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# Optimal Online Escape Path Against a Certificate*

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## Abstract

More than fifty years ago Bellman asked for the best escape path within a known fores t but for an unknown starting position. This deterministic finite path is the shortest path that leads out of a given environment from any starting point. There are some worst case positions where the full path length is required. Up to now such a fixed *ultimate optimal escape path* for a known shape for any starting position is only known for some special convex shapes (i.e., circles, strips of a given width, fat convex bodies, some isosceles triangles).

Therefore, we introduce a different, simple and intuitive escape path, the so-called *certificate path* which makes use of some additional information w.r.t. the starting point $s$. This escape path depends on the starting position $s$ and takes the distances from $s$ to the outer boundary of the environment into account. Because of this, in the above convex examples the certificate path *always* (for any position $s$) leaves the environment earlier than the ultimate escape path.

Next we assume that neither the precise shape of the environment nor the location of the starting point is known, we have much less information. For a class of environments (convex shapes and shapes with kernel positions) we design an *online* strategy that always leaves the environment. We show that the path length for leaving the environment is always shorter than 3.318764 the length of the corresponding certificate path. We also give a lower bound of 3.313126 which shows that for the above class of environments the factor 3.318764 is (almost) tight.

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## 1 Introduction

We consider the following motion planning task. Let us assume that we are given a simple polygon $P$ and a starting point $s$ inside $P$. We would like to design a simple path starting at $s$ that finally hits the boundary and leaves the polygon. In the sense of a game, we can choose a path but then an adversary can rotate the polygon $P$ around $s$ so that the path will leave the polygon very late.

Since we know the distances to the boundary we apply a simple intuitive strategy for this problem. The *certificate path* is the best combination of a line segment $l$ and an arc of length $l\alpha$ along the circle of radius $l$ around the starting point. So this path simply checks an

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angular portion of the environment for a distance $l$. For a given starting point the certificate path is the best (shortest) such path that guarantees to hit the boundary. Altogether the certificate path is a very simple *escape path* for given $s$ and $P$ (if an adversary can only rotate $P$ around $s$).

In turn, for any given unknown starting position $s$ inside an unknown polygon we would like to design an *online strategy* (with less information) that is never much worse than the length of the above certificate path. In this paper we show that for a class of environments there is a spiral strategy that leaves any such polygon and approximates the length of the certificate path within a ratio of 3.318674. We also prove that this is an (almost) tight bound. There is no other strategy that always attains a better ratio against the length of the certificate path.

This optimal online approximation is restricted to the following class of environments. We assume that in any direction from the unknown starting point only one boundary point exists. The distance to the boundary points still remains unknown. This subsumes any unknown convex environment (for any unknown starting position) and also unknown star-shaped environments (for any unknown starting point inside the kernel). The motivation of comparing an online escape path for special polygons (star-shaped) and special starting positions (inside the kernel) with a path that is computed with some additional but not complete information (certificate path) stems from the following observation.

For a given known polygonal shape and an unknown starting point we can also define the *ultimate optimal escape path.* This path will lead out of the environment for any starting point and any rotation of the polygon. The ultimate optimal escape path is the shortest finite path with this property. The clue is that only the polygon is known but neither the starting position nor the rotation around the starting position. The path is motivated by the situation of swimming in the fog in a pool whose shape is known. Because it is foggy, the starting point and the rotation around the starting point is not known. Unfortunately, ultimate optimal escape paths have been found only for a few special convex shapes (circles, strips of given width, fat convex bodies, isosceles triangles, ...). It is unrealistic to think that such paths will be found for more complicated convex or star-shaped environments.

Fortunately, for the few known ultimate optimal escape paths the above defined certificate path is not only a good approximation, we can even show that the certificate path *beats* the ultimate escape paths for *any* starting point in these examples. Therefore, we can argue that the certificate path for any star-shaped polygon and any starting point inside the kernel can serve as a substitute for the unknown ultimate optimal escape path.

The paper is organized as follows. In the next section we present the related work. After introducing the certificate path in Section 3 and showing different justifications for the measure in Section 4, we present and analyse a strategy with path length not larger than 3.318674 times the length of the certificate path in Section 5. The strategy is a logarithmic spiral attained by keeping aware of two extremes of the certificate. Optimizing the spiral for two extremes is also different from classical logarithmic spiral constructions where we normally optimize against a single distance (shortest path). In Section 6 we present a general lower bound that also proves that the given strategy is almost optimal for the restricted cases. No other strategy will have a better ratio than 3.313126 against the length of the certificate path. Proving lower bounds is a tedious task, the construction and the analysis might be interesting in its own right.

## 2 Related work

The *Swimming-in-the-fog* problem is a game where two players, a searcher and a hider, compete with each other. The searcher tries to reach the boundary of a known shape from its starting point along a single finite path, while the hider can rotate and translate the environment so that the path of the searcher will cross the boundary as late as possible. For a given shape the shortest finite path that always leads out of the given environment can be denoted as an *ultimate optimal escape* path as mentioned before.

Search games have been studied in many variations in the last 60 years since the first work by Koopman in 1946. The book by Gal [13] and the reissue by Alpern and Gal [1] gives a comprehensive overview of such search game problems also for unknown environments.

The above problem goes at least back to 1956 and to Bellman [3] who similarly asked for the shortest escape path within a known forest but for an unknown starting point. Since then the problem has attracted a lot of attention. Unfortunately until today, the problem could be solved only for very special convex environments (circles, strips of given width, rectangles, fat convex bodies, isosceles triangles); see for example the monograph of Finch and Wetzel [9].

For circles and fat convex bodies it was shown that the diameter is the ultimate optimal escape path; see Finch and Wetzel [9]. For the strip of width $l$ the ultimate optimal escape path was found by Zalgaller [24, 25]. For the simple equilateral triangle of side length 1 the Zig-Zag path of Besicovitch [4] of length $\approx 0.981981$ is optimal; see also [5]. Furthermore, in 1961 Gluss [14] introduced the problem of searching for a circle $C$ of given radius $s$ and given distance $r$ away from the start $A$. Two different cases can be considered, either $A$ is inside $C$ or not. Interestingly, in the latter case and for $s = 1$ a certificate path with length $l = r$ and an arc of length $2\pi \cdot l$ is the best one can do.

