1}K_h$ holds, it is clear that the inclusion $\{a^{-1}K_h \mid U(R_i) \subseteq K_h\} \subseteq \{K_k \mid a^{-1}U(R_i) \subseteq K_k\}$ holds. This implies that also the inclusion $\bigcap_{a^{-1}U(R_i) \subseteq K_k} K_k \subseteq \bigcap_{U(R_i) \subseteq K_h} a^{-1}K_h$ holds. Since $\bigcap_{U(R_i) \subseteq K_h} a^{-1}K_h = a^{-1} \bigcap_{U(R_i) \subseteq K_h} K_h$, we get that $max(a^{-1}U(R_i)) \subseteq a^{-1}max(U(R_i))$. ◀

Next, we will see that any maximal grid can be considered as a maximal set of atoms it involves.

▶ **Proposition 6.** *For any grid $g = P \times R$, $U(R) \subseteq \bigcap_{K_i \in P} K_i$ holds.*

Proof. For any grid $g = P \times R$, it holds that for every $K_i \in P$ and $A_j \in R$, $A_j \subseteq K_i$, implying that $U(R) \subseteq \bigcap_{K_i \in P} K_i$. ◀

▶ **Proposition 7.** *A grid $g = P \times R$ is horizontally maximal if and only if the equality $\bigcap_{K_i \in P} K_i = U(R)$ holds.*

Proof. For any grid $g = P \times R$, the inclusion $U(R) \subseteq \bigcap_{K_i \in P} K_i$ holds by Proposition 6. Let g be a horizontally maximal grid. Then there is no grid $g' = P \times R'$, such that $R \subset R'$. That is, there is no $A_l \in \{A_0, \dots, A_{m-1}\} \setminus R$, such that $U(R) \cup A_l \subseteq \bigcap_{K_i \in P} K_i$ would hold. Since every quotient is a disjoint union of atoms, every intersection of quotients is also a union of atoms. Therefore, the equality $\bigcap_{K_i \in P} K_i = U(R)$ holds.

Conversely, if $\bigcap_{K_i \in P} K_i = U(R)$, then for every grid $g' = P \times R'$, the inclusion $R' \subseteq R$ holds. Thus, the grid $g = P \times R$ is horizontally maximal. ◀

▶ **Corollary 8.** *A grid $g = P \times R$ is horizontally maximal if and only if the set R is maximal.*

Proof. Let $g = P \times R$ be a horizontally maximal grid. By Proposition 7, this means that the equality $\bigcap_{K_i \in P} K_i = U(R)$ holds. Since $max(U(R)) = \bigcap_{U(R) \subseteq K_i} K_i$, it is clear that $max(U(R)) = U(R)$. Thus, R is maximal. ◀

▶ **Corollary 9.** *A grid $g = P \times R$ is maximal if and only if P is a maximal set of quotients such that the equality $\bigcap_{K_i \in P} K_i = U(R)$ holds.*

According to Corollaries 8 and 9, any maximal grid involves a maximal set of atoms and the set of quotients, such that the intersection of these quotients is the union of the atoms involved.

As the main result of this section, we will prove the following theorem which shows how the construction of the NFA $\mathcal{N}_G$ can be expressed in terms of atoms:

▶ **Theorem 10.** *Let $G = \{g_0, \dots, g_{k-1}\}$ be a cover consisting of maximal grids $g_i = P_i \times R_i$, $i = 0, \dots, k-1$, and let $\mathcal{N}_G = (G, \Sigma, \eta_G, I_G, F_G)$ be the corresponding NFA, obtained by the intersection method. It holds that $g_i \in I_G$ if and only if $U(R_i) \subseteq L$, and $g_i \in F_G$ if and only if $\varepsilon \in U(R_i)$. For any $g_i, g_j \in G$ and $a \in \Sigma$, $g_j \in \eta_G(g_i, a)$ if and only if the inclusion $U(R_j) \subseteq a^{-1}U(R_i)$ holds.*

Proof. The set I_G of initial states of $\mathcal{N}_G$ consists of those grids that intersect the initial quotient $K_0 = L$. That is, for every grid $g_i \in G$, $g_i \in I_G$ if and only if $U(R_i) \subseteq L$ holds.

The set F_G of final states of $\mathcal{N}_G$ is the set of grids that intersect only the final quotients. Equivalently, for any $g_i \in G$, it holds that $g_i \in F_G$ if and only if $U(R_i)$ includes the final atom. The latter is equivalent to having $\varepsilon \in U(R_i)$.

Next, let $g_j \in \eta_G(g_i, a)$ for some $g_i, g_j \in G$ and $a \in \Sigma$. By the intersection rule, it holds that $\eta_G(g_i, a) = \bigcap_{K_h \in P_i} f(a^{-1}K_h)$. That is, $g_j \in \eta_G(g_i, a)$ if and only if $g_j \in f(a^{-1}K_h)$ for every $K_h \in P_i$. This implies that $g_j \in \eta_G(g_i, a)$ if and only if $a^{-1}K_h \in P_j$ holds for every $K_h \in P_i$.

