On the Skolem Problem for Continuous Linear Dynamical Systems

Ventsislav Chonev*[1], Joël Ouaknine†[2], and James Worrell‡[2]

1 **Institute of Science and Technology, Austria**
2 **University of Oxford, Oxford, United Kingdom**

Abstract

The Continuous Skolem Problem asks whether a real-valued function satisfying a linear differential equation has a zero in a given interval of real numbers. This is a fundamental reachability problem for continuous linear dynamical systems, such as linear hybrid automata and continuous-time Markov chains. Decidability of the problem is currently open – indeed decidability is open even for the sub-problem in which a zero is sought in a bounded interval. In this paper we show decidability of the bounded problem subject to Schanuel's Conjecture, a unifying conjecture in transcendental number theory. We furthermore analyse the unbounded problem in terms of the frequencies of the differential equation, that is, the imaginary parts of the characteristic roots. We show that the unbounded problem can be reduced to the bounded problem if there is at most one rationally linearly independent frequency, or if there are two rationally linearly independent frequencies and all characteristic roots are simple. We complete the picture by showing that decidability of the unbounded problem in the case of two (or more) rationally linearly independent frequencies would entail a major new effectiveness result in Diophantine approximation, namely computability of the Diophantine-approximation types of all real algebraic numbers.

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1 Introduction

The Continuous Skolem Problem is a fundamental decision problem concerning reachability in continuous-time linear dynamical systems. The problem asks whether a real-valued function satisfying an ordinary linear differential equation has a zero in a given interval of real numbers. More precisely, an instance of the problem comprises an interval $I \subseteq \mathbb{R}_{\geq 0}$ with rational endpoints, an ordinary differential equation

$$f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_0 f = 0 \tag{1}$$

whose coefficients are real algebraic, together with initial conditions $f(0), \ldots, f^{(n-1)}(0)$ that are also real algebraic numbers. Writing $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for the unique solution of the

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differential equation subject to the initial conditions, the question is whether there exists $t \in I$ such that $f(t) = 0$. Decidability of this problem is currently open. Decidability of the sub-problem in which the interval I is bounded, called the Bounded Continuous Skolem Problem, is also open [4, Open Problem 17].

The nomenclature *Continuous Skolem Problem* is based on an analogy with the Skolem Problem for linear recurrence sequences, which asks whether a given linear recurrence sequence has a zero term [12]. Whether the latter problem is decidable is an outstanding question in number theory and theoretical computer science; see, e.g., the exposition of Tao [20, Section 3.9].

The continuous dynamics of linear hybrid automata and the evolution of continuous-time Markov chains, amongst many other examples, are determined by linear differential equations of the form $\boldsymbol{x}'(t) = A\boldsymbol{x}(t)$, where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and A is an $n \times n$ matrix of real numbers [1]. A basic reachability question in this context is whether, starting from an initial state $\boldsymbol{x}(0)$, the system reaches a given hyperplane $\{\boldsymbol{y} \in \mathbb{R}^n : \boldsymbol{u}^T\boldsymbol{y} = 0\}$ with normal vector $\boldsymbol{u} \in \mathbb{R}^n$. For example, one can ask whether the continuous flow of a hybrid automaton leads to a particular transition guard being satisfied or an invariant being violated. Now the function $f(t) = \boldsymbol{u}^T\boldsymbol{x}(t)$ satisfies a linear differential equation of the form (1), and it turns out that the hyperplane reachability problem is inter-reducible with the Continuous Skolem Problem (see [4, Theorem 6] for further details). Moreover, under this reduction the Bounded Continuous Skolem Problem corresponds to a time-bounded version of the hyperplane reachability problem.

The *characteristic polynomial* of the differential equation (1) is

$$\chi(x) := x^n + a_{n-1}x^{n-1} + \ldots + a_0 \,.$$

Let $\lambda_1, \ldots, \lambda_m$ be the distinct roots of χ. Any solution of (1) has the form $f(t) = \sum_{j=1}^m P_j(t)e^{\lambda_j t}$, where the P_j are polynomials with algebraic coefficients that are determined by the initial conditions of the differential equation. We call a function f in this form an *exponential polynomial*. If the roots of χ are all simple then f can be written as an exponential polynomial in which the polynomials P_j are all constant.

The Continuous Skolem Problem can equivalently be formulated in terms of whether an exponential polynomial has a zero in a given interval of reals. If the characteristic roots have the form $\lambda_j = r_j + i\omega_j$, where $r_j, \omega_j \in \mathbb{R}$, then we can also write $f(t)$ in the form $f(t) = \sum_{j=1}^m e^{r_j t}(Q_{1,j}(t)\sin(\omega_j t) + Q_{2,j}(t)\cos(\omega_j t))$, where the polynomials $Q_{1,j}, Q_{2,j}$ have real algebraic coefficients. We call $\omega_1, \ldots, \omega_m$ the *frequencies* of f.

Our first result is to show decidability of the Bounded Continuous Skolem Problem subject to Schanuel's Conjecture, a unifying conjecture in transcendental number theory that plays a key role in the study of the exponential function over both the real and complex numbers [21, 22]. Intuitively, decidability of the Bounded Continuous Skolem Problem is non-trivial because an exponential polynomial can approach 0 tangentially. Assuming Schanuel's Conjecture, we show that any exponential polynomial admits a factorisation such that the zeros of each factor can be detected using finite-precision numerical computations. Our method, however, does not bound the precision required to find zeros, so we do not obtain a complexity bound for the procedure.

A celebrated paper of Macintyre and Wilkie [18] obtains decidability of the first-order theory of $\mathbb{R}_{\text{exp}} = (\mathbb{R}, 0, 1, <, \cdot, +, \exp)$ assuming Schanuel's Conjecture over $\mathbb{R}$. The proof of [17, Theorem 3.1] mentions an unpublished result of Macintyre and Wilkie that generalises [18] to obtain decidability when $\mathbb{R}_{\text{exp}}$ is augmented with the restricted functions $\sin\restriction_{[0,2\pi]}$ and $\cos\restriction_{[0,2\pi]}$, this time assuming Schanuel's Conjecture over $\mathbb{C}$. This result immediately implies (conditional) decidability of the Bounded Continuous Skolem Problem. However, decidability of the latter problem is simpler and, as we show below, can be established more directly.