It is unrealistic to think that such ultimate optimal escape paths will be achieved for more complicated environments. As an alternative in this paper for a given environment and a given starting point $s$ we have introduced a simple and natural *certificate path* that is computed individually for any starting point and takes the distance distribution from $s$ to the boundary into account. Fortunately, for the cases above we are able to show that our certificate path outperforms the corresponding ultimate escape paths for any possible starting point. The certificate path always leaves the environment earlier. The proofs are given in the full version of the paper; see [20].

The use of alternative comparison measures has some tradition. For example for the problem of searching for a point in a polygon and competing against the shortest path there is no competitive strategy. So for this case also other comparison measures have been suggested; see Fleischer et al. [11] or Koutsoupias et al. [18]. Additionally, comparing the online strategy to the shortest path to the boundary is often a very difficult task. For example, the spiral conjecture for searching for a single line or a single ray against the shortest path is still open. In this sense our result might be considered as an intermediate step.

We further assume that the precise shape of the environment and the position of the starting point is not known. We are searching for a good *online* approximation that competes with the certificate path which is computed with more information. We make use of the competitive framework. That is, we compare the length of the online escape path from a starting point to the boundary to the length of the certificate path to the boundary computed for the known environment and starting point. The competitive framework was introduced by Sleator and Tarjan [23], and used in many settings since then; see for example the survey by Fiat and Woeginger [7] or, for the field of online robot motion planning, see the surveys [15, 21].

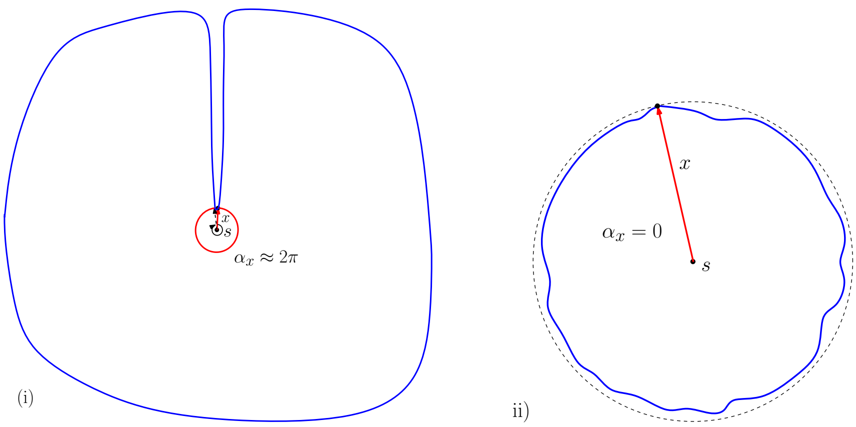


**Figure 1** Two extreme situation for reaching the boundary with a circular arc. On the right-hand side of the figure, the radial maximal distance from $s$ to the boundary is almost the same in any direction. So it suffices to move in an arbitrary direction of maximal distance, which is optimal. On the left-hand side of the figure only the distance to some few boundary points is very small, but much larger to most of the others. Therefore, a reasonable path checks the small distance with a circular arc of length approximately $2\pi$. In both cases $x(1 + \alpha_x)$ is minimal among all such circular strategies.

Our optimal online approximation is restricted to the following class of environments. We assume that in any direction from the unknown starting point only one boundary point exists. The distance to the boundary points still remains unknown. This subsumes any unknown convex environment (for any unknown starting position) and also unknown star-shaped environments (for any unknown starting point inside the kernel). In this sense the certificate is also a natural extension of the discrete performance measure Kirkpatrick [16] mentioned in the discrete case of searching for the end of a set of $m$ given lists of unknown length. In his setting it is sufficient to reach the end of only one list. In our configuration this means, that we have exactly $2\pi$ directions of unknown distance and it is sufficient to reach the shoreline in a single point. The corresponding relationship is shown in Section 4.

We will see that our solution is a specific logarithmic spiral. In general, logarithmic spirals are natural candidates for optimal competitive search strategies, but in almost all cases the optimality remains a conjecture; see [2, 6, 8, 10, 13]. In [19] the optimality of spiral search was shown for searching a point in the plane with a radar. Many other conjectures are still open. For example Finch and Zhu [10] considered the problem of searching for a line in the plane, the *relevant conjecture that the family of logarithmic spirals contains the minimal path remains open.*

## 3 The certificate path

Assume that you are located in an unknown environment and would like to reach its boundary. Formally, for the environment we consider a closed Jordan curve $B$ that subdivides the Euclidean plane into exactly two regions. The starting point $s$ lies inside the inner region, say $P$. The task is to reach a point on the boundary $B$ as soon as possible.

If you have some idea about the distance $x$ from $s$ to the boundary $B$ but nothing more, it is very intuitive to move along the circle of radius $x$ around the starting point. Therefore, a reasonable strategy moves toward this circle along a shortest path (by radius $x$) in some direction and then follows the circle in either clockwise or counterclockwise direction until the boundary is met. Let us denote this behaviour a *circular strategy.* If we hit the boundary after moving an arc $\alpha_x$ along the circle, the overall path length is given by $x(1 + \alpha_x)$.

We would like to use such a circular strategy of small path length. In the sense of a game, the adversary can only rotate the environment around the starting point and the certificate path guarantees to hit the boundary for any rotation.

## 3.1 Extreme cases and general definition

Let us first consider two somehow extreme examples of the above intuitive idea as given in Figure 1. If the distance from $s$ to the boundary is almost the same in any direction (similar to a circle), a line segment with maximal distance to the boundary (roughly the radius of the circle) will always hit the boundary and is indeed a very good escape path for any direction; see Figure 1(ii). The movement along an arc is not necessary in this case. In other words, $\alpha_x$ equals 0. We check a single direction for the largest distance.

