Structural Interactions and Absorption of Structural Rules in BI Sequent Calculus

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Abstract

Development of a contraction-free BI sequent calculus, be the contraction-freeness implicit or explicit, has not been successful in the literature. We address this problem by presenting such a sequent system. Our calculus involves no structural rules. It should be an insight into non-formula contraction absorption in other non-classical logics. Contraction absorption in sequent calculus is associated to simpler cut elimination and to efficient proof searches.

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1 Introduction

Propositional BI [22] is a combined logic formed from propositional intuitionistic logic IL and propositional multiplicative fragment of intuitionistic linear logic MILL. Recall that IL, and respectively MILL, have the following logical connectives: $\{\top_0, \bot_0, \wedge_2, \vee_2, \supset_2\}$ (Cf. any standard text on the mathematical logic for intuitionistic logic; [16] for instance), and respectively, $\{\mathbf{1}_0, \otimes_2, \multimap_2\}$ (Cf. [11] for linear logic).[1] A rough intuition about BI is that a BI expression is any expression that is constructable from $(\mathcal{P}, \{\top_0, \bot_0, \wedge_2, \vee_2, \supset_2, \mathbf{1}_0, \otimes_2, \multimap_2\})$. $\mathcal{P}$ denotes some set of propositional letters. Following the popular convention in BI, we use the symbol $*$ in place of $\otimes$, and $-\!\!*$ in place of $\multimap$. In place of $\mathbf{1}$, we use $^*\!\top$, emphasising some link of its to $\top$, as to be shortly stated. It holds true that what IL or MILL considers a theorem, BI also does [22]. To this extent BI is a conservative extension of the two propositional logics.

Now, one may contemplate the converse. Is it the case that what BI considers a theorem, IL or MILL also does, i.e. is it the case that every BI formula is reducible either into an IL formula or into a MILL formula? It is stated in [22] that that is not so.

Analysis of the way logics combine is itself an interest. When one combines two logics, it is possible - depending on how the chosen methodology combines the logics - that some logical connective in one of them collapses onto some logical connective in the other. A notable example is the case of classical logic and intuitionistic logic [6, 5]. There, intuitionistic implication can become classical implication. If another approach is chosen, classical implication can also become intuitionistic implication. In order to prevent these from occurring, one must prepare the combined logic domain in such a way that, within the domain, the classical logic domain is sufficiently independent of the intuitionistic logic domain. The reason pertains to the difference in their viewpoint of what an infinity is. Similarly in the combination of IL

[1] The subscripts denote the arity.

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with MILL, some sort of the merging of logical connectives could occur. In BI, one that is intentionally avoided is the conflict between the two implications. The following example in the BI proof theory is taken from [22].

$$\frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G} \supset R \quad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G} \mathrel{-\!\!*} R$$

F and G are assumed to be some arbitrary BI formula. The semi-colon and the comma are the two structural connectives acting as the structural counterparts of $\wedge$ and respectively $*$, which can nest over one another. Γ denotes some arbitrary BI structure.[2] BI achieves separation of the two implications by the two structural connectives.[3] Here the basic axioms of IL and MILL can be recalled: that if $(F \wedge G) \supset H$ for some formulas F, G and H is a theorem in IL, then so is $F \supset (G \supset H)$ (which structurally translates into $\supset R$ above); and that if $(F * G) \mathrel{-\!\!*} H$ is a theorem in MILL, then so is $F \mathrel{-\!\!*} (G \mathrel{-\!\!*} H)$ (which structurally translates into $\mathrel{-\!\!*} R$ above). On the other hand, there is certain glueing between $\top$ and $^*\top$: in BI, F is a true expression iff $F * {}^*\top$ is iff $F \wedge \top$ is. This connection is chosen not to be eliminated, although it could be eliminated if one so desires.

Under the particular combination that forms BI, there is no free distribution of ";" over "," or of "," over ";". This implies that a BI structure is, as we just stated, a nesting of structures in the form of $\Gamma_1; \Gamma_2$, called additive structures, and those in the form of Γ_1, Γ_2, called multiplicative structures. There is a proof theoretical asymmetry among them by the availability of structural inference rules. Consider for example the following familiar structural rules (in sequent calculi) that come from IL:

$$\frac{\Gamma(\Gamma_1; \Gamma_1) \vdash F}{\Gamma(\Gamma_1) \vdash F} \text{Contraction} \quad \frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1; \Gamma_2) \vdash F} \text{Weakening}$$

Here $\Gamma(\cdots)$ abstracts any other structures surrounding the focused ones in the sequents. These are available in BI sequent calculus LBI [24]. On the other hand, neither of the inferences below is - as a rule - permitted.

$$\frac{\Gamma(\Gamma_1, \Gamma_1) \vdash F}{\Gamma(\Gamma_1) \vdash F} \qquad \frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1, \Gamma_2) \vdash F}$$

'As a rule' because there are some exceptions to the guideline.

$$\frac{\Gamma(\Gamma_1, {}^*\top) \vdash F}{\Gamma(\Gamma_1) \vdash F} \qquad \frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1, {}^*\top) \vdash F}$$

1.1 Research problems and contributions

In $\Gamma_1; \Gamma_1$ on the premise of Contraction, or in $\Gamma_1; \Gamma_2$ on the conclusion of Weakening, neither Γ_1 nor Γ_2 must be additive. Consider then the following inferences, each of which is an instance of Contraction:

$$\frac{\Gamma((F; F), G) \vdash H}{\Gamma(F, G) \vdash H} Ctr_1 \quad \frac{\Gamma(F, (G; G)) \vdash H}{\Gamma(F, G) \vdash H} Ctr_2 \quad \frac{\Gamma((\Gamma_a, \Gamma_b); (\Gamma_a, \Gamma_b)) \vdash H}{\Gamma(\Gamma_a, \Gamma_b) \vdash H} Ctr_3$$

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[2] These and other orthodox proof-theoretical terms are assumed to be familiar to the readers. They are found for example in [25, 16]. The formal definitions that we will need for our technical discussions will be found in the next section.

[3] The need for more than one structural connectives in proof systems was recognised in display calculus [3] as well as in other studies, e.g. in the multi-modal categorial type logics [20] and in relevant logics [18, 19], which were developed prior to the appearance of BI.

Observe it is a formula that duplicates upwards in the first two inferences. These are simply adaptations of the usual structural contraction available in `G1i` [25], the standard `IL` sequent calculus. It is a well-known fact that, as far as `G1i` is concerned, elimination of the structural weakening requires hardly any effort, and that the structural contraction goes admissible once the left implication rule is modified in the weakening-free `IL` sequent calculus; Cf. [25, 16] for the results but also [13] for the idea of eliminating the structural contraction rule. Given that the same elimination technique has been shown to be applicable to many other extensions of `IL`, it is expected on a reasonable ground that handling these formula contractions (and weakenings) is straightforward also in `LBI`. As can be seen in Ctr_3, however, the scope of Contraction is not restricted to the formula contractions. The degree of the nesting of additive/multiplicative structures in Γ_1 in Contraction can be arbitrarily large.