In the unbounded case, we analyse exponential polynomials in terms of the number of rationally linearly independent frequencies. We show that the unbounded problem can be reduced to the bounded problem if there is at most one rationally linearly independent frequency, or if there are two rationally linearly independent frequencies and all characteristic roots are simple. These two reductions are unconditional and rely on the cell decomposition theorem for semi-algebraic sets [3] and Baker's Theorem on linear forms in logarithms of algebraic numbers [2].

In the full version on this paper [7] we complete the picture by showing that decidability of the unbounded problem in the case of two (or more) rationally linearly independent frequencies would entail a major new effectiveness result in Diophantine approximation – namely computability of the Diophantine-approximation types of all real algebraic numbers. As we discuss in [7], currently essentially nothing is known about Diophantine-approximation types of algebraic numbers of degree three or higher, and they are the subject of several longstanding open problems.

The question of deciding whether an exponential polynomial f has infinitely many zeros is investigated in [8]. There the problem is shown to be decidable if f satisfies a differential equation of order at most 7. This result does not rely on Schanuel's Conjecture. It is also shown in [8] that, as with the Continuous Skolem Problem, decidability of the Infinite Zeros Problem in the general case would entail significant new effectiveness results in Diophantine approximation.

2 Mathematical Background

2.1 Zero Finding

Let $f : [a, b] \to \mathbb{R}$ be a function defined on a closed interval of reals with endpoints $a, b \in \mathbb{Q}$. Suppose the following two conditions hold: (i) there exists $M > 0$ such that f is M-Lipschitz, i.e., $|f(s) - f(t)| \le M|s - t|$ for all $s, t \in [a, b]$; (ii) given $t \in [a, b] \cap \mathbb{Q}$ and positive error bound $\varepsilon \in \mathbb{Q}$, we can compute $q \in \mathbb{Q}$ such that $|f(t) - q| < \varepsilon$. Then given a positive rational number δ we can compute piecewise linear functions $f_\delta^+, f_\delta^- : [a, b] \to \mathbb{R}$ such that $f_\delta^-(t) \le f(t) \le f_\delta^+(t)$ and $f_\delta^+(t) - f_\delta^-(t) \le \delta$ for all $t \in [a, b]$. We do this as follows:

1. Pick $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\delta}{4(b-a)M}$ and consider sample points $s_j := a + \frac{(b-a)j}{N}$, $j = 0, \dots, N$, dividing the interval $[a, b]$ into N sub-intervals, each of length at most $\frac{\delta}{4M}$.
2. For each sample point s_j compute $q_j \in \mathbb{Q}$ such that $|q_j - f(s_j)| < \frac{\delta}{4}$, define $f_\delta^-(s_j) = q_j - \frac{\delta}{2}$, $f_\delta^+(s_j) = q_j + \frac{\delta}{2}$, and extend f_δ^- and f_δ^+ linearly between sample points.

Note that the Lipschitz condition on f ensures that $f_\delta^- \le f \le f_\delta^+$.

Now suppose that f satisfies the following additional conditions: (iii) $f(a) \neq 0, f(b) \neq 0$; (iv) for any $t \in (a, b)$ such that $f(t) = 0$, $f'(t)$ exists and is non-zero, i.e., f has no tangential zeros. Then we can decide the existence of a zero of f by computing upper and lower approximations f_δ^+ and f_δ^- for successively smaller values of δ. If $f_\delta^+(t) < 0$ for all t or $f_\delta^-(t) > 0$ for all t then we conclude that f has no zero on $[a, b]$; if $f_\delta^+(s) < 0$ and $f_\delta^-(t) > 0$ for some s, t then we conclude that f has a zero; otherwise we proceed to a smaller value of δ. This procedure terminates since by (iii) and (iv) either f has a zero in $[a, b]$ or it is bounded away from zero.

2.2 Number-Theoretic Algorithms

For the purposes of establishing decidability, we can assume that an instance of the Continuous Skolem Problem is a real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t}$, where $\lambda_1, \ldots, \lambda_m$ and the coefficients of the polynomials $P_1, \ldots, P_m$ are algebraic, see [4, Theorem 6].

For computational purposes we represent an algebraic number α by a polynomial P with rational coefficients such that $P(\alpha) = 0$, together with a numerical approximation $p + qi$, where $p, q \in \mathbb{Q}$, of sufficient accuracy to distinguish α from the other roots of P [9, Section 4.2.1]. Given this representation we can obtain numerical approximations of α with arbitrary precision [19]

Let K be the extension field of $\mathbb{Q}$ generated by $\lambda_1, \ldots, \lambda_m$ and the coefficients of the polynomials $P_1, \ldots, P_m$. Note that K is closed under complex conjugation. We can compute a primitive element of K, that is, an algebraic number θ such that $K = \mathbb{Q}(\theta)$, together with a representation of each characteristic root λ_j as a polynomial in θ with rational coefficients (see [9, Section 4.5]). From the representation of $\lambda_1, \ldots, \lambda_m$ as elements of $\mathbb{Q}(\theta)$, it is straightforward to determine maximal $\mathbb{Q}$-linearly independent subsets of $\{\mathrm{Re}(\lambda_j) : 1 \le j \le m\}$ and $\{\mathrm{Im}(\lambda_j) : 1 \le j \le m\}$ (see [14, Section 1]).

Let log denote the branch of the complex logarithm defined by $\log(re^{i\theta}) = \log(r) + i\theta$ for a positive real number r and $0 \le \theta < 2\pi$. Recall that one can compute $\log z$ and e^z to within arbitrarily small additive error given a sufficiently precise approximation of z [6].

2.3 Laurent Polynomials

Let K be a sub-field of $\mathbb{C}$ that has finite dimension over $\mathbb{Q}$ and is closed under complex conjugation. Fix non-negative integers r and s, and consider a single variable x and tuples of variables $\boldsymbol{y} = \langle y_1, \ldots, y_r \rangle$ and $\boldsymbol{z} = \langle z_1, \ldots, z_s \rangle$. Consider the ring of Laurent polynomials

$$\mathcal{R} := K[x, y_1, y_1^{-1}, \ldots, y_r, y_r^{-1}, z_1, z_1^{-1}, \ldots, z_s, z_s^{-1}],$$

which can be seen as a localisation[1] of the polynomial ring $\mathcal{A} := K[x, y_1, \ldots, y_r, z_1, \ldots, z_s]$ in the multiplicative set generated by the set of variables $\{y_1, \ldots, y_r\} \cup \{z_1, \ldots, z_s\}$. The multiplicative units of $\mathcal{R}$ are the non-zero monomials in variables $y_1, \ldots, y_r$ and $z_1, \ldots, z_s$. As the localisation of a unique factorisation domain, $\mathcal{R}$ is itself a unique factorisation domain [10, Theorem 10.3.7]. From the proof of this fact it moreover easily follows that $\mathcal{R}$ inherits from $\mathcal{A}$ computability of factorisation into irreducibles (e.g., using the algorithm of [16]).