On the other hand, if the distance to the boundary is very large w.r.t. almost all directions from $s$, but is relatively small (distance $x$) for some few directions, a segment of length $x$ and a circular arc of length $x\alpha_x$ with $\alpha_x \approx 2\pi$ will hit the boundary for any starting direction of the segment $x$; see Figure 1(i). The overall path length $x(1 + \alpha_x)$ is also comparatively small. The certificate path checks a small distance for many (almost all) directions.

Now, consider a more general environment modelled by a simple polygon $P$ and a fixed starting point $s$ in $P$ as given in Figure 2(ii). For convenience, we make use of an example, where any boundary point $b$ of $P$ is *visible* from $s$, i.e. the segment $sb$ lies fully inside $P$. Or the other way round, $s$ lies inside the kernel of $P$. Note, that the certificate can also be computed for more general polygons as shown in the full version of the paper [20].

For the polygon $P$ and for any radial direction $\phi \in [0, 2\pi]$ from $s$, we consider the boundary point $p_{s,\phi}$ on $P$ in direction $\phi$. This gives a *radial distance function* $f(\phi) := |s\, p_{s,\phi}|$ as depicted in Figure 2(i).

Now, let $p_{s,\phi}$ be a point with distance $x := |s\, p_{s,\phi}|$ in direction $\phi$. For any circle $C_s(x)$ with radius $x$ around $s$ such that $C_s(x)$ hits the boundary of $P$, there will be some maximal arc $\alpha_s(x)$ so that the above simple circular strategy is successful. Note, that this is independent from the starting direction for $x$. We are looking for the maximum circle segment of $C_s(x)$ that fully lies inside $P$.

Let $\Pi_s(x)$ denote the *certificate path for distance* $x$ of maximal length $x(1 + \alpha_s(x))$. The interpretation is that independent from the starting direction for $x$, this finite path will always touch the boundary. The adversary can only rotate the environment in order to attain a worst case length of $x(1 + \alpha_x)$. The path $\Pi_s(x)$ can be found in the plot of the radial distance function; see Figure 2(i). It consists of two segments, starting with a vertical segment of length $x$ and ending with a horizontal segment of length $\alpha_x$. For any starting angle this path will touch the boundary of the distance function.

In turn, the overall *certificate path* $\Pi_s$ in $P$ for a given starting point $s$ is the shortest certificate path $\Pi_s(x)$ among all distances $x$. That is, the certificate for $P$ and $s$ is:

$$\Pi_s := \min_x \Pi_s(x) = \min_x x(1 + \alpha_s(x)) \ .$$

For both extreme situations in Figure 1, the presented paths equal the overall certificate paths for the given environments.

If parts of the boundary are not visible from $s$, there is more than one boundary point in some directions. In this case we can also compute the radial distances in a continuous way

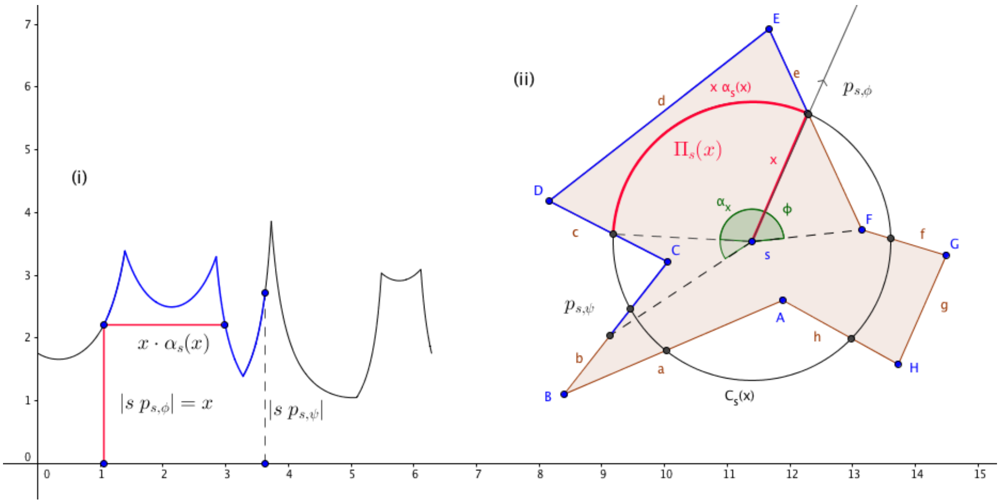


**Figure 2** (ii) Consider the polygon $P$ and a starting point $s$. Let us assume that we radially sweep the boundary of $P$ (starting from point $F$ with angle 0) in counterclockwise order and calculate the distance from the boundary to $s$ for any angle. (i) shows this *radial distance function* of the boundary of $P$ from $s$ in polar coordinates for the interval $[0, 2\pi]$. The blue sub-curve corresponds to the blue boundary part in (ii). The certificate path $\Pi_s(x)$ for distance $x$ is the longest path that successfully checks the distance $x$ by a circular strategy. This means that it hits the boundary for any starting direction $\phi$ of $x$ in $P$. In the polar-coordinate setting in (i) this is a path with two line segments of length $x$ and $\alpha_x$ that always hits the boundary of the radial distance function independent from the starting angle $\phi$. For the case where the boundary of $P$ is not totally visible an example is given in the full version of the paper [20].

and obtain a radial distance curve. The definition of the certificate path remains exactly the same. An example is given in the full version of the paper; see [20].

## 4 Justification of the certificate

The certificate path is an intuitive and simple way of leaving an environment and can be computed in polynomial time by computing a lower envelope of upper envelopes. We can interpret the certificate as a path that balances depth-first and breadth-first search for the starting position $s$ in a way that the resulting path is as short as possible. That way, it outperforms the ultimate optimal escape path at any given starting position for all known cases as proved in the full version of this paper; see [20].