One pertinent question to ask is if it is possible at all to eliminate the non-formula contractions from `LBI`, eliminating Contraction as the result. Actually, it is not very difficult to postpone answering this question, if replacement of Contraction with a set of alternative new structural rules is permitted. The Contraction can be then emulated in the new structural rules. Such replacement strategies work particularly well if one retains the cut rule in the sequent system. Knowing, however, that they rather relocate the issue that was expressed in the original question into the new structural rules, we may just as well strengthen the question and ask, instead, if a `BI` sequent calculus without structural rules is derivable at all, this way precluding any miscommunication.

In setting for the investigation, it seems there are two major sources of difficulty one must face. The first difficulty comes from the equivalences $\Gamma,^{*}\top = \Gamma = \Gamma;\top$, structural counterparts of the above-mentioned equivalences, which imply bidirectional inference rules.

$$\frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1;\top) \vdash F} \quad \frac{\Gamma(\Gamma_1;\top) \vdash F}{\Gamma(\Gamma_1) \vdash F} \quad \frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1,{}^{*}\top) \vdash F} \quad \frac{\Gamma(\Gamma_1,{}^{*}\top) \vdash F}{\Gamma(\Gamma_1) \vdash F}$$

As well as being obvious sources of non-termination, they obscure the core mechanism of the interactions between additive and multiplicative structures, since they imply a free transformation of an additive structure into a multiplicative one and vice versa. The second difficulty is the difficulty of isolating the effect of the structural contraction from that of the structural weakening. Donnelly *et al* [7] succeeded in eliminating structural weakening; however, they had to absorb contraction into the structural weakening as well as into logical rules. Absorption of one structural rule into another structural rule is a little problematic, since - as we have already mentioned - the former still occurs indirectly through the latter which is a structural rule. It is also not so straightforward to know whether either weakening or contraction is immune to the effect of the structural equivalences.

Despite the technical obstacles, we show the answer to the above-posed question to be in the affirmative by presenting a structural-rule-free `BI` sequent calculus. What it is to `LBI` is what `G3i` is to `G1i`. As far as can be gathered from the literature, the elimination of contraction from `BI` sequent calculus has not been previously successful, be the sense of contraction-freeness according to the sense in `G3i` or the sense in `G4i` [8]. The following are some motivations for presenting such a sequent calculus.

1. The current status of the knowledge of structural interactions within `BI` proof systems is not very satisfactory. From the perspective of theorem proving for example, the presence of the bidirectional rules and contraction as explicit structural rules in `LBI` means that it is difficult to actually prove that an invalid `BI` formula is underivable within the calculus. This is because `LBI` by itself does not provide termination conditions save when a (backward) derivation actually terminates: the only case in which no more backward derivation on a `LBI` sequent is possible is when the sequent is empty; the only case in

which it is empty is when it is the premise of an axiom. The contraction-free BI sequent calculus is a step forward in this respect.

2. There are other sequent calculi that necessarily require a non-formula structural contraction rule (or else alternative structural rules that emulate the effect). Sequent systems of the relevant logics closely related to BI [10] are good examples. Sequent systems of some constructive modal logics [23] also require non-formula contractions; Cf. [1]. It tends to be almost always the case that the presence of a structural contraction rule increases the technical complexity of a cut elimination proof (see the induction measure in [1]). The techniques to eliminate non-formula structural contraction rules are useful for simplifying the proof of cut admissibility in the sequent calculi of the existing or emerging non-classical logics.

This work has only a marginal technical dependency on earlier works: it suffices to have the knowledge of LBI [24]; and to understand [24], it suffices to have the basic knowledge of the structural proof theory [16, 25].

1.2 Structure of the remaining sections

In Section 2 we present technical preliminaries of BI proof theory. In Section 3 we introduce our BI calculus LBIZ with no structural rules. In Section 4 we show its main properties including admissibility of structural rules and its equivalence to LBI. We also show Cut admissibility in LBIZ. Section 5 concludes.

2 BI Proof Theory - Preliminaries

We assume availability of the following meta-logical notations. “If and only if” is abbreviated by “iff”.

▶ **Definition 1** (Meta-connectives). We denote logical conjunction (“and”) by $\wedge^\dagger$, logical disjunction (“or”) by $\vee^\dagger$, material implication (“implies”) by $\rightarrow^\dagger$, and equivalence by $\leftrightarrow^\dagger$. These follow the semantics of standard classical logic’s.

We denote the set of propositional variables by $\mathcal{P}$ and refer to an element of $\mathcal{P}$ by p or q with or without a subscript.

A BI formula $F(, G, H)$ with or without a subscript is constructed from the following grammar: $F := p \mid \top \mid \bot \mid {}^*\top \mid F \wedge F \mid F \vee F \mid F \supset F \mid F * F \mid F -\!\!* F$. The set of BI formulas is denoted by $\mathfrak{F}$.

▶ **Definition 2** (BI structures). BI structure $\Gamma(, Re)$ with or without a subscript/superscript, commonly referred to as a bunch [22], is defined by: $\Gamma := F \mid \Gamma ; \Gamma \mid \Gamma , \Gamma$. We denote by $\mathfrak{S}$ the set of BI structures.

We define the binding order to be $[\wedge, \vee, *] \gg [\supset, -\!\!*] \gg [;\ ,] \gg [\wedge^\dagger, \vee^\dagger] \gg [\rightarrow^\dagger, \leftrightarrow^\dagger]$ in a strictly decreasing precedence. Connectives in the same group have the same binding precedence.

Both of the structural connectives “;” and “,” are defined to be associative and commutative. On the other hand, we do not assume distributivity of “;” over ‘,’ or vice versa. A context “ $\Gamma(-)$” (with a hole “ $-$") takes the form of a tree because of the nesting of additive/multiplicative structures.

▶ **Definition 3** (Context). A context $\Gamma(-)$ is finitely constructed from the following grammar: $\Gamma(-) := - \mid \Gamma(-) ; \Gamma \mid \Gamma ; \Gamma(-) \mid \Gamma(-) , \Gamma \mid \Gamma , \Gamma(-)$.