We extend the operation of complex conjugation to a ring automorphism of $\mathcal{R}$ as follows. Given a polynomial

$$P = \sum_{j=1}^{n} a_j x^{u_j} {y_1}^{v_{j1}} \ldots {y_r}^{v_{jr}} {z_1}^{w_{j1}} \ldots {z_s}^{w_{js}},$$

where $a_1, \ldots, a_n \in K$, define its conjugate to be

$$\overline{P} := \sum_{j=1}^{n} \overline{a_j} x^{u_j} {y_1}^{v_{j1}} \ldots {y_r}^{v_{jr}} {z_1}^{-w_{j1}} \ldots {z_s}^{-w_{js}}.$$

[1] Recall that the *localisation* of a commutative ring $\mathcal{U}$ in a multiplicatively closed subset S such that $0_{\mathcal{U}} \notin S$ is the ring of formal fractions $\mathcal{U}_S = \{a/s : a \in \mathcal{U}, s \in S\}$, with addition and multiplication defined as usual.

This definition is motivated by thinking of the variables x and $y_1, \ldots, y_r$ as real-valued and the variables $z_1, \ldots, z_s$ as taking values in the unit circle in the complex plane.

We will need the following proposition characterising those polynomials in $P \in \mathcal{R}$ such that P and $\overline{P}$ are associates, i.e., such that $\overline{P}$ is equal to the product of P by a monomial. Here we use pointwise notation for exponentiation: given a tuple of integers $\boldsymbol{u} = \langle u_1, \ldots, u_s \rangle$, we write $\boldsymbol{z}^{\boldsymbol{u}}$ for the monomial $z_1^{u_1} \ldots z_s^{u_s}$. The proof of the proposition can be found in the full version [7].

▶ **Proposition 1.** *Let $P \in \mathcal{R}$ be such that $P = \boldsymbol{z}^{\boldsymbol{u}}\overline{P}$ for $\boldsymbol{u} \in \mathbb{Z}^s$. Then either (i) P has the form $P = \boldsymbol{z}^{\boldsymbol{u}}Q$ for some $Q \in \mathcal{R}$ with $Q = \overline{Q}$, or (ii) there exists $Q \in \mathcal{R}$ such that $P = Q + \boldsymbol{z}^{\boldsymbol{u}}\overline{Q}$ and P does not divide Q in $\mathcal{R}$.*

2.4 Transcendence Theory

We will use transcendence theory in our analysis of both the bounded and unbounded variants of the Continuous Skolem Problem. In the unbounded case we will use the following classical result.

▶ **Theorem 2** (Gelfond-Schneider). *Let a, b be algebraic numbers not equal to 0 or 1. Then for any branch of the logarithm function, $\frac{\log(b)}{\log(a)}$ is either rational or transcendental.*

In fact we will make use of the following corollary, which is obtained by applying Theorem 2 to the algebraic numbers $a = e^{i(\alpha_2 - \alpha_1)}$ and $b = e^{i(\beta_2 - \beta_1)}$.

▶ **Corollary 3.** *Let $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$ all lie in $[0, \pi]$ and suppose that $\cos(\alpha_1), \cos(\alpha_2), \cos(\beta_1)$ and $\cos(\beta_2)$ are algebraic. Then $\frac{\beta_2 - \alpha_2}{\beta_1 - \alpha_1}$ is either rational or transcendental.*

Our results in the bounded case depend on Schanuel's conjecture, a unifying conjecture in transcendental number theory [15], which, if true, greatly generalises many of the central results in the field (including the Gelfond-Schneider Theorem, above). Recall that a *transcendence basis* of a field extension L/K is a subset $S \subseteq L$ such that S is algebraically independent over K and L is algebraic over $K(S)$. All transcendence bases of L/K have the same cardinality, which is called the *transcendence degree* of the extension.

▶ **Conjecture 4** (Schanuel's Conjecture [15]). *Let $a_1, \ldots, a_n$ be complex numbers that are linearly independent over $\mathbb{Q}$. Then the field $\mathbb{Q}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n})$ has transcendence degree at least n over $\mathbb{Q}$.*

A special case of Schanuel's conjecture, that is known to hold unconditionally, is the Lindemann-Weierstrass Theorem [15]: if $a_1, \ldots, a_n$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent.

We apply Schanuel's conjecture via the following proposition.

▶ **Proposition 5.** *Given non-negative integers r and s, let $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_s\}$ be $\mathbb{Q}$-linearly independent sets of real algebraic numbers. Furthermore, let $P, Q \in \mathcal{R}$ be two polynomials that have algebraic coefficients and are coprime in $\mathcal{R}$. Then the equations*

$$P(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}) = 0 \quad (2)$$

$$Q(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}) = 0 \quad (3)$$

have no non-zero common solution $t \in \mathbb{R}$.

Proof. Consider a solution $t \neq 0$ of Equations (2) and (3). By passing to suitable associates, we may assume without loss of generality that P and Q lie in $\mathcal{A}$, i.e., that all variables in P and Q appear with non-negative exponent. Moreover, since P and Q are coprime in $\mathcal{R}$, their greatest common divisor R in $\mathcal{A}$ is a monomial. In particular,

$$R(t, e^{a_1 t}, \dots, e^{a_r t}, e^{ib_1 t}, \dots, e^{ib_s t}) \neq 0\,.$$

Thus, dividing P and Q by R, we may assume that P and Q are coprime in $\mathcal{A}$ and that Equations (2) and (3) still hold.

Since coprime univariate polynomials cannot have a common root, we may assume without loss of generality that $r + s \geq 1$. By Schanuel's conjecture, the extension

$$\mathbb{Q}(a_1 t, \dots, a_r t, ib_1 t, \dots, ib_s t, e^{a_1 t}, \dots, e^{a_r t}, e^{ib_1 t}, \dots, e^{ib_s t})/\mathbb{Q}$$

has transcendence degree at least $r + s$. Since $a_1, \dots, a_r$ and $b_1, \dots, b_s$ are algebraic over $\mathbb{Q}$, writing

$$S := \langle t, e^{a_1 t}, \dots, e^{a_r t}, e^{ib_1 t}, \dots, e^{ib_s t} \rangle\,,$$

it follows that the extension $\mathbb{Q}(S)/\mathbb{Q}$ also has transcendence degree at least $r + s$.