Furthermore, the certificate is closely related to a discrete cost measure that Kirkpatrick introduces in [16]. He analyses the problem of digging for oil at $m$ different locations $s_i$, where $|s_i|$ denotes the (unknown) distance to the source of the oil at the corresponding location. In this scenario, no extra costs arise for switching the location. The challenge is to find a strategy that reaches one source of oil while assuring a small overall digging effort.

At first, Kirkpatrick considers (partially informed) strategies. Those are given all distances from the top to the sources of oil, but not the corresponding location: In case the distances $|s_i|$ have about the same length at all locations, he states that a depth-first searching strategy is certainly effective. Thus, a single location can be chosen for digging, as Figure 3(i) indicates. Although at the chosen location, the distance to the source might be greatest, the digging costs are almost optimal. In case the distance to the source of a single location is significantly

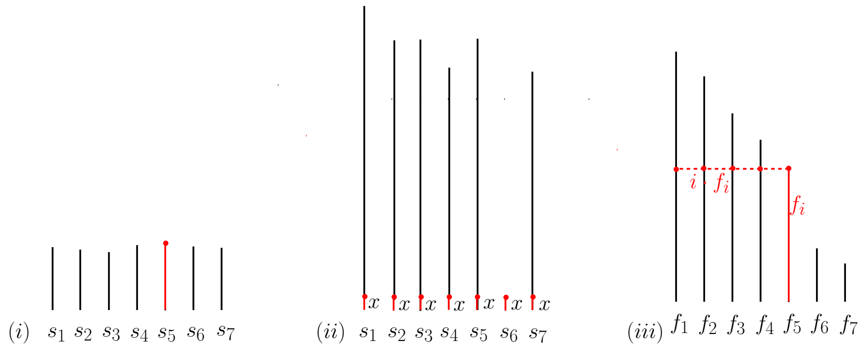


**Figure 3** Online searching for the end of a segment (or digging for oil) for $m = 7$ segments of unknown length. There are two extreme cases: (i) All segments have about the same length. It is reasonable to move along an arbitrary segment up to the end, which is almost optimal. (ii) One segment is significantly shorter than all other segments. One will find the end of a shortest segment by checking all segments with its length. (iii) In case the length of each segment is known, but not the corresponding number of segment. There is always an optimal strategy: Assume that $f_1 \geq f_2 \geq \cdots \geq f_m$ is the decreasing order of the length of all segments. An optimal strategy explores $i$ (arbitrary) segments up to depth $f_i$, where $i$ is chosen so that $i \cdot f_i = \min_{1 \leq k \leq m} k \cdot f_k$.

shorter than all others, a breadth-first searching strategy performs best. Figure 3(ii) shows that digging at every location with a certain effort $x$ still achieves a small overall effort of $x \cdot m$ in the worst case. These two extreme situations are similar to the cases outlined in Section 3.1 and depicted in Figure 1. For the general case, Kirkpatrick suggests to use a hybrid strategy. If $f_1 \geq f_2 \geq \cdots \geq f_m$ denotes the sorted set of distances, he suggests to choose $i$ so that $i \cdot f_i$ is minimal. The hybrid strategy digs at $i$ (arbitrarily chosen) locations up to the same depth $f_i$. In the worst case, this strategy reaches a source at the last location with a final effort of $i \cdot f_i$; see Figure 3(iii). Among all such partially informed strategies, this hybrid strategy is certainly optimal and achieves a maximum digging effort of $\lambda := i \cdot f_i$. Similar to this hybrid strategy, we defined the certificate path in the previous section. The certificate path can also be considered as a mixture of depth-first and breadth-first searching. However, the certificate path models a motion. The effort of the digging strategy to explore a certain depth depends on the product of the number chosen locations and the digging depth. In contrast to this, the effort of the certificate path depends on the sum of the searching depth and width. Consequently, the certificate path is a stronger cost measure than the equivalent of the hybrid digging strategy in the plane.

During the further analysis, Kirkpatrick compares a totally uninformed digging strategy to the optimal hybrid strategy. He proves that this strategy approximates the hybrid strategy in $O(\lambda \log(\min(m, \lambda))$ and shows that this factor is tight. Similar to his approach, we compare the certificate path to a totally uninformed spiral strategy and obtain a constant competitive ratio. David Kirkpatrick [17] brought up the question what happens in a continuous setting. Note, that the game is a quite different in this case, because we have to take the movements in the plane into account and we also require a starting orientation.

## 5 Online approximation of the certificate path

We are searching for a reasonable escape strategy in an unknown environment. As shown in the previous section, the certificate path and its length is a reasonable candidate for