It is clear that if $a^{-1}K_h \in P_j$ holds for every $K_h \in P_i$, then the inclusion $\bigcap_{K_k \in P_j} K_k \subseteq \bigcap_{K_h \in P_i} a^{-1}K_h$ holds. And conversely, if $\bigcap_{K_k \in P_j} K_k \subseteq \bigcap_{K_h \in P_i} a^{-1}K_h$, then since by Corollary 9, P_j is a maximal set of quotients such that the equality $\bigcap_{K_k \in P_j} K_k = U(R_j)$ holds, it must be that $a^{-1}K_h \in P_j$ for $K_h \in P_i$. Thus, $a^{-1}K_h \in P_j$ for $K_h \in P_i$ if and only if the inclusion $\bigcap_{K_k \in P_j} K_k \subseteq \bigcap_{K_h \in P_i} a^{-1}K_h$ holds. Because of the equality $\bigcap_{K_h \in P_i} a^{-1}K_h = a^{-1} \bigcap_{K_h \in P_i} K_h$, we get the equivalent condition $\bigcap_{K_k \in P_j} K_k \subseteq a^{-1} \bigcap_{K_h \in P_i} K_h$. Using Corollary 9, we get the inclusion $U(R_j) \subseteq a^{-1}U(R_i)$. ◀

Theorem 10 provides another way of constructing the NFA $\mathcal{N}_G$ from a given set of maximal grids covering the quotient-atom matrix: instead of applying the intersection rule to get transitions of $\mathcal{N}_G$, one can use the sets of atoms corresponding to the grids, and apply quotients of unions of the atoms involved. This can be done using the transition function of the átomaton.

We mention that basically the same approach to the Kameda-Weiner method has been presented by Champarnaud and Coulon [6]. They used projections of grids, consisting of subsets of the state set of the DFA $\mathcal{D}^{RDT}$, to construct an NFA similarly as in Theorem 10.

In the next section, we will generalize this idea of using sets of atoms (or unions of atoms) of a language L, to construct NFAs for L.

4 Generating Automata by a Set of Languages

In this section, we introduce a method to generate NFAs from a set of languages.

Let L be a regular language, and let $K = \{K_0, \dots, K_{n-1}\}$ be the set of quotients of L. A set $\{L_0, \dots, L_{k-1}\}$ of languages is a *cover* of the quotients of L, or simply, a cover for L, if every quotient K_j of L is a union of some L_i's. We note that since L is the quotient of itself by the empty word ε, L is a union of some L_i's.

We define the NFA based on a cover $\{L_0, \dots, L_{k-1}\}$ as follows:

▶ **Definition 11.** The NFA generated by a cover $\{L_0, \dots, L_{k-1}\}$ for L is defined by $\mathcal{G} = (Q, \Sigma, \delta, I, F)$, where $Q = \{q_0, \dots, q_{k-1}\}$, $I = \{q_i \mid L_i \subseteq L\}$, $F = \{q_i \mid \varepsilon \in L_i\}$, and $q_j \in \delta(q_i, a)$ if and only if $L_j \subseteq a^{-1}L_i$ for all $q_i, q_j \in Q$ and $a \in \Sigma$.

▶ **Lemma 12.** *For all states q_i, q_j of NFA $\mathcal{G}$ and for any word $w \in \Sigma^+$, $q_j \in \delta(q_i, w)$ if and only if $L_j \subseteq w^{-1}L_i$.*

Proof. We prove the statement by induction on the length of w. If $w = a$ for some $a \in \Sigma$, then the lemma holds by Definition 11.

Now, let $w = ua$, where $u \in \Sigma^+$ and $a \in \Sigma$, and assume that the lemma holds for u, that is, for all states q_i, q_j of $\mathcal{G}$, $q_j \in \delta(q_i, u)$ if and only if $L_j \subseteq u^{-1}L_i$. Consider a state q_i, and let $q_k \in \delta(q_i, ua)$. Then there is a state q_j, such that $q_j \in \delta(q_i, u)$ and $q_k \in \delta(q_j, a)$. Equivalently, by the induction assumption and Definition 11, respectively, the inclusions $L_j \subseteq u^{-1}L_i$ and $L_k \subseteq a^{-1}L_j$ hold. Hence $L_k \subseteq a^{-1}L_j \subseteq a^{-1}(u^{-1}L_i) = (ua)^{-1}L_i$. Thus, $q_k \in \delta(q_i, ua)$ if and only if $L_k \subseteq (ua)^{-1}L_i$. ◀