From Equations (2) and (3) we can regard S as specifying a common root of P and Q. Pick some variable $\sigma \in \{x, y_j, z_j : 1 \leq i \leq r, 1 \leq j \leq s\}$ that has positive degree in P. Then the component of S corresponding to σ is algebraic over the remaining components of S. We claim that the remaining components of S are algebraically dependent and thus S comprises at most $r + s - 1$ algebraically independent elements, contradicting Schanuel's conjecture. The claim clearly holds if σ does not appear in Q. On the other hand, if σ has positive degree in Q then, since P and Q are coprime in $\mathcal{A}$, the multivariate resultant $\mathrm{Res}_\sigma(P, Q)$ is a non-zero polynomial in the set of variables $\{x, y_j, z_j : 1 \leq i \leq r, 1 \leq j \leq s\} \setminus \{\sigma\}$ which has a root at S (see, e.g., [11, Page 163]). Thus the claim also holds in this case. In either case we obtain a contradiction to Schanuel's conjecture and we conclude that Equations (2) and (3) have no non-zero solution $t \in \mathbb{R}$. ◀

3 Decidability of the Bounded Continuous Skolem Problem

Given non-negative integers r and s, suppose that $\{a_1, \dots, a_r\}$ and $\{ib_1, \dots, ib_s\}$ are $\mathbb{Q}$-linearly independent sets of real and imaginary numbers respectively. Let the ring of Laurent polynomials $\mathcal{R}$ be as in Section 2.3 and consider the exponential polynomial

$$f(t) = P(t, e^{a_1 t}, \dots, e^{a_r t}, e^{ib_1 t}, \dots, e^{ib_s t})\,, \tag{4}$$

where $P \in \mathcal{R}$ is irreducible. We say that f is a *type-1* exponential polynomial if P and $\overline{P}$ are not associates in $\mathcal{R}$, we say that f is *type-2* if $P = \alpha\overline{P}$ for some $\alpha \in K$, and we say that f is *type-3* if $P = U\overline{P}$ for some non-constant unit $U \in \mathcal{R}$.

▶ **Example 6.** The simplest example of a type-3 exponential polynomial is $g(t) = 1 + e^{it}$. Here $g(t) = P(e^{it})$, where $P(z) = 1 + z$ is an irreducible polynomial that is associated with its conjugate $\overline{P}(z) = 1 + z^{-1}$. Note that the exponential polynomial $f(t) = 2 + 2\cos(t)$, which has infinitely many tangential zeros, factors as the product of two type-3 exponential polynomials $f(t) = g(t)\overline{g(t)}$.

In the case of a type-2 exponential polynomial $P = \alpha\overline{P}$ it is clear that we must have $|\alpha| = 1$. Moreover, by replacing P by βP, where $\beta^2 = \overline{\alpha}$, we may assume without loss of generality that $P = \overline{P}$. Similarly, in the case of a type-3 exponential polynomial, we can assume without loss of generality that $P = \boldsymbol{z}^{\boldsymbol{u}}\overline{P}$ for some non-zero vector $\boldsymbol{u} \in \mathbb{Z}^s$.

Now consider an arbitrary exponential polynomial $f(t) := \sum_{j=1}^{m} P_j(t)e^{\lambda_j t}$. Assume that the coefficient field K of $\mathcal{R}$ contains the coefficients of $P_1, \ldots, P_m$. Let $\{a_1, \ldots, a_r\}$ be a basis of the $\mathbb{Q}$-vector space spanned by $\{\mathrm{Re}(\lambda_j) : 1 \le j \le m\}$ and let $\{b_1, \ldots, b_s\}$ be a basis of the $\mathbb{Q}$-vector space spanned by $\{\mathrm{Im}(\lambda_j) : 1 \le j \le m\}$. Without loss of generality we may assume that each characteristic root λ is an integer linear combination of $a_1, \ldots, a_r$ and $ib_1, \ldots, ib_s$. Then $e^{\lambda t}$ is a product of positive and negative powers of $e^{a_1 t}, \ldots, e^{a_r t}$ and $e^{ib_1 t}, \ldots, e^{ib_s t}$, and hence there is a Laurent polynomial $P \in \mathcal{R}$ such that

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}) . \tag{5}$$

Since P can be written as a product of irreducible factors in $\mathcal{R}$, it follows that f can be written as product of type-1, type-2, and type-3 exponential polynomials, and moreover this factorisation can be computed from f. Thus it suffices to show how to decide the existence of zeros of these three special forms of exponential polynomial. We will handle all three cases using Schanuel's conjecture.

Writing the exponential polynomial $f(t)$ in (5) in the form $f(t) = \sum_{j=1}^{m} Q_j(t)e^{\lambda_j t}$, it follows from the irreducibility of P that the polynomials $Q_1, \ldots, Q_m$ have no common root. But then by the Lindemann-Weierstrass Theorem any zero of f must be transcendental (see [4, Theorem 8]).

▶ **Theorem 7.** *The Bounded Continuous Skolem Problem is decidable subject to Schanuel's conjecture.*

Proof. Consider an exponential polynomial

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}) , \tag{6}$$

where $P \in \mathcal{R}$ is irreducible. Suppose that $\{a_1, \ldots, a_r\}$ and $\{ib_1, \ldots, ib_s\}$ are $\mathbb{Q}$-linearly independent sets of, respectively, real and imaginary numbers lying in the coefficient field K of $\mathcal{R}$. We show how to decide whether f has a zero in a bounded interval $I \subseteq \mathbb{R}_{\ge 0}$, considering separately the case of type-1, type-2, and type-3 exponential polynomials.

Case (i): f is a type-1 exponential polynomial

Note that P and $\overline{P}$ are coprime in $\mathcal{R}$ since, by assumption, they are both irreducible and are not associates. We claim that in this case the equation $f(t) = 0$ has no solution $t \in \mathbb{R}$. Indeed $f(t) = 0$ implies

$$\begin{aligned} P(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}) &= 0 \\ \overline{P}(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}) &= 0 , \end{aligned}$$

and the non-existence of a zero of f follows immediately from Proposition 5.