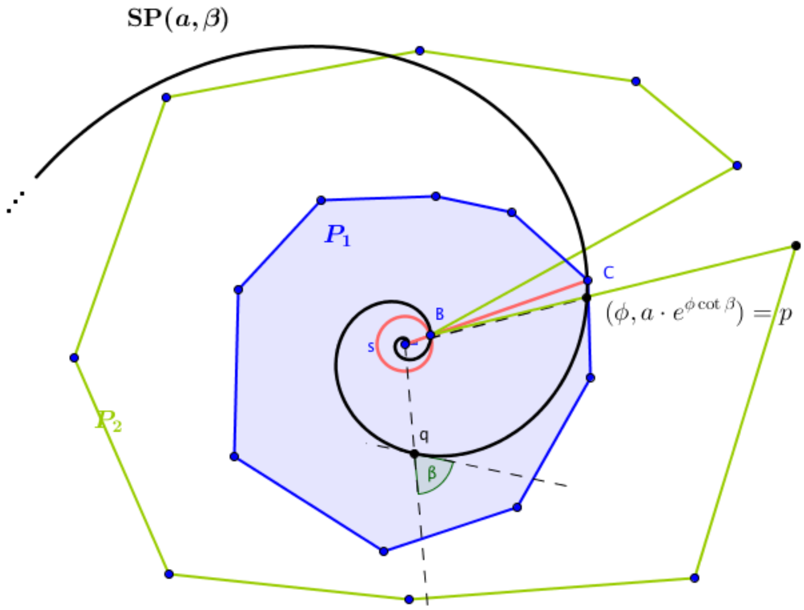


**Figure 4** We apply a spiral strategy for unknown polygons and an unknown starting point $s$ in the kernel. The eccentricity $\beta$ is chosen so that the two extreme cases have the same ratio. For both polygons $P_1$ and $P_2$, the strategy passes the boundary at point $p = (\phi, a \cdot e^{\phi \cot \beta})$ close to $C$. The path length of the strategy for leaving the polygons is roughly the same. The certificate for $P_1$ has length $|s\,C|$ (checking the maximal distance to the boundary of $P_1$), whereas the certificate for $P_2$ has length $|s\,B|(1 + 2\pi)$ with $|s\,C| = e^{2\pi \cot \beta}|s\,B|$ (checking the smallest distance to the boundary of $P_2$ with a full circle). We can construct such examples for any point $p$ on the spiral.

comparisons. Let us assume that $x(1 + \alpha_x)$ is the length of the certificate for some polygon $P$ and for an arbitrary distance $x$. We can assume that $\alpha_x \in [0, 2\pi]$. This holds since the shortest distance $d_s$ from $s$ to the boundary always results in a candidate $d_s(1 + 2\pi)$. All other reasonable distances $x$ are larger than $d_s$ and $\alpha_x \le 2\pi$ holds for the optimal $x$.

Similar to the considerations of Kirkpatrick (see Section 4), we would like to guarantee that we leave the polygon $P$ if we have overrun the distance $x$ more than $\alpha_x$ times. This means that the boundary should not wind arbitrarily around $s$. Therefore, we restrict our consideration to a position $s$ in the kernel of a star-shaped polygon so that there is a single (unknown) distance to the outer boundary in any direction.

In this case we apply the following logarithmic spiral strategy. A logarithmic spiral can be defined by polar coordinates $(\varphi, a \cdot e^{\varphi \cot(\beta)})$ for $\varphi \in (-\infty, \infty)$, a constant $a$ and an eccentricity $\beta$ as shown in Figure 4. For an angle $\phi$, the path length of the spiral up to point $(\phi, a \cdot e^{\phi \cot(\beta)})$ is given by $\frac{a}{\cos \beta} e^{\phi \cot(\beta)}$.

For our purpose we choose $\beta$ so that the two extreme cases of the certificate attain the same ratio; see Figure 4. We can assume that the certificate of the environment is $x(1 + \alpha_x)$ for an arbitrary distance $x$ and an angle $\alpha_x \in [0, 2\pi]$. Since the spiral strategy checks the distances in a monotonically increasing and periodical way, there has to be some angle $\phi$ so that $x = a \cdot e^{(\phi - \alpha_x)\cot(\beta)}$ holds. This means that in the worst case, the spiral strategy will leave the environment at point $p = (\phi, a \cdot e^{\phi \cot(\beta)})$ with path length $\frac{a}{\cos \beta} \cdot e^{\phi \cot(\beta)}$. Exactly $\alpha_x$ distances of length $x$ have been exceeded, which means that the boundary has been reached. (Note that, this might not hold for points outside the kernel.)

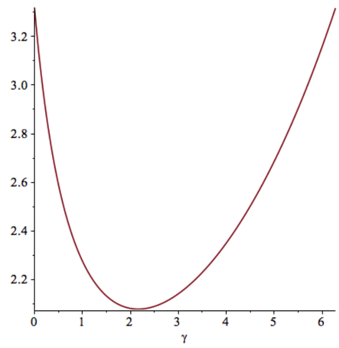


**Figure 5** The graph of the ratio function $f$ of Equation (3) for the spiral strategy with eccentricity $\beta \approx 1.26471$. The two extreme cases 0 and $2\pi$ have the same ratio $\approx 3.318674$ and all other ratios are strictly smaller.

We would like to choose $\beta$ so that the two extreme cases $\alpha_x = 0$ and $\alpha_x = 2\pi$ have the same ratio. Thus, we are searching for an angle $\beta$ so that

$$\frac{\frac{a}{\cos\beta} \cdot e^{\phi \cot\beta}}{a \cdot e^{\phi \cot\beta}(1+0)} = \frac{\frac{a}{\cos\beta} \cdot e^{\phi \cot\beta}}{a \cdot e^{(\phi-2\pi)\cot\beta}(1+2\pi)} \Leftrightarrow \quad (1)$$

$$1 = \frac{e^{2\pi \cot\beta}}{1+2\pi} \quad (2)$$

holds. The right-hand side of Equation (1) shows the case where $x_2 = a \cdot e^{(\phi-2\pi)\cot(\beta)}$ and $\alpha_{x_2} = 2\pi$ gives the certificate and the left-hand side shows the case that $x_1 = a \cdot e^{\phi \cot(\beta)}$ and $\alpha_{x_1} = 0$ gives the certificate $x_i(1+\alpha_{x_i})$, respectively. In both cases the spiral will detect the boundary latest at point $p = (\phi, a \cdot e^{\phi \cot(\alpha)})$, because the spiral checks $2\pi$ distances larger than or equal to $x_2$ and at least one distance $x_1$. Figure 4 shows the construction of corresponding polygons $P_1$ and $P_2$.

The solution of Equation (2) gives $\beta = \operatorname{arccot}\left(\frac{\ln(2\pi+1)}{2\pi}\right) = 1.264714\ldots$ and the ratio is $\frac{1}{\cos\beta} = 3.3186738\ldots$. Fortunately, for all other values $x = a \cdot e^{(\phi-\gamma)\cot\beta}$ and $\alpha_x = \gamma$ for $\gamma \in (0, 2\pi)$ the ratio is smaller than these two extremes. The overall ratio function is

$$f(\gamma) = \frac{\frac{a}{\cos\beta} \cdot e^{\phi \cot\beta}}{a \cdot e^{(\phi-\gamma)\cot\beta}(1+\gamma)} = \frac{e^{\gamma \cot\beta}}{\cos\beta(1+\gamma)} \text{ for } \gamma \in [0, 2\pi] \quad (3)$$

and Figure 5 shows the plot of all possible ratios of the spiral strategy with eccentricity $\beta$.