▶ **Proposition 13.** *The following properties hold for NFA $\mathcal{G}$:*
1. *$L_{q_i,F}(\mathcal{G}) \subseteq L_i$ for every $q_i \in Q$.*
2. *$L(\mathcal{G}) \subseteq L$.*

Proof. 1. Consider a state q_i of $\mathcal{G}$. Let $w \in L_{q_i,F}(\mathcal{G})$. If $w = \varepsilon$, then $q_i \in F$, and $\varepsilon \in L_i$ by Definition 11. If $w \in \Sigma^+$, then there is some q_j such that $q_j \in F$ and $q_j \in \delta(q_i, w)$. By Lemma 12, $L_j \subseteq w^{-1}L_i$ and $\varepsilon \in L_j$ implying that $w \in L_i$.

2. Since $L(\mathcal{G})$ is the union of right languages of the initial states of $\mathcal{G}$, the claim follows from Definition 11 and Part 1. ◀

▶ **Lemma 14.** *If $a^{-1}L_i$ is a union of L_j's for every L_i and $a \in \Sigma$, then $w^{-1}L_i$ is a union of L_j's for every L_i and $w \in \Sigma^+$.*

Proof. Let $a^{-1}L_i$ be a union of L_j's for every L_i and $a \in \Sigma$, that is, $a^{-1}L_i = \bigcup_{j \in J_{i,a}} L_j$ for some $J_{i,a} \subseteq \{0, \dots, k-1\}$. We prove the statement by induction on the length of w. If $w = a$ for some $a \in \Sigma$, then the lemma trivially holds.

Now, let $w = ua$, where $u \in \Sigma^+$ and $a \in \Sigma$, and assume that the lemma holds for u, that is, $u^{-1}L_i = \bigcup_{j \in J_{i,u}} L_j$ for some $J_{i,u} \subseteq \{0, \dots, k-1\}$. Then $(ua)^{-1}L_i = a^{-1}(u^{-1}L_i) = a^{-1}(\bigcup_{j \in J_{i,u}} L_j) = \bigcup_{j \in J_{i,u}} a^{-1}L_j = \bigcup_{j \in J_{i,u}} \bigcup_{h \in J_{j,a}} L_h$. Thus, $(ua)^{-1}L_i$ is a union of L_h's. ◀

▶ **Proposition 15.** *Let $\mathcal{G} = (Q, \Sigma, \delta, I, F)$ be the NFA generated by a cover $\{L_0, \dots, L_{k-1}\}$ for L. The equality $L_{q_i,F}(\mathcal{G}) = L_i$ holds for every $q_i \in Q$ if and only if $a^{-1}L_i$ is a union of L_j's for every L_i and $a \in \Sigma$.*

Proof. First, let the equality $L_{q_i,F}(\mathcal{G}) = L_i$ hold for every $q_i \in Q$. Let us consider any L_i and $a \in \Sigma$. Then it holds that $a^{-1}L_i = a^{-1}L_{q_i,F}(\mathcal{G}) = \bigcup_{q_j \in \delta(q_i,a)} L_{q_j,F}(\mathcal{G}) = \bigcup_{L_j \subseteq a^{-1}L_i} L_j$.

Conversely, assume that $a^{-1}L_i$ is a union of L_j's for every L_i and $a \in \Sigma$. Let us consider any state q_i of $\mathcal{G}$. By Proposition 13, the inclusion $L_{q_i,F}(\mathcal{G}) \subseteq L_i$ holds, so we only have to show that $L_i \subseteq L_{q_i,F}(\mathcal{G})$. Let w be any word in L_i. If $w = \varepsilon$, then $q_i \in F$, and so $w \in L_{q_i,F}(\mathcal{G})$. If $w \in \Sigma^+$, then by Lemma 14, $w^{-1}L_i$ is a union of L_j's. Since $w \in L_i$, there must be some L_j such that $L_j \subseteq w^{-1}L_i$ and $\varepsilon \in L_j$. By Lemma 12, there is some $q_j \in F$ such that $q_j \in \delta(q_i, w)$. Therefore $w \in L_{q_i,F}(\mathcal{G})$, and we conclude that $L_{q_i,F}(\mathcal{G}) = L_i$. ◀

▶ **Proposition 16.** *Let $\mathcal{G} = (Q, \Sigma, \delta, I, F)$ be the NFA generated by a cover $\{L_0, \dots, L_{k-1}\}$ for L. If $a^{-1}L_i$ is a union of L_j's for every L_i and $a \in \Sigma$, then $\mathcal{G}$ accepts L.*

Proof. If $a^{-1}L_i$ is a union of L_j's for every L_i and $a \in \Sigma$, then by Proposition 15, $L_{q_i,F}(\mathcal{G}) = L_i$ holds for every $q_i \in Q$. Since $L(\mathcal{G}) = \bigcup_{q_i \in I} L_{q_i,F}(\mathcal{G}) = \bigcup_{L_i \subseteq L} L_i = L$, the equality $L(\mathcal{G}) = L$ holds. ◀