Case (ii): f is a type-2 exponential polynomial

In this case we have $P = \overline{P}$ and so f is real-valued. Our aim is to use the procedure of Section 2.1 to determine whether or not f has a zero in $[c, d]$, where $c, d \in \mathbb{Q}$. To this end, notice first that $f(c), f(d) \neq 0$ since any root of f must be transcendental. Moreover, since f' is bounded on $[c, d]$, f is Lipschitz on $[c, d]$. It remains to verify that the equations $f(t) = 0, f'(t) = 0$ have no common solution $t \in [c, d]$.

We can write $f'(t)$ in the form

$$f'(t) = Q(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}),$$

where Q is the polynomial

$$Q = \frac{\partial P}{\partial x} + \sum_{j=1}^{r} a_j y_j \frac{\partial P}{\partial y_j} + \sum_{j=1}^{s} i b_j z_j \frac{\partial P}{\partial z_j}.$$

We claim that P and Q are coprime in $\mathcal{R}$. Indeed, since P is irreducible, P and Q can only fail to be coprime if P divides Q.

If P has strictly positive degree k in x then Q has degree $k-1$ in x and thus P cannot divide Q. (Recall that all polynomials in $\mathcal{R}$ have non-negative degree in the variable x.) On the other hand, if P has degree 0 in x then Q is obtained from P by multiplying each monomial $\boldsymbol{y}^{\boldsymbol{u}}\boldsymbol{z}^{\boldsymbol{v}}$ appearing in P by the complex-number constant $\sum_{j=1}^{r} a_j u_j + i \sum_{j=1}^{s} b_j v_j$. Moreover, by the assumption of linear independence of $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_s\}$, each monomial in P is multiplied by a different constant. Since P is not a unit, it has at least two different monomials and so P is not a constant multiple of Q. Furthermore, for each variable $\sigma \in \{y_j, y_j^{-1} : 1 \leq j \leq r\} \cup \{z_j, z_j^{-1} : 1 \leq j \leq s\}$, its degree in P is equal to its degree in Q. Thus P cannot be a multiple of Q by a non-constant polynomial either.

We conclude that P does not divide Q and hence P and Q are coprime. It now follows from Proposition 5 that the equations $f(t) = f'(t) = 0$ have no solution $t \in \mathbb{R}$.

Case (iii): f is a type-3 exponential polynomial

Suppose that f is a type-3 exponential polynomial. Then in (6) we have that $P = \boldsymbol{z}^{\boldsymbol{u}}\overline{P}$ for some non-zero vector $\boldsymbol{u} \in \mathbb{Z}^s$. By Proposition 1 we can write $P = Q + \boldsymbol{z}^{\boldsymbol{u}}\overline{Q}$ for some polynomial $Q \in \mathcal{R}$ that is coprime with P.

Now define

$$g_1(t) := Q(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t})$$

and $g_2(t) := e^{ib_1 u_1} \cdots e^{ib_s u_s}\overline{g_1(t)}$, so that $f(t) = g_1(t) + g_2(t)$ for all t.

We show that $g_2(t) \neq 0$ for all $t \in \mathbb{R}$. Indeed if $g_2(t) = 0$ for some t then we also have $g_1(t) = 0$ and hence $f(t) = 0$. For such a t it follows that

$$\begin{aligned} P(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}) &= 0 \\ Q(t, e^{a_1 t}, \ldots, e^{a_r t}, e^{ib_1 t}, \ldots, e^{ib_s t}) &= 0. \end{aligned}$$

But P and Q are coprime and so these two equations cannot both hold by Proposition 5. Not only do we have $g_2(t) \neq 0$ for all $t \in \mathbb{R}$, but, applying the sampling procedure in Section 2.1 we can compute a strictly positive lower bound on $|g_2(t)|$ over the interval $[c, d]$.

Since $g_2(t) \neq 0$ for all $t \in \mathbb{R}$ we may define the function $h : [c, d] \to \mathbb{R}$ by

$$h(t) := \pi + i \log\left(\frac{g_1(t)}{g_2(t)}\right).$$

Notice that $h(t) = 0$ if and only if $f(t) = 0$. Our aim is to use the procedure of Section 2.1 to decide the existence of a zero of h in the interval $[c, d]$, and thus decide whether f has a zero in $[c, d]$.

Let $t \in (c,d)$ be such that $h(t) = 0$. Then $g_1(t) = -g_2(t)$ and so $\frac{g_1(t)}{g_2(t)} = -1$ does not lie on the branch cut of the logarithm function. It follows that h is differentiable at t and

$$\begin{aligned}
h'(t) = 0 \quad & \text{iff} \quad \frac{g_2(t)}{g_1(t)} \frac{g_1'(t)g_2(t) - g_2'(t)g_1(t)}{g_2(t)^2} = 0 \\
& \text{iff} \quad g_1'(t)g_2(t) - g_2'(t)g_1(t) = 0 \quad (\text{since } |g_1(t)| = |g_2(t)| \neq 0) \\
& \text{iff} \quad g_1'(t)g_2(t) + g_2'(t)g_2(t) = 0 \quad (\text{since } g_1(t) = -g_2(t)) \\
& \text{iff} \quad g_1'(t) + g_2'(t) = 0 \\
& \text{iff} \quad f'(t) = 0 \, .
\end{aligned}$$

Thus $h(t) = h'(t) = 0$ implies $f(t) = f'(t) = 0$. But the proof in Case (ii) shows that $f(t) = f'(t) = 0$ is impossible. (Nothing in that argument hinges on f being real-valued.) Thus h has no tangential zeros in (c,d).

We cannot directly use the procedure in Section 2.1 to decide whether h has a zero in $[c,d]$ since h is not necessarily continuous: its value can jump from $-\pi$ to π (or *vice versa*) due to the branch cut of the logarithm along the positive real axis. However, due to the strictly positive lower bound on $|g_2(t)|$, the function $|h|$ *is* Lipschitz on $[c,d]$. Thus, applying the sampling procedure in Section 2.1 for computing lower and upper bounds of Lipschitz functions we can compute a set $E \subseteq [c,d]$ such that E is a finite union of intervals with rational endpoints, $|f(t)| \leq \frac{2\pi}{3}$ for $t \in E$, and $|f(t)| \geq \frac{\pi}{3}$ for $t \notin E$. In particular, E contains all zeros of f in $[c,d]$ and f is Lipschitz on E. Thus we can apply the zero-finding procedure from Section 2.1 to the restriction $h \restriction E$ and thereby decide whether h has a zero on $[c,d]$. ◀

4 The Unbounded Case

In this section we consider the unbounded case of the Continuous Skolem Problem. For our analysis it is convenient to present exponential polynomials in the form

$$f(t) = \sum_{j=1}^{n} e^{r_j t} \left(P_{1,j}(t) \cos(\omega_j t) + P_{2,j}(t) \sin(\omega_j t) \right) , \tag{7}$$

where r_j, ω_j are real algebraic numbers and $P_{1,j}, P_{2,j}$ are polynomials with real algebraic coefficients for $j = 1, \ldots, n$. Our aim is to classify the difficulty of the problem in terms of the number of rationally linear independent frequencies $\omega_1, \ldots, \omega_n$.