$$\frac{}{F \vdash F}\ \text{id} \qquad \frac{\Gamma_1 \vdash G \qquad \Gamma(G) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ \text{Cut} \qquad \frac{}{\Gamma(\bot) \vdash H}\ \bot L$$

$$\frac{}{\Gamma \vdash \top}\ \top R \qquad \frac{}{{}^{*}\top \vdash {}^{*}\top}\ {}^{*}\top R \qquad \frac{\Gamma(F;G) \vdash H}{\Gamma(F \wedge G) \vdash H}\ \wedge L$$

$$\frac{\Gamma(F) \vdash H \qquad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\ \vee L \qquad \frac{\Gamma_1 \vdash F \qquad \Gamma(\Gamma_1; G) \vdash H}{\Gamma(\Gamma_1; F \supset G) \vdash H}\ \supset L$$

$$\frac{\Gamma(F,G) \vdash H}{\Gamma(F * G) \vdash H}\ *L \qquad \frac{\Gamma_1 \vdash F \qquad \Gamma(G) \vdash H}{\Gamma(\Gamma_1, F \mathrel{-\!\!*} G) \vdash H}\ \mathrel{-\!\!*}L \qquad \frac{\Gamma \vdash F \qquad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\ \wedge R$$

$$\frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ \vee R \qquad \frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G}\ \supset R \qquad \frac{\Gamma_1 \vdash F \qquad \Gamma_2 \vdash G}{\Gamma_1, \Gamma_2 \vdash F * G}\ *R$$

$$\frac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G}\ \mathrel{-\!\!*}R \qquad \frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1; \Gamma_2) \vdash H}\ \text{Wk L} \qquad \frac{\Gamma(\Gamma_1; \Gamma_1) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ \text{Ctr L}$$

$$\frac{\Gamma(\Gamma_1; \top) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ EqAnt_1 \qquad \frac{\Gamma(\Gamma_1, {}^{*}\top) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ EqAnt_2$$

Figure 1 `LBI`: a `BI` sequent calculus. Inference rules with a double-dotted line are bidirectional. $i \in \{1, 2\}$. Structural connectives are fully associative and commutative.

We assume that a `BI` structure Γ_2 replaces $-$ in a context $\Gamma_1(-)$ as $\Gamma_1(\Gamma_2)$ which, we again assume, is a `BI` structure.

▶ **Definition 4** (Sequents)**.** The set of `BI` sequents $\mathfrak{D}$ is defined by:
$\mathfrak{D} := \{\Gamma \vdash F \mid \Gamma \in \mathfrak{S} \wedge^{\dagger} F \in \mathfrak{F}\}$.
We call the left hand side of $\vdash$ antecedent, and the right hand side of $\vdash$ consequent.

A variant of the first `BI` sequent calculus `LBI` [24] is found in Figure 1. Notice that we do not use the nullary structural connectives used in the reference. All the additive inference rules, by which we mean all the inference rules that originate in `IL`, share contexts. Consider $\vee L$ for example. In the inference rule the same context in the conclusion propagates onto both premises. Multiplicative inference rules, by which we mean the inference rules that originate in `MILL`, are context-free [25] or resource sensitive. A good example to illustrate this is $*R$: both Γ_1 and Γ_2 in the conclusion sequent are viewed as resources for the inference rule, and are split into the premises of the rule. Note again our assumption of commutativity of "," here. `Cut` is admissible in [`LBI- Cut`].

▶ **Lemma 5** (Cut admissibility in LBI - Cut)**.** *There is a direct cut elimination procedure which proves admissibility of* `Cut` *in [*`LBI- Cut`*] (sketched in [24]; corrected in [2]).*

The following derivation highlights a simple additive/multiplicative interaction in `BI`.

$$\cfrac{\cfrac{\cfrac{}{F \vdash F}\ \text{id} \qquad \cfrac{}{F \vdash F}\ \text{id}}{F \vdash F \wedge F}\ \wedge R}{\vdash F \mathrel{-\!\!*} F \wedge F}\ \mathrel{-\!\!*}L$$

This shows that $F \mathrel{-\!\!*} F \wedge F$ is provable in `LBI`. Further, given that semantics is given to `LBI` [24], it is a valid `BI` formula. Any others that are provable in `LBI` are valid. We assume that readers are familiar with provability or derivability (found in standard proof theory texts), and with validity or satisfiability (found in Wikipedia).

3 LBIZ: A Structural-Rule-Free BI Sequent Calculus

In this section we present a new BI sequent calculus LBIZ (Figure 2) in which no structural rules appear. We first introduce notations necessary for reading inference rules in the calculus. From this point on, whenever we write $\widetilde{\Gamma}$ for any BI structure, it shall be agreed that it may be empty. The emptiness is in the following sense: $\widetilde{\Gamma_1}; \Gamma_2 = \Gamma_2$ if Γ_1 is empty; and $\widetilde{\Gamma_1}, \Gamma_2 = \Gamma_2$ if Γ_1 is empty. Apart from this, we use two other notations.

3.1 Essence of antecedent structures

Co-existence of IL and MILL in BI calls for new contraction-absorption techniques. We need to consider possible interferences to one structural rule from the others. To illustrate the technical difficulty, $EqAnt_{2\,\mathtt{LBI}}$ for instance interacts directly with $WkL_{\mathtt{LBI}}$. When $WkL_{\mathtt{LBI}}$ is absorbed into the rest, the effect propagates to one direction of $EqAnt_{2\,\mathtt{LBI}}$, resulting in;

$$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1, (^*\!\top; \widetilde{\Gamma_2})) \vdash H}\ EA_2$$

Hence absorption of $WkL_{\mathtt{LBI}}$ must involve analysis of $EqAnt_{2\,\mathtt{LBI}}$ as well. To solve this particular problem we define a new notation: 'essence' of BI structures.

▶ **Definition 6** (Essence of BI structures). Let Γ_1 be a BI structure. Then we have a set of its essences as defined in the following inductive rules.

- Γ_2 is an essence of Γ_1 if $\Gamma_1 = \Gamma_2$.[4]
- $\Gamma(\Gamma', (^*\!\top; \widetilde{\Gamma_2}))$[5] is an essence of Γ_1 if $\Gamma(\Gamma')$ is an essence of Γ_1.

By $\mathbb{E}(\Gamma_1)$ we denote an essence of Γ_1.

The essence takes care of an arbitrary number of EA_2 applications, while nicely retaining a compact representation of a sequent (see the calculus). In each of $\supset L$ and $-\!\!*L$, the essence in the premise(s) and that in the conclusion are the same and identical BI structure. Specifically, the use of $\mathbb{E}(\Gamma)$ in multiple sequents in a derivation tree signifies the same BI structure.

▶ **Example 7.** A LBIZ-derivation:

$$\dfrac{\dfrac{}{F_1; ((^*\!\top; \Gamma_1), F_1 \supset F_2) \vdash F_1}\,id \qquad \dfrac{}{F_2; F_1; ((^*\!\top; \Gamma_1), F_1 \supset F_2) \vdash F_2}\,id}{F_1; ((^*\!\top; \Gamma_1), F_1 \supset F_2) \vdash F_2}\supset L$$

can be alternatively written down as:

$$\dfrac{\dfrac{}{\mathbb{E}(F_1; F_1 \supset F_2) \vdash F_1}\,id \qquad \dfrac{}{F_2; \mathbb{E}(F_1; F_1 \supset F_2) \vdash F_2}\,id}{\mathbb{E}(F_1; F_1 \supset F_2) \vdash F_2}\supset L$$

if $\mathbb{E}(F_1; F_1 \supset F_2) = F_1; ((^*\!\top; \Gamma_1), F_1 \supset F_2)$.