We present four examples of covers for the language L and the corresponding NFAs generated by these covers, where the condition of Proposition 16 holds, ensuring that the generated NFA accepts L:

▶ **Example 17.** Consider the set $K = \{K_0, \dots, K_{n-1}\}$ of quotients of L as a cover for L. Let $\mathcal{G}_K$ be the NFA generated by the set K. Since for every quotient K_i and $a \in \Sigma$ there exists some quotient K_j such that $a^{-1}K_i = K_j$, we know by Proposition 16 that $\mathcal{G}_K$ accepts L. It is well known that the states of the minimal DFA correspond to the quotients of L. However, the NFA $\mathcal{G}_K$ is isomorphic to the *saturated* version [7] of the minimal DFA of L.

▶ **Example 18.** Consider the set $K' \subseteq K$ of *prime* quotients of L, that is, those non-empty quotients of L which are not unions of other quotients, as a cover for L. Let $\mathcal{G}_{K'}$ be the NFA generated by the set K'. Since every quotient of L is a union of some prime quotients of L, it is clear that for every prime quotient K'_i and $a \in \Sigma$, $a^{-1}K'_i$ is a union of prime quotients. Thus, $\mathcal{G}_{K'}$ accepts L by Proposition 16. The NFA $\mathcal{G}_{K'}$ is known as the *canonical residual finite state automaton (canonical RFSA)* [7] of L.

▶ **Example 19.** Consider the set $A = \{A_0, \dots, A_{m-1}\}$ of atoms of L. The set of atoms is a cover for L, because every quotient of L is a union of atoms [5]. The NFA $\mathcal{G}_A$, generated by the set A, is the átomaton of L (cf. Definition 1). It is known that for every atom A_i and $a \in \Sigma$, $a^{-1}A_i$ is a union of atoms [5]. Thus, the condition of Proposition 16 holds, and $\mathcal{G}_A$ accepts L.

▶ **Example 20.** Let $A_0, \dots, A_{m-1}$ be the atoms of L. Consider the set $M = \{M_0, \dots, M_{m-1}\}$ of the maximized versions of atoms, that is, $M_i = max(A_i)$ for $i = 0, \dots, m-1$. Clearly, if $A_i \subseteq K_j$ for some atom A_i and quotient K_j, then the inclusion $M_i \subseteq K_j$ holds by Definiton 3. Thus, the set M is a cover for L. The NFA $\mathcal{G}_M$, generated by the set M, is the *maximized átomaton* [13] of L. Since for any $M_i \in M$ and $a \in \Sigma$, $a^{-1}M_i = \bigcup_{A_j \subseteq M_i} a^{-1}A_j$, and because $a^{-1}A_j$ is a union of atoms [5], we get that $a^{-1}M_i$ is a union of atoms. By [13, Proposition 2, Part 4], the inclusion $A_j \subseteq a^{-1}M_i$ holds if and only if $M_j \subseteq a^{-1}M_i$ holds. We conclude that $a^{-1}M_i$ is a union of M_j's, and by Proposition 16, $\mathcal{G}_M$ accepts L.

However, we note that the condition of Proposition 16 is not necessary for the generated NFA to accept L.

▶ **Proposition 21.** *If $\mathcal{N}$ is a trim NFA accepting L, with the set $\{L_0, \dots, L_{k-1}\}$ of the right languages of its states, then this set is a cover for L.*

Proof. By determinizing $\mathcal{N}$, the quotients of L are formed as unions of some L_i's. ◀

▶ **Proposition 22.** *Let $\mathcal{N} = (Q, \Sigma, \delta, I, F)$ be a trim NFA of L, with the set $\{L_0, \dots, L_{k-1}\}$ of the right languages of its states, and let $\mathcal{G} = (Q', \Sigma, \delta', I', F')$ be the NFA generated by the set $\{L_0, \dots, L_{k-1}\}$. Let $\varphi : Q \to Q'$ be the mapping assigning to every state q of $\mathcal{N}$, the state q'_i of $\mathcal{G}$, such that $L_i = L_{q,F}(\mathcal{N})$. Then φ is a morphism from $\mathcal{N}$ into $\mathcal{G}$.*

Proof. Let $\mathcal{N} = (Q, \Sigma, \delta, I, F)$ be a trim NFA accepting L, with the set $\{L_0, \dots, L_{k-1}\}$ of the right languages of its states. By Proposition 21, the set $\{L_0, \dots, L_{k-1}\}$ is a cover for L. Let $\mathcal{G} = (Q', \Sigma, \delta', I', F')$ be the NFA generated by the set $\{L_0, \dots, L_{k-1}\}$, with $Q' = \{q'_0, \dots, q'_{k-1}\}$. Let $\varphi : Q \to Q'$ be the mapping assigning to every state q of $\mathcal{N}$, the state q'_i of $\mathcal{G}$, such that $L_i = L_{q,F}(\mathcal{N})$. We note that there may be some states p and q of $\mathcal{N}$, such that $p \neq q$ and $L_{p,F}(\mathcal{N}) = L_{q,F}(\mathcal{N})$, so φ is a many-to-one correspondence. We show that φ is a morphism from $\mathcal{N}$ into $\mathcal{G}$.