Recall that in Section 3 we have shown the bounded problem to be decidable subject to Schanuel's Conjecture. In the full version of this paper [7] we give a reduction of the unbounded problem to the bounded problem in case the set of frequencies spans a one-dimensional vector space over $\mathbb{Q}$. In the present section we give a reduction of the unbounded problem to the bounded problem in case the set of frequencies spans a two-dimensional vector space over $\mathbb{Q}$ and the polynomials $P_{1,j}$ and $P_{2,j}$ are all constant. (This last condition is equivalent to the assumption that $f(t)$ is simple.) The argument in the two-dimensional case is a more sophisticated version of that in the one-dimensional case, although the result is not more general due the assumption of simplicity.

In the full version [7] we present a family of instances showing that obtaining decidability of the unbounded problem in the two-dimensional case without the assumption of simplicity would require much finer Diophantine-approximation bounds than are currently known.

4.1 Background on Semi-Algebraic Sets

A subset of $\mathbb{R}^n$ is *semi-algebraic* if it is defined by a Boolean combination of constraints of the form $P(x_1, \ldots, x_n) > 0$, where P is a polynomial with real algebraic coefficients. A partial function $f : \mathbb{R}^n \to \mathbb{R}$ is semi-algebraic if its graph is a semi-algebraic subset of $\mathbb{R}^{n+1}$. The Tarski-Seidenberg Theorem [5, Section 1] states that the semi-algebraic sets are closed under projection and are therefore precisely the first-order definable sets over the structure $(\mathbb{R}, <, +, \cdot, 0, 1)$.

Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones of length $n \geq 1$. An $(i_1, \ldots, i_n)$*-cell* is a subset of $\mathbb{R}^n$, defined by induction on n as follows:

(i) A (0)-cell is a singleton subset of $\mathbb{R}$ and a (1)-cell is an open interval $(a, b) \subseteq \mathbb{R}$.

(ii) Let $X \subseteq \mathbb{R}^n$ be a $(i_1, \ldots, i_n)$-cell and $f : X \to \mathbb{R}$ a continuous semi-algebraic function. Then $\{(\boldsymbol{x}, f(\boldsymbol{x})) \in \mathbb{R}^{n+1} : \boldsymbol{x} \in X\}$ is a $(i_1, \ldots, i_n, 0)$-cell, while $\{(\boldsymbol{x}, y) \in \mathbb{R}^{n+1} : \boldsymbol{x} \in X \wedge y < f(\boldsymbol{x})\}$ and $\{(\boldsymbol{x}, y) \in \mathbb{R}^{n+1} : \boldsymbol{x} \in X \wedge y > f(\boldsymbol{x})\}$ are both $(i_1, \ldots, i_n, 1)$-cells.

(iii) Let $X \subseteq \mathbb{R}^n$ be a $(i_1, \ldots, i_n)$-cell and $f, g : X \to \mathbb{R}$ continuous semi-algebraic functions such that $f(\boldsymbol{x}) < g(\boldsymbol{x})$ for all $\boldsymbol{x} \in X$. Then $\{(\boldsymbol{x}, y) \in \mathbb{R}^{n+1} : f(\boldsymbol{x}) < y < g(\boldsymbol{x})\}$ is a $(i_1, \ldots, i_n, 1)$-cell.

A cell in $\mathbb{R}^n$ is a $(i_1, \ldots, i_n)$-cell for some (necessarily unique) sequence $(i_1, \ldots, i_n)$.

A fundamental result about semi-algebraic sets, that we will use below, is the Cell-Decomposition Theorem [3]: given a semi-algebraic set $E \subseteq \mathbb{R}^n$ one can compute a partition of E as a disjoint union of cells $E = C_1 \cup \ldots \cup C_m$.

We will also need the following result, proved in [7].

▶ **Lemma 8.** *Let $D \subseteq \mathbb{R}^n$ be a semi-algebraic set, $g : D \to \mathbb{R}$ a bounded semi-algebraic function, and $r_1, \ldots, r_n$ real algebraic numbers. Define $S = \{t \in \mathbb{R}_{\geq 0} : (e^{r_1 t}, \ldots, e^{r_n t}) \in D\}$. Then*

1. *It is decidable whether or not S is bounded. If S is bounded then we can compute $T_0 \in \mathbb{N}$ such that $S \subseteq [0, T_0]$ and if S is unbounded then we can compute $T_0 \in \mathbb{N}$ such that $(T_0, \infty) \subseteq S$.*
2. *If S is unbounded then the limit $g^* = \lim_{t\to\infty} g(e^{r_1 t}, \ldots, e^{r_n t})$ exists, is an algebraic number, and there are effective constants $T_1, \varepsilon > 0$ such that $|g(e^{r_1 t}, \ldots, e^{r_n t}) - g^*| < e^{-\varepsilon t}$ for all $t > T_1$.*

4.2 Two Linearly Independent Frequencies

The following lemma, which is a reformulation of [4, Lemma 13], plays an instrumental role in this section. The lemma itself relies on a powerful quantitative result in transcendence theory – Baker's Theorem on linear forms in logarithms of algebraic numbers [2].

▶ **Lemma 9.** *Let b_1, b_2 be real algebraic numbers, linearly independent over $\mathbb{Q}$. Furthermore, let φ_1, φ_2 be real numbers such that $e^{i\varphi_1}$ and $e^{i\varphi_2}$ are algebraic. Then there exist effectively computable constants $N, T > 0$ such that for all $t \geq T$ and all $k_1, k_2 \in \mathbb{Z}$, at least one of $|b_1 t - \varphi_1 - 2k_1\pi| > 1/t^N$ and $|b_2 t - \varphi_2 - 2k_2\pi| > 1/t^N$ holds.*

The main result of the section is the following.