$\mathbb{E}'(\Gamma)$ (or $\mathbb{E}_1(\Gamma)$ or any essence that differs from $\mathbb{E}$ by the presence of a subscript, a superscript or both) in the same derivation tree does not have to be coincident with the BI structure that the $\mathbb{E}(\Gamma)$ denotes. However, we do - for prevention of inundation of many superscripts and subscripts - make an exception. In the cases where no ambiguity is likely to arise such as in the following:

[4] For some Γ_2. The equality is of course up to associativity and commutativity.
[5] For some $\widetilde{\Gamma_2}$; similarly in the rest.

$$\frac{}{\mathbb{E}(\widetilde{\Gamma}; p) \vdash p}\ id \qquad \frac{}{\Gamma(\bot) \vdash F}\ \bot L \qquad \frac{}{\Gamma \vdash \top}\ \top R \qquad \frac{}{\mathbb{E}(\widetilde{\Gamma}; {}^{*}\top) \vdash {}^{*}\top}\ {}^{*}\top R$$

$$\frac{\Gamma(F; G) \vdash H}{\Gamma(F \wedge G) \vdash H}\ \wedge L \qquad \frac{\Gamma \vdash F \qquad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\ \wedge R$$

$$\frac{\Gamma(F) \vdash H \qquad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\ \vee L \qquad \frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ \vee R$$

$$\frac{\mathbb{E}(\widetilde{\Gamma_1}; F \supset G) \vdash F \qquad \Gamma(G; \mathbb{E}(\widetilde{\Gamma_1}; F \supset G)) \vdash H}{\Gamma(\mathbb{E}(\widetilde{\Gamma_1}; F \supset G)) \vdash H}\ \supset L \qquad \frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G}\ \supset R$$

$$\frac{\Gamma(F, G) \vdash H}{\Gamma(F * G) \vdash H}\ *L \qquad \frac{Re_i \vdash F_1 \qquad Re_j \vdash F_2}{\Gamma' \vdash F_1 * F_2}\ *R$$

$$\frac{Re_i \vdash F \qquad \Gamma((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F \mathrel{-\!\!*} G))) \vdash H}{\Gamma(\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F \mathrel{-\!\!*} G)) \vdash H}\ \mathrel{-\!\!*} L \qquad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G}\ \mathrel{-\!\!*} R$$

Figure 2 `LBIZ`: a `BI` sequent calculus with zero occurrence of explicit structural rules. $i, j \in \{1, 2\}$. $i \neq j$. Structural connectives are fully associative and commutative. In $*R$ and $\mathrel{-\!\!*} L$, if Γ' is not empty, $(Re_1, Re_2) \in$ `Candidate`(Γ'); otherwise, $Re_i = {}^{*}\top$ and Re_j is empty. Both $\mathbb{E}$ and `Candidate` are as defined in the main text.

$$\frac{\Gamma(\mathbb{E}(\Gamma_1; F; G)) \vdash H}{\Gamma(\mathbb{E}(\Gamma_1; F \wedge G)) \vdash H}\ \wedge L$$

we assume that the essence in the conclusion is the same antecedent structure as the essence in the premise(s) except what the inference rule modifies.

3.2 Correspondence between Re_i/Re_j and Γ'

▶ **Definition 8** (Relation $\preceq$)**.** We define a binary relation $\preceq$: $\mathfrak{S} \times \mathfrak{S}$ as follows.

- $\Gamma_1 \preceq \Gamma_2$ if $\Gamma_1 = \Gamma_2$.
- $\Gamma(\Gamma_1) \preceq \Gamma(\Gamma_1; \Gamma')$.
- $[\Gamma_1 \preceq \Gamma_2] \wedge^{\dagger} [\Gamma_2 \preceq \Gamma_3] \rightarrow^{\dagger} [\Gamma_1 \preceq \Gamma_3]$.

Intuitively if $\Gamma_1 \preceq \Gamma_2$, then there exists a `LBI`-derivation:

$$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_2) \vdash H}\ WkL$$

for any $\Gamma(\Gamma_1)$ and any H. Here and elsewhere a double line indicates zero or more derivation steps.

▶ **Definition 9** (Candidates)**.** Let Γ be a `BI` structure, then any of the following pairs is a candidate of Γ.

- $(\Gamma_x, {}^{*}\top)$ if $\Gamma_x \preceq \Gamma$.
- (Γ_x, Γ_y) if $\Gamma_x, \Gamma_y \preceq \Gamma$.

We denote the set of candidates of Γ by `Candidate`(Γ).

Now we see the connection between Re_i/Re_j and Γ' in the two rules $*R/\mathrel{-\!\!*} L$.

▶ **Definition 10** (Re_i/Re_j in $*R/\mathrel{-\!\!*} L$)**.** In $*R$ and $\mathrel{-\!\!*} L$, if Γ' is empty (this case applies to $\mathrel{-\!\!*} L$ only), $Re_i = {}^{*}\top$ and Re_j is empty. If it is not empty, then $(Re_1, Re_2) \in$ `Candidate`(Γ').

`Candidate` allows for absorption of an arbitrary number of Wk L applications in the two inference rules. The sequent: $D : p_1;((p_2;p_3),(p_4;p_5)) \vdash p_2 * p_5$, illustrates why it is used. It is clearly `LBI`-derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_2 \vdash p_2}\,\text{id} \qquad \dfrac{}{p_5 \vdash p_5}\,\text{id}}{p_2, p_5 \vdash p_2 * p_5}\,{*R}}{p_2, (p_4;p_5) \vdash p_2 * p_5}\,\text{Wk L}}{(p_2;p_3), (p_4;p_5) \vdash p_2 * p_5}\,\text{Wk L}}{D : p_1;((p_2;p_3),(p_4;p_5)) \vdash p_2 * p_5}\,\text{Wk L}$$

However, $*R$ in `LBI` does not apply immediately to D. Hence $*R$ in `LBIZ` must absorb Wk L.

With the two notations we have introduced, what the inference rules in `LBIZ` are doing should be clear. There are no structural rules. Implicit contraction occurs only in $\supset L$ and $-\!\!*L$.[6] In both of the inference rules, a structure rather than a formula duplicates upwards. This is necessary, for we have the following observation.

▶ **Observation 11** (Non-formula contractions are not admissible)**.** *There exist sequents* $\Gamma \vdash F$ *which are derivable in* `LBI - Cut` *but not derivable in* `LBI - Cut` *without structural contraction.*

Proof. For $-\!\!*L$ use a sequent $\top -\!\!* p_1, \top -\!\!*(p_1 \supset p_2) \vdash p_2$ and assume that every propositional variable is distinct. Then without contraction, there are several derivations. Two sensible ones are shown below (the rest similar). Here and elsewhere we may label a sequent by D with or without a subscript/superscript just so that we may refer to it by the name.