Altogether, we have the following result.

▶ **Theorem 1.** *There is a spiral strategy for any unknown starting point $s$ inside the kernel of an unknown environment $P$ that always hits the boundary with path length smaller than* 3.318674 *times the length of the corresponding certificate for $s$ and $P$.*

**Proof.** Assume that the certificate of $P$ and $s$ is given by $x(1+\alpha_x)$. We can set $\gamma := \alpha_x$ and we will also find an angle $\phi$ so that $x = a \cdot e^{(\phi-\gamma)\cot\beta}$ holds. At point $p = (\phi, a \cdot e^{\phi \cot\beta})$ the spiral has subsumed an arc of angle $\gamma$ with distances $x$, so the spiral strategy will leave $P$ at $p$ in the worst case. (Note that, if the start point is not inside the kernel, this might not be true!) The ratio is given by $f(\gamma)$ as in (3) and Figure 5. In the worst case for the strategy $\gamma$ is either 0 or $2\pi$ for the ratio 3.318674, respectively. ◀

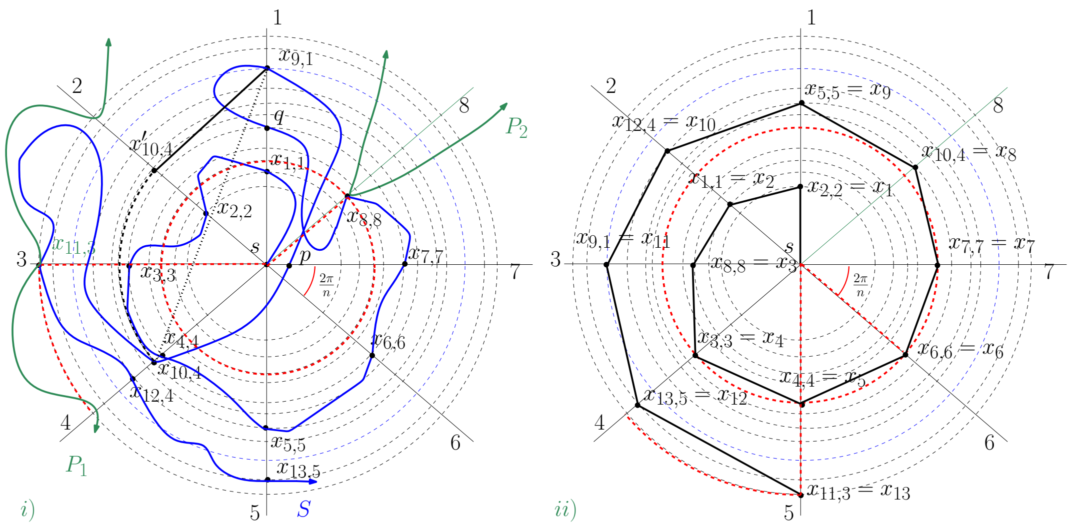


**Figure 6** i) The strategy $S$ results in a sequence $S'$ that represents the visits on $n$ rays successively. There will be a next entry $x_{i,j_i}$ if the strategy exceeds the distance on ray $j_i$. For two successive extensions on the same ray only the last entry is registered in $S'$. For the subsequence $S'_{13} = (x_{1,1}, x_{2,2}, \ldots, x_{13,5})$ there will be a last visit on each ray, a minimal distance $x_m = x_{8,8}$ on ray 8 and a maximal distance $x_M = x_{11,3}$ on ray 3. These values gives rise to the construction of certificates for $S$ as sketched by the polygons $P_1$ and $P_2$. There are polygons $P_1$ and $P_2$ with certificates $x_M(1 + \frac{2\pi}{n})$ and $x_m(1 + 2\pi)$ and so far $S$ has not been escaped from neither $P_1$ nor $P_2$. In $S'_{13}$ the direct distance between two successive points, for example $x_{9,1}$ and $x_{10,4}$, is shorter than the original path length on $S$ and we can further shorten the distance by assuming that $x_{10,4}$ is on the neighbouring ray as depicted by $x'_{10,4}$.

ii) We sort the entries of $S'_k$ into a sequence $X_k$ and visit the rays in increasing distance and periodic order. The path length of $X_k$ is not larger than the original path length of $S_k$ and the corresponding certificates for the maximal and minimal value $x'_M = x_M$ and $x'_m > x_m$ are not smaller. Thus, the sum of the corresponding ratios gives a lower bound for the sum of the original ratios.

We have designed a spiral strategy for some reasonable environments. In the next section, we give a lower bound that shows that this strategy is (almost) optimal for these environments.

Note that a spiral strategy for the online approximation of the certificate path of an arbitrary unknown polygon and position cannot be competitive in general. The polygon might itself wind around the spiral. The ratio against the certificate might be arbitrarily large, consequently. In more general environments other online strategies have to be applied. A potential strategy might be a connected sequence of circles $C_i$ with exponentially increasing radii $r^i$. This online strategy should result in a constant competitive ratio. Obtaining the optimal strategy for such cases might be complicated and gives rise to future work.

## 6 Lower bound construction: Online strategy against the certificate

▶ **Theorem 2.** *Any strategy that escapes from an unknown environment $P$ in unknown position $s$ will achieve a competitive factor of at least* 3.313126 *against the length of a corresponding certificate for $s$ and $P$ in the worst-case.*

**Proof.** Let us assume that a strategy $S$ is given that attains a better ratio $C$ in the worst case. We consider a bunch of $n$ rays emanating from $s$ with equidistant angle $\frac{2\pi}{n}$ as depicted in Figure 6 for $n = 8$.