▶ **Theorem 10.** *Let $f(t) = \sum_{j=1}^{n} e^{r_j t} \left(a_{1,j} \cos(\omega_j t) + a_{2,j} \sin(\omega_j t)\right)$ be an exponential polynomial where $r_j, a_{1,j}, a_{2,j}, \omega_j$ are real algebraic numbers and the $\mathbb{Q}$-span of $\{\omega_1, \ldots, \omega_n\}$ has dimension two as a $\mathbb{Q}$-vector space. Then we can decide whether or not $\{t \in \mathbb{R}_{\geq 0} : f(t) = 0\}$ is bounded and, if bounded, we can compute an integer T such that $\{t \in \mathbb{R}_{\geq 0} : f(t) = 0\} \subseteq [0, T]$.*

Proof. Let b_1, b_2 be real algebraic numbers, linearly independent over $\mathbb{Q}$, such that ω_j is an integer linear combination of b_1 and b_2 for $j = 1, \ldots, n$. For each $n \in \mathbb{Z}$, $\sin(nb_1 t)$ and $\cos(nb_1 t)$ can be written as polynomials in $\sin(b_1 t)$ and $\cos(b_1 t)$ with integer coefficients, and similarly for b_2. It follows that we can write f in the form

$$f(t) = Q(e^{r_1 t}, \ldots, e^{r_n t}, \cos(b_1 t), \sin(b_1 t), \cos(b_2 t), \sin(b_2 t))$$

for some polynomial Q with real algebraic coefficients that is computable from f.

Write $R_{++} = \{t \geq 0 : \sin(b_1 t) \geq 0 \wedge \sin(b_2 t) \geq 0\}$, $R_{+-} = \{t \geq 0 : \sin(b_1 t) \geq 0 \wedge \sin(b_2 t) \leq 0\}$, and likewise define R_{-+}, R_{--} for the two remaining sign conditions on $\sin(b_1 t)$ and $\sin(b_2 t)$. We show how to decide boundedness of $\{t \in R_{++} : f(t) = 0\}$. (The cases for R_{+-}, R_{-+}, and R_{--} follow *mutatis mutandis*.) The idea is to compute a partition of $\{t \in R_{++} : f(t) = 0\}$ into components $Z_1, \ldots, Z_m$ and to separately decide boundedness of each component Z_j.

Define a semi-algebraic set

$$E = \left\{(\boldsymbol{u}, x_1, x_2) \in \mathbb{R}^{n+2} : \exists y_1, y_2 \geq 0 \left(x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1 \wedge Q(\boldsymbol{u}, x_1, y_1, x_2, y_2) = 0\right) \right\}.$$

Then for $t \in R_{++}$ we have $f(t) = 0$ if and only if $(e^{\boldsymbol{r}t}, \cos(b_1 t), \cos(b_2 t)) \in E$, where $\boldsymbol{r} = (r_1, \ldots, r_n)$. Now consider a cell decomposition $E = C_1 \cup \ldots \cup C_m$ for cells $C_1, \ldots, C_m \subseteq \mathbb{R}^{n+2}$, and define

$$Z_j = \{t \in R_{++} : (e^{\boldsymbol{r}t}, \cos(b_1 t), \cos(b_2 t)) \in C_j\}, \qquad j = 1, \ldots, m, \tag{8}$$

Then $\{t \in R_{++} : f(t) = 0\} = Z_1 \cup \ldots \cup Z_m$.

Now fix $j \in \{1, \ldots, m\}$. We show how to decide boundedness of Z_j. To this end, write $D_j \subseteq \mathbb{R}^n$ for the projection of the corresponding cell $C_j \subseteq \mathbb{R}^{n+2}$ on the first n coordinates.

First suppose that $\{t \in \mathbb{R} : e^{\boldsymbol{r}t} \in D_j\}$ is bounded. Then by Lemma 8 we can compute an upper bound T of this set. But $Z_j \subseteq \{t \in \mathbb{R}_{\geq 0} : e^{\boldsymbol{r}t} \in D_j\}$ and so $Z_j \subseteq [0, T]$.

On the other hand, suppose that $\{t \in \mathbb{R} : e^{\boldsymbol{r}t} \in D_j\}$ is unbounded. Then, by Lemma 8, this set contains an unbounded interval (T, ∞) for some $T \in \mathbb{N}$. Write $I = [-1, 1]$ and define functions $g_1, g_2, h_1, h_2 : D_j \to \mathbb{R}$ by

$$g_1(\boldsymbol{u}) = \inf\{x \in I : \exists y\, (\boldsymbol{u}, x, y) \in C_j\} \qquad g_2(\boldsymbol{u}) = \inf\{y \in I : \exists x\, (\boldsymbol{u}, x, y) \in C_j\} \tag{9}$$

$$h_1(\boldsymbol{u}) = \sup\{x \in I : \exists y\, (\boldsymbol{u}, x, y) \in C_j\} \qquad h_2(\boldsymbol{u}) = \sup\{y \in I : \exists x\, (\boldsymbol{u}, x, y) \in C_j\} \tag{10}$$

These functions are all semi-algebraic by quantifier elimination. Hence by Lemma 8 the limits $g_i^* = \lim_{t\to\infty} g_i(e^{\boldsymbol{r}t})$ and $h_i^* = \lim_{t\to\infty} h_i(e^{\boldsymbol{r}t})$ exist for $i = 1, 2$ and are algebraic numbers. Clearly we have $g_1^* \leq h_1^*$ and $g_2^* \leq h_2^*$. We now consider three cases according to the strictness of these inequalities.

Case I: $g_1^* = h_1^*$ and $g_2^* = h_2^*$

We show that Z_j is bounded and that we can compute T_2 such that $Z_j \subseteq [0, T_2]$.