First, if $q \in I$ is an initial state of $\mathcal{N}$, then there is some L_i, such that $L_i = L_{q,F}(\mathcal{N})$ and $L_i \subseteq L$, which implies that the corresponding state q'_i of $\mathcal{G}$ is also initial, that is, $\varphi(q) \in I'$.

Similarly, if $q \in F$, then there is some L_i, such that $L_i = L_{q,F}(\mathcal{N})$ and $\varepsilon \in L_i$, implying that $q'_i \in F'$, that is, $\varphi(q) \in F'$.

If $q \in \delta(p, a)$ holds for some states $p, q \in Q$ and $a \in \Sigma$, then there are some L_i and L_j, such that $L_i = L_{p,F}(\mathcal{N})$, $L_j = L_{q,F}(\mathcal{N})$, and $L_j \subseteq a^{-1}L_i$. It is implied that $q'_j \in \delta'(q'_i, a)$, that is, $\varphi(q) \in \delta'(\varphi(p), a)$. We conclude that φ is a morphism from $\mathcal{N}$ into $\mathcal{G}$. ◀

▶ **Theorem 23.** *If there is a trim NFA accepting L, with the set $\{L_0, \dots, L_{k-1}\}$ of the right languages of its states, then the NFA generated by the cover $\{L_0, \dots, L_{k-1}\}$ for L is such an NFA.*

Proof. Let $\mathcal{N} = (Q, \Sigma, \delta, I, F)$ be a trim NFA accepting L, with the set $\{L_0, \dots, L_{k-1}\}$ of the right languages of its states. By Proposition 21, the set $\{L_0, \dots, L_{k-1}\}$ is a cover for L. Let $\mathcal{G} = (Q', \Sigma, \delta', I', F')$ be the NFA generated by the set $\{L_0, \dots, L_{k-1}\}$, with $Q' = \{q'_0, \dots, q'_{k-1}\}$. By Proposition 22, there is a morphism $\varphi : Q \to Q'$ from $\mathcal{N}$ into $\mathcal{G}$, such that $\varphi(q) = q'_i$ for some $q \in Q$ and $q'_i \in Q'$ if and only if $L_{q,F}(\mathcal{N}) = L_i$.

The morphism φ implies that for every state $q \in Q$, with its right language $L_{q,F}(\mathcal{N}) = L_i$ for some L_i, the inclusion $L_{q,F}(\mathcal{N}) \subseteq L_{q'_i,F'}(\mathcal{G})$, that is, $L_i \subseteq L_{q'_i,F'}(\mathcal{G})$ holds. Since by Proposition 13, Part 1, the inclusion $L_{q'_i,F'}(\mathcal{G}) \subseteq L_i$ holds, the equality $L_{q'_i,F'}(\mathcal{G}) = L_i$ must hold. Also, the morphism φ implies that the inclusion $L(\mathcal{N}) \subseteq L(\mathcal{G})$ holds. Since by Proposition 13, Part 2, $L(\mathcal{G}) \subseteq L$, and we assumed that $L(\mathcal{N}) = L$, we conclude that $\mathcal{G}$ accepts L. ◀

Theorem 23 shows that our method to generate NFAs from a set of languages is indeed general. That is, if one is interested in finding an NFA for a given language, such that the states of that NFA correspond to certain languages, this method can be used to generate such an NFA if it exists. If the generated NFA is not such an NFA, then it does not exist.

To conclude this section, we point out three cases which can occur if a cover $\{L_0, \dots, L_{k-1}\}$ for L is used to generate an NFA $\mathcal{G}$:

First, the NFA $\mathcal{G}$ accepts L, and the right language of every state q_i of $\mathcal{G}$ is L_i. This case is described by Propositions 15 and 16.

In the second case, the NFA $\mathcal{G}$ accepts L, but the right language of some state q_i of $\mathcal{G}$ is not L_i. The third case is when $\mathcal{G}$ does not accept L. Characterization of the last two cases is an interesting problem for further study.

5 Generating Automata by Atomic Languages

Let L be a regular language, with its quotients $K_0, \dots, K_{n-1}$ and atoms $A_0, \dots, A_{m-1}$.

▶ **Definition 24.** A language L_i is *atomic* with regard to L if L_i is a union of atoms of L.