The strategy $S$ will successively extend the distances from $s$ also along the rays. Let the sequence $S' = (x_{1,j_1}, x_{2,j_2}, x_{3,j_3}, \ldots)$ describe successive visits of the $n$ rays by the strategy. In $x_{i,j_i}$ the entry $i$ stands for the order and $j_i$ stands for the ray. In $S'$ we only register a visit on ray $j_i$, if it exceeds the previous visit on the ray $j_i$. Furthermore, if the distance at ray $j_i$ is exceeded in two successive entries we do not register the first visit in the sequence $S'$.

In Figure 6(i) we have registered 13 successive visits $x_{i,j_i}$. Here for example the visit of ray 7 at point $q$ between $x_{9,1}$ and $x_{10,4}$ was not registered in $S'$ because it does not improve the distance of the former visit $x_{7,7}$. Additionally, the visit of ray 1 at point $q$ just before $x_{9,1}$ improved the distance $x_{1,1}$ but it was further improved on $x_{9,1}$ and in between no other ray was improved. For any continuous strategy $S$ we will find such an infinite sequence $S'$. Let $x_{i,j_i}$ denote the visit and also the distance to the starting point $s$.

Let us assume that we stop the strategy $S$ at of some ray $j_k$ where the distance was just exceeded on ray $j_k$, so $S'$ has $k$ steps. Let $S_k$ denote the sub-strategy of $S$ and $S'_k$ the corresponding subsequence. There will be at least two ratios for $S_k$ that correspond to values of $S'_k$ as follows. In $S'_k$ we consider the last visits on each ray which gives the corresponding maximal visited distance to $s$ for each ray. There will be an overall maximal distance $x_M$ on some ray $M_j$ and a minimal distance $x_m$ on some other ray $m_j$. In Figure 6(i) we have stopped the strategy $S$ at $x_{13,5}$ on ray 5 and in $S'_{13}$ the minimal distance for the last *round* is given by $x_m = x_{8,8}$ on ray 8 and the maximal distance is given by $x_M = x_{11,3}$ on ray 3.

We can construct polygons $P_1$ and $P_2$ so that $x_M(1 + \frac{2\pi}{n})$ is a certificate for $P_1$ and $x_m(1 + 2\pi)$ is a certificate for $P_2$. In the first case all other rays have been visited with depth smaller or equal to $x_M$ and we build a polygon $P_1$ outside $S_k$ that visits any ray at $x_M - \epsilon$ and ray $M_j$ at $x_M$. This means that a circular strategy with $x_M$ and an arc of length $x_M \frac{2\pi}{n}$ will be sufficient and gives the certificate for $P_1$ (or at least an upper bound for the certificate of $P_1$). See for example the polygon $P_1$ sketched in Figure 6(i) for the maximal visit $x_M = x_{13,5}$. On the other hand for the minimal value $x_m$ we construct a polygon $P_2$ that hits $x_m$ on ray $m_j$ but runs arbitrarily far away from $S_k$ in any other direction. Thus, $x_m(1 + 2\pi)$ gives the certificate (or at least an upper bound for the certificate of $P_2$). See for example the polygon $P_2$ sketched in Figure 6(i) for the minimal visit $x_m = x_{8,8}$. We do not expect that $S_k$ has already detected these polygons but $S$ finally will. So the ratio of the path length $|S_k|$ over $x_m(1 + 2\pi)$ and also the ratio of the path length $|S_k|$ over $x_M(1 + \frac{2\pi}{n})$ give lower bounds for the strategy $S$. Note that the half of the sum of the two ratios cannot exceed $C$ because otherwise at least one has to be greater than $C$.

For any such stop we will sort the values of $S'_k$ in a sequence $X_k$ and we will visit the $n$ rays in a monotone and periodic way by sequence $X_k$ connecting the points by line segments; see Figure 6(ii). We can prove that the overall path length of $X_k$ cannot be larger than $|S_k|$.

This can be seen as follows. Successively visiting the points of $S'_k$ in a polygonal chain is already a short cut for $S_k$. This polygonal path for $S'_k$ might move between two successive values $x_{i,j_i}$ to $x_{i+1,j_{i+1}}$ where $j_i$ and $j_{i+1}$ are not neighbouring rays. In this case we can further short cut the length of the chain of $S'_k$ by just counting a movement from $x_{i,j_i}$ to the distance $x_{i+1,j_{i+1}}$ on one of the directly neighbouring rays. For example in Figure 6(i) the segment from $x_{9,1}$ and $x'_{10,4}$ improves the path length from $x_{9,1}$ and $x_{10,4}$ but passes ray 2 and 3. We only count the distance between $x_{9,1}$ and $x'_{10,4}$ on the neighbouring ray which further improves the length. This means that for a lower bound on the overall path length we can also consider a path that visits two neighbouring rays with angle $\frac{2\pi}{n}$ successively from one to the other with the corresponding depth values $x_{i,j_i}$ stemming from $S'_k$. By triangle inequality it can be shown that the shortest path that visit all the depth of a sequence $S'_k$ on two rays by changing from one ray to the other in any step, visits the two rays successively in an increasing order. A similar argument was applied by one author of this article in [19] where a detailed proof of this property is given in the Appendix of [19]. Finally, we can rearrange the path of $S'_k$ to a path that visits the rays in a periodic and monotone way.

Altogether, we have translated the strategy $S_k$ in a discrete strategy $X_k = (x_1, x_2, \ldots, x_k)$ with $k$ entries on $n$ rays that visit the rays in a periodic order such that $x_i$ visits ray $i \bmod n$ and with overall shorter path length; see Figure 6(ii). Consider the corresponding certificates of this new strategy in comparison to the original strategy. For the smallest value on the last round and the largest value on the last round we will obtain a certificate path $x_k(1+\frac{2\pi}{n})$ which is the same for the previous maximal value $x_M = x_k$ and a certificate path $x_{k-n+1}(1+2\pi)$ which is never smaller than $x_m(1+2\pi)$ for the minimal value $x_m \leq x_{n-k+1}$. The minimal value can only increase since we sorted the values of $S'_k$. For example in Figure 6(ii) we have $k = 13$ and the minimal value in the last round is given by $x_6 = x_{6,6}$ which is larger than $x_m = x_{8,8}$. Altogether, the sum of these two ratios in the periodic and monotone setting is always smaller than the sum of the ratios in the original setting.