1.

$$\dfrac{\dfrac{}{\top -\!\!*(p_1 \supset p_2) \vdash \top}\,\top R \qquad p_1 \vdash p_2}{D : \top -\!\!* p_1, \top -\!\!*(p_1 \supset p_2) \vdash p_2}\,{-\!\!*L}$$

2.

$$\dfrac{\dfrac{}{\top -\!\!* p_1 \vdash \top}\,\top R \qquad \dfrac{\dfrac{\top \vdash p_1 \qquad \dfrac{}{p_2 \vdash p_2}\,id}{\top; p_1 \supset p_2 \vdash p_2}\,\supset L}{p_1 \supset p_2 \vdash p_2}\,EqAnt_1 L}{D : \top -\!\!* p_1, \top -\!\!*(p_1 \supset p_2) \vdash p_2}\,{-\!\!*L}$$

In both of the derivation trees above, one branch is open. Moreover, such holds true when only formula-level contraction is permitted in `LBI`. The sequent D cannot be derived under the given restriction. If non-formula contractions are available, there is another construction leading to a closed derivation tree:

$$\dfrac{\dfrac{\Pi(D_1) \qquad \Pi(D_2)}{(\top -\!\!* p_1, \top -\!\!*(p_1 \supset p_2)); (\top -\!\!* p_1, \top -\!\!*(p_1 \supset p_2)) \vdash p_2}\,{-\!\!*L}}{D : \top -\!\!* p_1, \top -\!\!*(p_1 \supset p_2) \vdash p_2}\,CtrL$$

where $\Pi(D_1)$ and $\Pi(D_2)$ are:

$\Pi(D_1)$:

$$\dfrac{}{\top -\!\!*(p_1 \supset p_2) \vdash \top}\,\top R$$

$\Pi(D_2)$:

[6] Implicit weakening and others occur also in other inference rules; but they are not very relevant in backward theorem proving.

$$\dfrac{\dfrac{}{\top \mathbin{-\!\!*} p_1 \vdash \top}\,\top R \qquad \dfrac{\dfrac{}{p_1 \vdash p_1}\,id \qquad \dfrac{\dfrac{}{p_2 \vdash p_2}\,id}{p_1; p_2 \vdash p_2}\,WkL}{p_1; p_1 \supset p_2 \vdash p_2}\,\supset L}{p_1; (\top \mathbin{-\!\!*} p_1, \top \mathbin{-\!\!*} (p_1 \supset p_2)) \vdash p_2}\,\mathbin{-\!\!*} L$$

All the derivation tree branches are closed.

For $\supset L$, use $({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2$. Without non-formula contractions we have (only two sensible ones are shown; the rest similar):

1.
$$\dfrac{{}^*\top \vdash p_1 \qquad \dfrac{\dfrac{\dfrac{}{p_2 \vdash p_2}\,id}{{}^*\top; p_2 \vdash p_2}\,WkL}{({}^*\top; p_1), ({}^*\top; p_2) \vdash p_2}\,EA_2}{D : ({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2}\,\supset L$$

2.
$$\dfrac{\dfrac{p_1 \vdash p_2}{{}^*\top; p_1 \vdash p_2}\,WkL}{D : ({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2}\,EA_2$$

In the presence of structural contraction, there is a closed derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_1 \vdash p_1}\,id}{{}^*\top; p_1; {}^*\top \vdash p_1}\,WkL \qquad \dfrac{\dfrac{}{p_2 \vdash p_2}\,id}{{}^*\top; p_1; {}^*\top; p_2 \vdash p_2}\,WkL}{{}^*\top; p_1; {}^*\top; p_1 \supset p_2 \vdash p_2}\,\supset L}{(({}^*\top; p_1), ({}^*\top; p_1 \supset p_2)); (({}^*\top; p_1), ({}^*\top; p_1 \supset p_2)) \vdash p_2}\,EA_2}{D : ({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2}\,CtrL$$

◀

We list LBIZ derivations of the two examples in the observation for easy comparisons. We assume that $\Gamma = (\top \mathbin{-\!\!*} p_1, \top \mathbin{-\!\!*} (p_1 \supset p_2))$. Also, by $\Pi(D)$ we denote a derivation tree of a sequent D. We assume that $\Pi(D)$ is always closed: every derivation branch of the tree has an empty sequent as the leaf node (the premise of an axiom).

$$\dfrac{\dfrac{}{\top \mathbin{-\!\!*} p_1 \vdash \top}\,\top R \qquad \dfrac{\dfrac{}{\top \mathbin{-\!\!*} (p_1 \supset p_2) \vdash \top}\,\top R \qquad \Pi(({}^*\top, p_1); ({}^*\top, p_1 \supset p_2); \Gamma \vdash p_2)}{({}^*\top, p_1); \Gamma \vdash p_2}\,\mathbin{-\!\!*} L}{\Gamma \vdash p_2}\,\mathbin{-\!\!*} L$$

$\Pi(({}^*\top, p_1); ({}^*\top, p_1 \supset p_2); \Gamma \vdash p_2)$ is as follows.

$$\dfrac{\dfrac{}{({}^*\top, p_1); ({}^*\top, p_1 \supset p_2); \Gamma \vdash p_1}\,\text{id} \qquad \dfrac{}{p_2; ({}^*\top, p_1); ({}^*\top, p_1 \supset p_2); \Gamma \vdash p_2}\,\text{id}}{({}^*\top, p_1); ({}^*\top, p_1 \supset p_2); \Gamma \vdash p_2}\,\supset L$$

For the other sequent, we have:

$$\dfrac{\dfrac{}{({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_1}\,\text{id} \qquad \dfrac{}{p_2; ({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2}\,\text{id}}{({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2}\,\supset L$$

4 Main Properties of LBIZ

In this section we show the main properties of LBIZ such as admissibility of weakening, that of EA_2, that of both $EqAnt_{1\,\texttt{LBI}}$ and $EqAnt_{2\,\texttt{LBI}}$, that of contraction, and its equivalence to LBI. Cut is also admissible. We will refer to the notion of derivation depth very often.

▶ **Definition 12** (Derivation depth). Let $\Pi(D)$ be a derivation tree. Then the derivation depth of D', a node in $\Pi(D)$, is:

- 1 if D' is the conclusion node of an axiom inference rule.
- 1 + (derivation depth of D_1) if $\Pi(D')$ looks like:

$$\frac{\Pi(D_1)}{D'}$$

- 1 + (the larger of the derivation depths of D_1 and D_2) if $\Pi(D')$ looks like:

$$\frac{\Pi(D_1) \qquad \Pi(D_2)}{D'}$$

4.1 Admissibility of weakening and EA_2

Admissibilities of both weakening and EA_2 are proved depth-preserving. This means in case of weakening that if a sequent $\Gamma(\Gamma_1) \vdash H$ is derivable with derivation depth of k, then $\Gamma(\Gamma_1; \Gamma_2) \vdash H$ is derivable with derivation depth of l such that $l \leq k$.

▶ **Proposition 13** (LBIZ weakening admissibility). *If a sequent $D : \Gamma(\Gamma_1) \vdash F$ is* LBIZ-*derivable, then so is $D' : \Gamma(\Gamma_1; \Gamma_2) \vdash F$ depth-preserving.*

Proof. By induction on derivation depth of D. ◀

▶ **Proposition 14** (Admissibility of EA_2). *If a sequent $D : \Gamma(\Gamma_1) \vdash F$ is* LBIZ-*derivable, then so is $D' : \Gamma(\mathbb{E}(\Gamma_1)) \vdash F$ depth-preserving.*

Proof. By induction on derivation depth of D. ◀

4.2 Inversion lemma

The inversion lemma below is important in simplification of the subsequent discussion.