Let $\mathcal{N} = (Q, \Sigma, \delta, I, F)$ be a trim NFA accepting L, with $Q = \{q_0, \dots, q_{k-1}\}$. For every state q_i of $\mathcal{N}$, we define an atomic language $B_i = \bigcup_{L_{q_i,F}(\mathcal{N}) \cap A_h \neq \emptyset} A_h$ as the union of all atoms of L which intersect with the right language of q_i. In other words, B_i is the smallest atomic language that contains the right language of state q_i. Clearly, if $L_{q_i,F}(\mathcal{N}) \subseteq K_j$ holds for some quotient K_j of L, then, because every quotient is a union of atoms, $B_i \subseteq K_j$ holds as well. Since by Proposition 21, the set of right languages of the states of $\mathcal{N}$ forms a cover for L, the set of B_i's has the same property. We note that there may be some states q_i and q_j of $\mathcal{N}$, such that $q_i \neq q_j$, but $B_i = B_j$. Let the set of distinct B_i's be B.

Let $\mathcal{G}_B = (Q_B, \Sigma, \delta_B, I_B, F_B)$ be the NFA generated by the cover B for the language L. We note that $|Q_B| \leqslant |Q|$. Let $\varphi_{atom} : Q \to Q_B$ be the mapping assigning to state q_i of $\mathcal{N}$, the state q_{B_i} of $\mathcal{G}_B$, such that $B_i = \bigcup_{L_{q_i,F}(\mathcal{N}) \cap A_h \neq \emptyset} A_h$.

▶ **Proposition 25.** *The mapping φ_{atom} is a morphism from $\mathcal{N}$ into $\mathcal{G}_B$.*

Proof. First, if $q_i \in I$ is initial, then $L_{q_i,F}(\mathcal{N}) \subseteq L$, and since L is a union of (initial) atoms, the inclusion $B_i \subseteq L$ holds, implying that q_{B_i} is also initial, that is, $\varphi_{atom}(q_i) \in I_B$.

Similarly, if $q_i \in F$, then $\varepsilon \in L_{q_i,F}(\mathcal{N})$, implying that $\varepsilon \in B_i$, and thus $q_{B_i} \in F_B$, that is, $\varphi_{atom}(q_i) \in F_B$.

It remains to be shown that for all states $q_i, q_j \in Q$ and $a \in \Sigma$, if $q_j \in \delta(q_i, a)$ holds, then $\varphi_{atom}(q_j) \in \delta_B(\varphi_{atom}(q_i), a)$ holds as well. Let $q_j \in \delta(q_i, a)$ for some $q_i, q_j \in Q$ and $a \in \Sigma$. Then the inclusion $L_{q_j,F}(\mathcal{N}) \subseteq a^{-1}L_{q_i,F}(\mathcal{N})$ holds. Because of $L_{q_i,F}(\mathcal{N}) \subseteq B_i$, the inclusion $L_{q_j,F}(\mathcal{N}) \subseteq a^{-1}B_i$ holds. Since it is known that any quotient of a union of atoms is some union of atoms, $a^{-1}B_i$ is a union of atoms. Consequently, $L_{q_j,F}(\mathcal{N}) \subseteq B_j \subseteq a^{-1}B_i$ holds, implying that $q_{B_j} \in \delta_B(q_{B_i}, a)$, that is, $\varphi_{atom}(q_j) \in \delta_B(\varphi_{atom}(q_i), a)$.

We conclude that φ_{atom} is a morphism from $\mathcal{N}$ into $\mathcal{G}_B$. ◀

▶ **Corollary 26.** *For every state q_i of $\mathcal{N}$, the inclusion $L_{q_i,F}(\mathcal{N}) \subseteq L_{q_{B_i},F_B}(\mathcal{G}_B)$ holds. Also, $L(\mathcal{G}_B) = L$.*

Proof. The morphism $\varphi_{atom} : Q \to Q_B$ implies that for every $q_i \in Q$, the inclusion $L_{q_i,F}(\mathcal{N}) \subseteq L_{q_{B_i},F_B}(\mathcal{G}_B)$ holds, and also that $L(\mathcal{N}) \subseteq L(\mathcal{G}_B)$ holds.

Since $L(\mathcal{N}) = L$, and we know by Proposition 13 that $L(\mathcal{G}_B) \subseteq L$, we conclude that $L(\mathcal{G}_B) = L$. ◀

▶ **Corollary 27.** *If there is a one-to-one correspondence between the sets Q and B, then the NFA $\mathcal{N}$ is isomorphic to a subautomaton of $\mathcal{G}_B$.*

Next, for every atomic language B_i we consider its maximized version, the language $C_i = max(B_i) = \bigcap_{B_i \subseteq K_j} K_j$. Clearly, C_i is also atomic, and $B_i \subseteq C_i$. If the inclusion $B_i \subseteq K_j$ holds for some quotient K_j, then by the definition of C_i, $C_i \subseteq K_j$ holds as well. Since the set of B_i's forms a cover for L, so does the set of corresponding C_i's. We note that there may be some B_i and B_j, such that $B_i \neq B_j$, but $C_i = C_j$. Let the set of distinct C_i's be C.