Finally, we would like to find a periodic and monotone strategy that optimizes the sum of exactly such ratios in this discrete version. This optimal strategy will perform at least as good as any strategy obtained by the above reconstruction. Thus, the optimal value for the sum is a lower bound for the sum of two ratios in the original setting.

For optimizing the sum for an arbitrary strategy we use an infinite sequence of values $X = (x_1, x_2, \ldots)$ and we define the following functionals

$$F_k^1(X) = \frac{\sum_{i=1}^{k-1} \sqrt{x_i^2 - 2\cos\left(\frac{2\pi}{n}\right) x_i x_{i+1} + x_{i+1}^2}}{x_k(1+\frac{2\pi}{n})} \tag{4}$$

and

$$F_k^2(X) = \frac{\sum_{i=1}^{k-1} \sqrt{x_i^2 - 2\cos\left(\frac{2\pi}{n}\right) x_i x_{i+1} + x_{i+1}^2}}{x_{k-n+1}(1+2\pi)} \tag{5}$$

that represent the ratios. We are looking for a sequence $X$ so that

$$\inf_Y \sup_k F_k^1(Y) + F_k^2(Y) = D \text{ and } \sup_k F_k^1(X) + F_k^2(X) = D$$

holds which shows that $D$ is the best sum ratio that we can achieve.

Optimizing such discrete functionals can be done by the method proposed by Gal; see also Gal [12, 13], Alpern and Gal [1], and an adaption of Schuierer [22]. It is shown that under certain prerequisites there will be an optimal exponential strategy $x_i = a^i$. The main requirement is that the functional has to fulfil a unimodality property. This means that the piecewise sum of two strategies $X$ and $Y$ is never worse than one of the single strategies. This should also hold for a scalar multiplication of a single strategy. So any linear combination of strategies that are bounded by a constant will remain bounded by the maximal bound. The proof of Gal shows that in this case we can always combine bounded strategies so that we finally get arbitrarily close to an exponential strategy that has the same bound; see the full proof of Gal in [13] Appendix 2, Theorem 1.

We can easily show that the requirements for the main Theorem of Gal are fulfilled for both functionals $F_k^1(X)$ and $F_k^2(X)$. For a similar functional a detailed proof of this property was given in the Appendix of [19].

Now let us assume that we have an optimal strategy $X$ for the sum, say $F_k^1(X) + F_k^2(X)$. This means that both functionals will also be bounded by constants $D_1$ and $D_2$ w.r.t. $X$. We make use of linear combination of $X$ but apply them independently to the functionals $F_k^1(X)$

and $F_k^2(X)$. The Theorem of Gal shows that we will get arbitrarily close to an exponential strategy $x_i = a^i$ that is not worse than $X$ for both $F_k^1(X)$ and $F_k^2(X)$. This means that $x_i = a^i$ is also not worse than $X$ for the sum functional.

Altogether, it is allowed to search for the best strategy $x_i = a^i$ and we have to optimize

$$\frac{\sum_{i=1}^{k-1} \sqrt{a^{2i} - 2\cos\left(\frac{2\pi}{n}\right) a^{2i+1} + a^{2i+2}}}{a^k(1+\frac{2\pi}{n})} + \frac{\sum_{i=1}^{k-1} \sqrt{a^{2i} - 2\cos\left(\frac{2\pi}{n}\right) a^{2i+1} + a^{2i+2}}}{a^{k-n+1}(1+2\pi)} \Longleftrightarrow$$

$$\sum_{i=1}^{k-1} a^i \left( \frac{\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right) a + a^2}}{a^k(1+\frac{2\pi}{n})} \right) + \sum_{i=1}^{k-1} a^i \left( \frac{\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right) a + a^2}}{a^{k-n+1}(1+2\pi)} \right) . \tag{6}$$

For Equation (6) we resolve the geometric serie part and simplify the expression to the minimization of

$$g_n(a) := \frac{1}{a-1} \left( \frac{\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right) a + a^2}}{(1+\frac{2\pi}{n})} \right) + \frac{a^{n+1}}{a-1} \left( \frac{\sqrt{1 - 2\cos\left(\frac{2\pi}{n}\right) a + a^2}}{(1+2\pi)} \right) . \tag{7}$$

We minimize Equation (6) by numerical means. For any number of rays $n$ a minimal value of $g_n(a)$ gives a lower bound on the sum of two ratios in the original problem. So we can choose $n$ as large as we want. We minimize $g_n(a)$ by numerical means using Maple. For example for $n = 28000000000$ we obtain $a = 1.0000000006809\ldots$ and $g(a) = 6.62521\ldots$ This means that for an arbitrary strategy of the original problem there will always be at least one ratio larger than $\frac{6.6252}{2} = 3.313126$ which finishes the proof. ◀

## 7 Conclusion

We have introduced a new, simple and intuitive performance measure for the comparison against an online escape path for an unknown environment. The measure outperforms the (few) known ultimate optimal escape paths of convex environments and is also sort of a generalization of a discrete list searching approach by Kirkpatrick.

For a more general class of environments, we presented an online spiral strategy that approximates the measure within an (almost) optimal factor of $\approx 3.318674$. Different to classical results the spiral optimizes against two extremes. It was shown that the factor is almost tight by constructing a lower bound that also holds for arbitrary environments. Additionally, one of the very few cases where the optimality of spiral search is verified.

Future work can be done by considering randomization. Additionally, we would like to prove the strong conjecture that the certificate path is indeed always better than the shortest escape path for all environments (even when the optimal escape path is not known).

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