▶ **Lemma 15** (Inversion lemma for LBIZ). *For the following sequent pairs, if the sequent on the left is* LBIZ-*derivable at most with the derivation depth of k, then so is (are) the sequent(s) on the right.*

$$\begin{array}{rl}
\Gamma(F \wedge G) \vdash H, & \Gamma(F; G) \vdash H \\
\Gamma(F_1 \vee F_2) \vdash H, & \textit{both } \Gamma(F_1) \vdash H \textit{ and } \Gamma(F_2) \vdash H \\
\Gamma(F * G) \vdash H, & \Gamma(F, G) \vdash H \\
\Gamma(\Gamma_1; \top) \vdash H, & \Gamma(\Gamma_1) \vdash H \\
\Gamma(\Gamma_1, {}^{*}\!\top) \vdash H, & \Gamma(\Gamma_1) \vdash H \\
\Gamma \vdash F \wedge G, & \textit{both } \Gamma \vdash F \textit{ and } \Gamma \vdash G \\
\Gamma \vdash F \supset G, & \Gamma; F \vdash G \\
\Gamma \vdash F -\!\!* G, & \Gamma, F \vdash G
\end{array}$$

Proof. By induction on derivation depth. ◀

4.3 Admissibility of $EqAnt_{1,2}$

▶ **Proposition 16** (Admissibility of $EqAnt_{1,2}$). *$EqAnt_{1\,\texttt{LBI}}$ and $EqAnt_{2\,\texttt{LBI}}$ are depth-preserving admissible in* LBIZ.

Proof. Follows from inversion lemma,[7] Proposition 13 and Proposition 14. ◀

[7] Inversion lemma proves one direction.

4.4 Preparation for contraction admissibility in $*R/\mathrel{-\!\!*}L$ cases

We dedicate one subsection here to prepare for the main proof of contraction admissibility. Based on Proposition 13, we make an observation about the set of candidates. The discovery, which is to be stated in Proposition 18, led to the solution to the problem of the elimination of `LBI` structural contraction.

▶ **Definition 17** (Representing candidates). Let $\hat{\precsim} : \mathfrak{S} \times \mathfrak{S}$ be a binary relation satisfying:

- $\Gamma_1 \hat{\precsim} \Gamma_2$ if $\Gamma_1 = \Gamma_2$.
- $\Gamma_1 \hat{\precsim} \Gamma_1; \Gamma_3$.
- $[\Gamma_1 \hat{\precsim} \Gamma_2] \wedge^{\dagger} [\Gamma_2 \hat{\precsim} \Gamma_3] \to^{\dagger} [\Gamma_1 \hat{\precsim} \Gamma_3]$.
- $\Gamma_1, \Gamma_2 \hat{\precsim} \Gamma_1, (\Gamma_2; \Gamma_3)$.

Now let Γ be a `BI` structure. Then any of the following pairs is a representing candidate of Γ.

- $(\Gamma_x, {}^{*}\top)$ if $\Gamma_x \hat{\precsim} \Gamma$.
- (Γ_x, Γ_y) if $\Gamma_x, \Gamma_y \hat{\precsim} \Gamma$.

We denote the set of representing candidates of Γ by $\texttt{RepCandidate}(\Gamma)$.

We trivially have that $\texttt{RepCandidate}(\Gamma) \subseteq \texttt{Candidate}(\Gamma)$ for any Γ. More can be said.

▶ **Proposition 18** (Sufficiency of `RepCandidate`). `LBIZ` *with* `RepCandidate` *instead of* `Candidate` *for* (Re_1, Re_2) *is as expressive as* `LBIZ` *(with* `Candidate`*).*

Proof. The only inference rules in `LBIZ` that use `Candidate` are $*R$ and $\mathrel{-\!\!*}L$. So it suffices to consider only those.

For $*R$, suppose by way of showing contradiction that `LBIZ` with `RepCandidate` is not as expressive as `LBIZ`, then there exists some `LBIZ` derivation tree $\Pi(D)$:

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ D_1 : Re_i \vdash F_1 & & D_2 : Re_j \vdash F_2\end{matrix}}{D : \Gamma' \vdash F_1 * F_2}\ *R$$

such that (Re_1, Re_2) must be in $\texttt{Candidate}(\Gamma') \backslash \texttt{RepCandidate}(\Gamma')$. Now, without loss of generality assume $(i, j) = (1, 2)$. Then $D_1' : Re_i' \vdash F_1$ and $D_2' : Re_j' \vdash F_2$ for $(Re_i', Re_j') \in \texttt{RepCandidate}(\Gamma')$ are also `LBIZ` derivable (by Proposition 13). But this means that we can choose the (Re_i', Re_j') for (Re_1, Re_2), a direct contradiction to the supposition. Similarly for $\mathrel{-\!\!*}L$. ◀

▶ **Theorem 19** (Contraction admissibility in `LBIZ`). *If* $D : \Gamma(\Gamma_a; \Gamma_a) \vdash F$ *is* `LBIZ`*-derivable, then so is* $D' : \Gamma(\Gamma_a) \vdash F$*. The derivation depth is preserved.*

Proof. By induction on derivation depth. The base cases are when it is 1, *i.e.* when D is the conclusion sequent of an axiom. Consider which axiom has applied. If it is $\top R$, then it is trivial to show that if $\Gamma(\Gamma_a; \Gamma_a) \vdash \top$, then so is $\Gamma(\Gamma_a) \vdash \top$. Also for $\bot L$, a single occurrence of $\bot$ on the antecedent part of D suffices for the $\bot L$ application, and the current theorem is trivially provable in this case, too. For both id and ${}^{*}\top R$, $\Pi(D)$ looks like:

$$\dfrac{}{\mathbb{E}(\widetilde{\Gamma_1}; \alpha) \vdash \alpha}$$

where α is $p \in \mathcal{P}$ for id, ${}^{*}\top$ for ${}^{*}\top R$ and $\Gamma(\Gamma_a; \Gamma_a) = \mathbb{E}(\widetilde{\Gamma_1}; \alpha)$. If α is not a sub-structure of either of the occurrences of Γ_a, then D' is trivially derivable. Otherwise, assume that the focused α in $\mathbb{E}(\widetilde{\Gamma_1}; \alpha)$ is a sub-structure of one of the occurrences of Γ_a in $\Gamma(\Gamma_a; \Gamma_a)$.

Then there exists some Γ_2 and $\widetilde{\Gamma_3}$ such that $\mathbb{E}(\widetilde{\Gamma_1};\alpha) = \mathbb{E}(\Gamma_2;\widetilde{\Gamma_3};\alpha) = \mathbb{E}_1(\Gamma_2);\mathbb{E}_2(\widetilde{\Gamma_3};\alpha)$ and that Γ_a is an essence of $\widetilde{\Gamma_3};\alpha$. But then $D' : \Gamma(\Gamma_a)$ is still an axiom.