Let $\mathcal{G}_C = (Q_C, \Sigma, \delta_C, I_C, F_C)$ be the NFA generated by the cover C for the language L. We note that $|Q_C| \leqslant |Q_B|$. Let $\varphi_{max} : Q_B \to Q_C$ be the mapping assigning to state q_{B_i} of $\mathcal{G}_B$, the state q_{C_i} of $\mathcal{G}_C$.

▶ **Proposition 28.** *The mapping φ_{max} is a morphism from $\mathcal{G}_B$ into $\mathcal{G}_C$.*

Proof. First, if $q_{B_i} \in I_B$, then $B_i \subseteq L$. Since C_i is a subset of the same quotients as B_i, $C_i \subseteq L$, implying that $q_{C_i} \in I_C$. If $q_{B_i} \in F_B$, then $\varepsilon \in B_i$, and since $B_i \subseteq C_i$, it holds that $\varepsilon \in C_i$, so we get $q_{C_i} \in F_C$.

We also have to show that if $q_{B_j} \in \delta_B(q_{B_i}, a)$ holds for some states q_{B_i} and q_{B_j} of $\mathcal{G}_B$ and $a \in \Sigma$, then $q_{C_j} \in \delta_C(q_{C_i}, a)$ for the corresponding states q_{C_i} and q_{C_j} of $\mathcal{G}_C$. Indeed, if $q_{B_j} \in \delta_B(q_{B_i}, a)$, then $B_j \subseteq a^{-1}B_i$. By Proposition 5, Part 1, we know that $max(B_j) \subseteq max(a^{-1}B_i)$, and by Part 2, the inclusion $max(a^{-1}B_i) \subseteq a^{-1}max(B_i)$ holds. Since $C_i = max(B_i)$ and $C_j = max(B_j)$, we get that $C_j \subseteq a^{-1}C_i$ holds. Thus, $q_{C_j} \in \delta_C(q_{C_i}, a)$.

We conclude that φ_{max} is a morphism from $\mathcal{G}_B$ into $\mathcal{G}_C$. ◀

▶ **Corollary 29.** *For every state q_{B_i} of $\mathcal{G}_B$, the inclusion $L_{q_{B_i}, F_B}(\mathcal{G}_B) \subseteq L_{q_{C_i}, F_C}(\mathcal{G}_C)$ holds. Also, $L(\mathcal{G}_C) = L$.*

Proof. The morphism $\varphi_{max} : Q_B \to Q_C$ implies that for every $q_{B_i} \in Q_B$, the inclusion $L_{q_{B_i}, F_B}(\mathcal{G}_B) \subseteq L_{q_{C_i}, F_C}(\mathcal{G}_C)$ holds, and also that $L(\mathcal{G}_B) \subseteq L(\mathcal{G}_C)$ holds.

Since $L(\mathcal{G}_B) = L$ by Corollary 26, and $L(\mathcal{G}_C) \subseteq L$ by Proposition 13, we conclude that $L(\mathcal{G}_C) = L$. ◀

▶ **Corollary 30.** *If there is a one-to-one correspondence between the sets B and C, then the NFA $\mathcal{G}_B$ is isomorphic to a subautomaton of $\mathcal{G}_C$.*

Based on the results above, we can state the following theorem:

▶ **Theorem 31.** *There is a morphism $\varphi_{max} \circ \varphi_{atom}$ from a trim NFA $\mathcal{N}$ into the NFA $\mathcal{G}_C$, generated by the set C of languages $C_i = max(\bigcup_{L_{q_i,F}(\mathcal{N}) \cap A_h \neq \emptyset} A_h)$, where q_i is a state of $\mathcal{N}$, with $L(\mathcal{G}_C) = L(\mathcal{N})$. Moreover, if there is a one-to-one correspondence between the states of $\mathcal{N}$ and $\mathcal{G}_C$, then $\mathcal{N}$ is isomorphic to a subautomaton of $\mathcal{G}_C$.*

The following theorem shows that the NFA minimization method presented by Kameda and Weiner is a special case of generating an NFA:

▶ **Theorem 32.** *Let $G = \{g_0, \dots, g_{k-1}\}$ be a set of maximal grids, with $g_i = P_i \times R_i$, forming a cover of the quotient-atom matrix of L. The NFA $\mathcal{N}_G$, obtained by the Kameda-Weiner method using G, is isomorphic to the NFA $\mathcal{G}_C$, generated by the set $C = \{C_0, \dots, C_{k-1}\}$ of languages $C_i = U(R_i)$, $i = 0, \dots, k-1$.*