By Lemma 8 there exist $T_1, \varepsilon > 0$ such that for all $t > T_1$ and $i = 1, 2$,

$$|g_i(e^{\boldsymbol{r}t}) - g_i^*| < e^{-\varepsilon t} \text{ and } |h_i(e^{\boldsymbol{r}t}) - h_i^*| < e^{-\varepsilon t}. \tag{11}$$

Then for $t \in R_{++}$ such that $t > T_1$ we have

$$\begin{aligned}
t \in Z_j \quad &\Longleftrightarrow \quad \left(e^{\boldsymbol{r}t}, \cos(b_1 t), \cos(b_2 t)\right) \in C_j \quad \text{(by (8))} \\
&\Longrightarrow \quad g_1(e^{\boldsymbol{r}t}) \leq \cos(b_1 t) \leq h_1(e^{\boldsymbol{r}t}) \text{ and } g_2(e^{\boldsymbol{r}t}) \leq \cos(b_2 t) \leq h_2(e^{\boldsymbol{r}t}) \quad \text{(by (9)(10))} \\
&\Longrightarrow \quad |\cos(b_1 t) - g_1^*| < e^{-\varepsilon t} \text{ and } |\cos(b_2 t) - g_2^*| < e^{-\varepsilon t} \quad \text{(by (11))} \qquad (12)
\end{aligned}$$

Write $g_1^* = \cos(\varphi_1)$ and $g_2^* = \cos(\varphi_2)$ for some $\varphi_1, \varphi_2 \in [0, \pi]$. Since $|\cos(\varphi_1 + x) - \cos(\varphi_1)| \geq x^3/3$ for all x sufficiently small (by a Taylor expansion), the inequality (12) implies that for some $k_1, k_2 \in \mathbb{Z}$,

$$|b_1 t - \varphi_1 - 2k_1\pi| < 3e^{-\varepsilon t/3} \text{ and } |b_2 t - \varphi_2 - 2k_2\pi| < 3e^{-\varepsilon t/3}\,. \tag{13}$$

Combining the upper bounds in (13) with the polynomial lower bounds $|b_1 t - \varphi_1 - 2k_1\pi| > 1/t^N$ and $|b_2 t - \varphi_2 - 2k_2\pi| > 1/t^N$ from Lemma 9 we obtain an effective bound T_2 for which $t \in Z_j$ implies $t < T_2$.

Case II: $g_1^* < h_1^*$

In this case we show that Z_j is unbounded. The geometric intuition is as follows. We imagine a particle in the plane whose position at time t is $(\cos(b_1 t), \cos(b_2 t))$, together with a "moving target" whose extent at time t is $\Gamma_t = \{(x, y) : (e^{\boldsymbol{r}t}, x, y) \in C_j\}$. Below we essentially argue that such a particle is bound to hit Γ_t at some time t since its orbit is dense in $[-1, +1]^2$ and Γ_t has positive dimension in the limit.

Proceeding formally, first notice that C_j cannot be a $(\ldots, 0, 1)$-cell or a $(\ldots, 0, 0)$-cell, for then we would have $g_1(\boldsymbol{u}) = h_1(\boldsymbol{u})$ for all $\boldsymbol{u} \in D_j$ and hence $g_1^* = h_1^*$. Thus C_j must either be a $(\ldots, 1, 0)$-cell or a $(\ldots, 1, 1)$-cell. In either case, C_j includes a cell of the form $\{(\boldsymbol{u}, x, \xi(\boldsymbol{u}, x)) : \boldsymbol{u} \in D, g_1(\boldsymbol{u}) < x < h_1(\boldsymbol{u})\}$ for some semi-algebraic function ξ.

Let c, d be real algebraic numbers such that $g_1^* < c < d < h_1^*$. Write $c = \cos(\psi')$ and $d = \cos(\psi)$ for $0 \leq \psi < \psi' \leq \pi$. By Lemma 8 the limits $\lim_{t\to\infty} \xi(e^{\boldsymbol{r}t}, c)$ and $\lim_{t\to\infty} \xi(e^{\boldsymbol{r}t}, d)$ exist and are algebraic numbers in the interval $[-1, 1]$. Let $\theta, \theta' \in [0, \pi]$ be such that $\cos(\theta) = \lim_{t\to\infty} \xi(e^{\boldsymbol{r}t}, d)$ and $\cos(\theta') = \lim_{t\to\infty} \xi(e^{\boldsymbol{r}t}, c)$.

By Corollary 3 we know that $\frac{\theta'-\theta}{\psi'-\psi}$ is either rational or transcendental. In particular we know that it is not equal to $\frac{b_2}{b_1}$, which is algebraic and irrational. Let us suppose that $\frac{\theta'-\theta}{\psi'-\psi} > \frac{b_2}{b_1}$ (the converse case is almost identical). Then there exists θ'' with $\theta < \theta'' < \theta'$, such that

$$\theta < \theta'' + \frac{b_2}{b_1}(\psi' - \psi) < \theta'\,. \tag{14}$$

Since $2\pi, b_1, b_2$ are linearly independent over $\mathbb{Q}$ it follows from Kronecker's approximation theorem that $\{(b_1 t, b_2 t) \bmod 2\pi : t \in \mathbb{R}_{\geq 0}\}$ is dense in $[0, 2\pi)^2$ (see [13, Chapter 23]). Thus there is an increasing sequence $t_1 < t_2 < \ldots$, with $b_1 t_n \equiv \psi \bmod 2\pi$ for all n, such that $b_2 t_n \bmod 2\pi$ converges to θ''. Then, defining $s_1 < s_2 < \ldots$ by $s_n = t_n + \frac{\psi'-\psi}{b_1}$, we have $b_1 s_n \equiv \psi' \bmod 2\pi$ for all n and, by (14),

$$\lim_{n\to\infty} b_2 s_n = \lim_{n\to\infty} b_2 t_n + \frac{b_2}{b_1}(\psi' - \psi) = \theta'' + \frac{b_2}{b_1}(\psi' - \psi) < \theta' \pmod{2\pi}\,.$$

Let $\eta(t) = \xi(e^{\boldsymbol{r}t}, \cos(b_1 t)) - \cos(b_2 t)$. Then for $t \in R_{++}$ such that $g(e^{\boldsymbol{r}t}) < \cos(b_1 t) < h(e^{\boldsymbol{r}t})$,

$$\begin{aligned} \eta(t) = 0 \quad &\Longrightarrow \quad \cos(b_2 t) = \xi(e^{\boldsymbol{r}t}, \cos(b_1 t)) \\ &\Longrightarrow \quad (e^{\boldsymbol{r}t}, \cos(b_1 t), \cos(b_2 t)) \in C_j \\ &\Longrightarrow \quad t \in Z_j \text{ (by (8))}\,. \end{aligned}$$

Now $\lim_{n\to\infty} \eta(t_n) = \cos(\theta) - \cos(\theta'') > 0$ and $\lim_{n\to\infty} \eta(s_n) < \cos(\theta') - \cos(\theta') = 0$. Moreover for n sufficiently large we have $[t_n, s_n] \subseteq R_{++}$. It follows that $\eta(t)$ has a zero in every interval $[t_n, s_n]$ for n large enough. We conclude that Z_j is unbounded.

Case III: $g_2^* < h_2^*$

This case is symmetric to Case II and we omit details. ◀

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