For inductive cases, suppose that the current theorem holds true for any derivation depth of up to k. We must demonstrate that it still holds for the derivation depth of $k+1$. Consider what the LBIZ inference rule applied last is, and, in case of a left inference rule, consider where the active structure Γ_b of the inference rule is in $\Gamma(\Gamma_a;\Gamma_a)$.

1. $\wedge L$, and Γ_b is $F_1 \wedge F_2$: if Γ_b does not appear in Γ_a, induction hypothesis on the premise sequent concludes. Otherwise, $\Pi(D)$ looks like:

$$\dfrac{\vdots \atop D_1 : \Gamma(\Gamma'_a(F_1;F_2);\Gamma'_a(F_1 \wedge F_2)) \vdash H}{D : \Gamma(\Gamma'_a(F_1 \wedge F_2);\Gamma'_a(F_1 \wedge F_2)) \vdash H} \wedge L$$

$D'_1 : \Gamma(\Gamma'_a(F_1;F_2);\Gamma'_a(F_1;F_2)) \vdash H$ is LBIZ-derivable (inversion lemma);
$D''_1 : \Gamma(\Gamma'_a(F_1;F_2)) \vdash H$ is also LBIZ-derivable (induction hypothesis); then $\wedge L$ on D''_1 concludes.

2. $\supset L$, and Γ_b is $\mathbb{E}(\widetilde{\Gamma'};F \supset G)$: if Γ_b does not appear in Γ_a, then the induction hypothesis on both of the premises concludes. If it is entirely in Γ_a, then $\Pi(D)$ looks either like:

$$\dfrac{\vdots \atop D_1 : \mathbb{E}(\widetilde{\Gamma'};F \supset G) \vdash F \qquad \vdots \atop D_2}{D : \Gamma(\Gamma'_a(\mathbb{E}(\widetilde{\Gamma'};F \supset G));\widetilde{\Gamma'_a}(\mathbb{E}(\widetilde{\Gamma'};F \supset G))) \vdash H} \supset L$$

where $D_2 : \Gamma(\Gamma'_a(G;\mathbb{E}(\widetilde{\Gamma'};F{\supset}G));\Gamma'_a(\mathbb{E}(\widetilde{\Gamma'};F{\supset}G))) \vdash H$, or, in case Γ_a is $\Gamma'_a;F{\supset}G$, like:

$$\dfrac{\vdots \atop D_1 : \Gamma'_a;F{\supset}G;\Gamma'_a;F{\supset}G \vdash F \qquad \vdots \atop D_2}{D : \Gamma(\Gamma'_a;F{\supset}G;\Gamma'_a;F{\supset}G) \vdash H} \supset L$$

where $D_2 : \Gamma(G;\Gamma'_a;F{\supset}G;\Gamma'_a;F{\supset}G) \vdash H$.
In the former case,
$D'_2 : \Gamma(\Gamma'_a(G;\mathbb{E}(\widetilde{\Gamma'};F{\supset}G));\Gamma'_a(G;\mathbb{E}(\widetilde{\Gamma'};F{\supset}G))) \vdash H$ (weakening admissibility);
$D''_2 : \Gamma(\Gamma'_a(G;\mathbb{E}(\widetilde{\Gamma'};F{\supset}G))) \vdash H$ (induction hypothesis);
then $\supset L$ on D_1 and D''_2 concludes. In the latter, induction hypothesis on D_1 and on D_2; then via $\supset L$ for a conclusion. Finally, if only a substructure of Γ_b is in Γ_a with the rest spilling out of Γ_a, then if the principal formula $F \supset G$ does not occur in Γ_a, then straightforward; otherwise similar to the latter case.

3. $*R$: $\Pi(D)$ looks like:

$$\dfrac{\vdots \atop D_1 : Re_i \vdash F_1 \qquad \vdots \atop D_2 : Re_j \vdash F_2}{D : \Gamma(\Gamma_a;\Gamma_a) \vdash F_1 * F_2} *R$$

By Proposition 18, assume that $(Re_1, Re_2) \in$ `RepCandidate`$(\Gamma(\Gamma_a;\Gamma_a))$ without loss of generality. Then by the definition of $\hat{\preceq}$ it must be that either (1) $\Gamma_a;\Gamma_a$ preserves completely in Re_1 or Re_2, or (2) it remains neither in Re_1 nor in Re_2. If $\Gamma_a;\Gamma_a$ is preserved in Re_1 (or Re_2), then induction hypothesis on the premise that has Re_1 (or Re_2) and then $*R$ conclude; otherwise, it is trivial to see that only a single Γ_a needs to be present in D.

4. $-\!\!*L$, and Γ_b is $\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1};F{-\!\!*}G)$: if Γ_b is not in Γ_a, then induction hypothesis on the right premise sequent concludes. If it is in Γ_a, $\Pi(D)$ looks like:

$$\dfrac{\vdots \atop D_1 : Re_i \vdash F \qquad \vdots \atop D_2}{D : \Gamma(\Gamma'_a(\widetilde{\Gamma'},\mathbb{E}(\widetilde{\Gamma_1};F{-\!\!*}G));\Gamma'_a(\widetilde{\Gamma'},\mathbb{E}(\widetilde{\Gamma_1};F{-\!\!*}G))) \vdash H} -\!\!*L_1$$

where D_2 is:

$$\Gamma(\Gamma'_a((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G))); \Gamma'_a(\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G))) \vdash H$$

$D'_2 : \Gamma(\Gamma'_a((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G))); \Gamma'_a((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G)))) \vdash H$ via Proposition 13 is also LBIZ-derivable. $D''_2 : \Gamma(\Gamma'_a((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G)))) \vdash H$ via induction hypothesis. Then ${-\!\!*}L$ on D_1 and D''_2 concludes. If, on the other hand, Γ_a is in Γ_b, then it is either in Γ_1 or in Γ'. But if it is in Γ_1, then it must be weakened away, and if it is in Γ', similar to the $*R$ case.

5. Other cases are similar to one of the cases already examined. ◀

4.5 Equivalence of LBIZ to LBI

▶ **Theorem 20** (Equivalence between LBIZ and LBI). *$D : \Gamma \vdash F$ is LBIZ-derivable if and only if it is LBI-derivable.*

Proof. Into the *only if* direction, assume that D is LBIZ-derivable, and then show that there is a LBI-derivation for each LBIZ derivation. But this is obvious because each LBIZ inference rule is derivable in LBI.[8]

Into the *if* direction, assume that D is LBI-derivable, and then show that there is a corresponding LBIZ-derivation to each LBI derivation by induction on the derivation depth of D.