Proof. Let $G = \{g_0, \ldots, g_{k-1}\}$ be a set of maximal grids $g_i = P_i \times R_i$, forming a cover of the quotient-atom matrix of L. Let $\mathcal{N}_G = (G, \Sigma, \eta_G, I_G, F_G)$ be the NFA obtained by the intersection method using G, and let $\mathcal{G}_C = (Q_C, \Sigma, \delta_C, I_C, F_C)$ be the NFA generated by the set $C = \{C_0, \ldots, C_{k-1}\}$, where $C_i = U(R_i)$ for $i = 0, \ldots, k-1$. We show that $\mathcal{N}_G$ is isomorphic to $\mathcal{G}_C$ by applying Theorem 10.

First, by Theorem 10, for every grid $g_i \in G$, it holds that $g_i \in I_G$ if and only if the inclusion $U(R_i) \subseteq L$ holds, that is, $C_i \subseteq L$. Since this is equivalent to the condition $q_{C_i} \in I_C$, there is a one-to-one correspondence between the sets I_G and I_C.

Also, it holds that $g_i \in F_G$ if and only if $\varepsilon \in U(R_i)$, that is, $\varepsilon \in C_i$. This is equivalent to the condition $q_{C_i} \in F_C$. Thus, there is a one-to-one correspondence between the sets F_G and F_C.

It remains to show that $g_j \in \eta_G(g_i, a)$ if and only if $q_{C_j} \in \delta_C(q_{C_i}, a)$ for all $g_i, g_j \in G$ and $a \in \Sigma$. Indeed, by Theorem 10, $g_j \in \eta_G(g_i, a)$ holds if and only if the inclusion $U(R_j) \subseteq a^{-1}U(R_i)$ holds, that is, $C_j \subseteq a^{-1}C_i$. On the other hand, by Definition 11, $q_{C_j} \in \delta_C(q_{C_i}, a)$ if and only if $C_j \subseteq a^{-1}C_i$, where $q_{C_i}, q_{C_j} \in Q_C$ and $a \in \Sigma$. Therefore, $g_j \in \eta_G(g_i, a)$ holds if and only if $q_{C_j} \in \delta_C(q_{C_i}, a)$ holds for all $g_i, g_j \in G$ and $a \in \Sigma$. ◀

▶ **Corollary 33.** *There exists an atomic NFA with the right languages $C_0, \ldots, C_{k-1}$, such that the set of atoms contained in every C_i is maximal, if and only if the Kameda-Weiner method finds it.*

Proof. Follows from Theorem 23 and Theorem 32. ◀

▶ **Corollary 34.** *There is a morphism from any trim NFA $\mathcal{N}$ into the NFA $\mathcal{N}_G$ obtained by the Kameda-Weiner method using the set G of maximal grids, corresponding to the maximal sets of atoms associated to the right languages of $\mathcal{N}$.*

Proof. Follows from Theorem 31 and Theorem 32. ◀

As a special case, if $\mathcal{N}$ is a minimal NFA, then by Theorem 31, $\mathcal{N}$ is isomorphic to a subautomaton of the NFA $\mathcal{G}_C$ generated by the set C of the maximized atomic languages of the right languages of $\mathcal{N}$, or equivalently, as by Theorem 32, of the NFA $\mathcal{N}_G$ obtained by the Kameda-Weiner method, using the corresponding maximal grids.

This indeed ensures that if one considers covers of the quotient-atom matrix, starting from the smallest cover, and produces NFAs according to the Kameda-Weiner method, or equivalently, generates NFAs, using unions of atoms corresponding to the grids in the cover, the first obtained NFA which accepts the given language, is a minimal NFA.

As we mentioned earlier, Champarnaud and Coulon [6] have presented an approach to the Kameda-Weiner method which, similarly to our method, finds NFAs corresponding to grid covers, using projections of grids (corresponding to sets of atoms). We note that they also used grid extensions and automaton morphisms, similarly to our theory. However, we point out that our theory explicitly shows that atoms of regular languages have an important role in the Kameda-Weiner method.

We also mention that Sengoku's method [12] of constructing NFAs is related to atoms; it yields atomic NFAs. However, we note that by a result proved in [5], not every language has an atomic minimal NFA.

6 Conclusions

We presented a reinterpretation of the Kameda-Weiner method for NFA minimization, and generalized it by introducing a method to generate NFAs by certain sets of languages. We hope that our contributions provide a useful insight into the difficult problem of NFA minimization, to obtain a better understanding of this problem.

We also think that the introduced method of generating NFAs is of interest on its own as exemplified in Section 4. This method provides a unified view of the construction of several known NFAs, including the canonical RFSA and the átomaton of the language.

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