If it is 1, *i.e.* if D is the conclusion sequent of an axiom, we note that $\bot L_{\texttt{LBI}}$ is identical to $\bot L_{\texttt{LBIZ}}$; $id_{\texttt{LBI}}$ and $^*\!\top R_{\texttt{LBI}}$ via $id_{\texttt{LBIZ}}$ and resp. $^*\!\top R_{\texttt{LBIZ}}$ with Proposition 13 and Proposition 14; and $\top R_{\texttt{LBI}}$ is identical to $\top R_{\texttt{LBIZ}}$. For inductive cases, assume that the *if* direction holds true up to the LBI-derivation depth of k, then it must be demonstrated that it still holds true for the LBI-derivation depth of $k+1$. Consider what the LBI rule applied last is:

1. $\supset L_{\texttt{LBI}}$: $\Pi_{\texttt{LBI}}(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : \Gamma_1 \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2 : \Gamma(\Gamma_1; G) \vdash H\end{matrix}}{D : \Gamma(\Gamma_1; F{\supset}G) \vdash H} \supset L_{\texttt{LBI}}$$

By induction hypothesis, both D_1 and D_2 are also LBIZ-derivable. Proposition 13 on D_1 in LBIZ-space results in $D'_1 : \Gamma_1; F{\supset}G \vdash F$, and on D_2 results in $D'_2 : \Gamma(\Gamma_1; G; F{\supset}G) \vdash H$. Then an application of $\supset L_{\texttt{LBIZ}}$ on D'_1 and D_2 concludes in LBIZ-space.

2. ${-\!\!*}L_{\texttt{LBI}}$: $\Pi_{\texttt{LBI}}(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : \Gamma_1 \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2 : \Gamma(G) \vdash H\end{matrix}}{D : \Gamma(\Gamma_1, F{-\!\!*}G) \vdash H} {-\!\!*}L_{\texttt{LBI}}$$

By induction hypothesis, D_1 and D_2 are also LBIZ-derivable.

   a. If $\Gamma(G)$ is G, *i.e.* if the antecedent part of D_2 is a formula (G), then Proposition 13 on D_2 results in $D'_2 : G; (\Gamma_1, F{-\!\!*}G) \vdash H$ in LBIZ-space. Then ${-\!\!*}L_{\texttt{LBIZ}}$ on D_1 and D'_2 leads to $D' : \Gamma_1, F{-\!\!*}G \vdash H$ as required.

   b. If $\Gamma(G)$ is $\Gamma'(\Gamma'', G)$, then Proposition 13 on D_2 leads to $D'_2 : \Gamma'((\Gamma'', G); (\Gamma'', \Gamma_1, F{-\!\!*}G)) \vdash H$. Then ${-\!\!*}L_{\texttt{LBIZ}}$ on D_1 and D'_2 leads to $D' : \Gamma'(\Gamma'', \Gamma_1, F{-\!\!*}G) \vdash H$ as required.

[8] Note that EA_2 is LBI-derivable with $WkL_{\texttt{LBI}}$ and $EqAnt_{2\,\texttt{LBI}}$.

c. Finally, if $\Gamma(G)$ is $\Gamma'(\Gamma''; G) \vdash H$, then Proposition 13 on D_2 gives $D_2' : \Gamma'(\Gamma''; G; (\Gamma_1, F \text{-}\!\!* G)) \vdash H$. Then $\text{-}\!\!* L_{\texttt{LBIZ}}$ on D_1 and D_2' leads to $D' : \Gamma'(\Gamma''; (\Gamma_1, F \text{-}\!\!* G)) \vdash H$ as required.

3. $WkL_{\texttt{LBI}}$: Proposition 13.
4. $CtrL_{\texttt{LBI}}$: Theorem 19.
5. $EqAnt_{1\,\texttt{LBI}}$: Proposition 16.
6. $EqAnt_{2\,\texttt{LBI}}$: Proposition 16.
7. The rest: straightforward. ◀

4.6 `LBIZ` Cut Elimination

Cut is admissible in `LBIZ`. As a reminder (although already stated under Figure 1) `Cut` is the following rule:

$$\frac{\Gamma_1 \vdash F \qquad \Gamma_2(F) \vdash G}{\Gamma_2(\Gamma_1) \vdash G}\,\texttt{Cut}$$

Just as in the case of intuitionistic logic, cut admissibility proof for a contraction-free `BI` sequent calculus is simpler than that for `LBI` [2]. Since we have already proved depth-preserving weakening admissibility, the following context sharing cut, $\texttt{Cut}_{CS}$, is easily verified derivable in `LBIZ` + `Cut`:

$$\frac{\widetilde{\Gamma_3}; \Gamma_1 \vdash F \qquad \Gamma_2(F; \Gamma_1) \vdash H}{\Gamma_2(\widetilde{\Gamma_3}; \Gamma_1) \vdash H}\,\texttt{Cut}_{CS}$$

where Γ_1 appears on both of the premises. F in the above cut rule appearing on both premises is called the cut formula. The use of $\texttt{Cut}_{CS}$ simplifies the cut elimination proof a little.

We recall the standard notations of the cut rank and the cut level.

▶ **Definition 21** (Cut level/rank)**.** Given a cut instance in a closed derivation:

$$\frac{D_1 : \Gamma_1 \vdash F \qquad D_2 : \Gamma_2(F) \vdash H}{D_3 : \Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}$$

The level of the cut instance is: $\texttt{der\_depth}(D_1) + \texttt{der\_depth}(D_2)$, where $\texttt{der\_depth}(D)$ denotes derivation depth of D. The rank of the cut instance is the size of the cut formula F, $\texttt{f\_size}(F)$, which is defined as follows:

- it is 1 if F is a nullary logical connective or a propositional variable.
- it is $\texttt{f\_size}(F_1) + \texttt{f\_size}(F_2) + 1$ if F is in the form: $F_1 \bullet F_2$ for $\bullet \in \{\wedge, \vee, \supset, *, \text{-}\!\!*\}$.

▶ **Theorem 22** (Cut admissibility in `LBIZ`)**.** `Cut` *is admissible in* `LBIZ`.

Proof. By induction on the cut rank and a sub-induction on the cut level. We make use of $\texttt{Cut}_{CS}$. In this proof (X, Y) for some `LBIZ` inference rules X and Y means that one of the premises has been just derived with X and the other with Y. $\Gamma(\Gamma_1)(\Gamma_2)$ abbreviates $(\Gamma(\Gamma_1))(\Gamma_2)$. In pairs of derivations below, the first is the derivation tree to be permuted and the second is the permuted derivation tree.

$(\boldsymbol{id}, \boldsymbol{id})$:

1.

$$\frac{\overline{\mathbb{E}(\widetilde{\Gamma_1}; p) \vdash p}\,id \qquad \overline{\mathbb{E}'(\widetilde{\Gamma_2}; p) \vdash p}\,id}{\mathbb{E}'(\widetilde{\Gamma_2}; \mathbb{E}(\widetilde{\Gamma_1}; p)) \vdash p}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\mathbb{E}'(\widetilde{\Gamma_2};\mathbb{E}(\widetilde{\Gamma_1};p)) \vdash p}\; id$$

Of course, for the above permutation to be correct, we must be able to demonstrate the fact that the antecedent structure $\mathbb{E}''(\widetilde{\Gamma_2};\widetilde{\Gamma_1};p)$ is such that $[\mathbb{E}''(\widetilde{\Gamma_2};\widetilde{\Gamma_1};p)] = [\mathbb{E}'(\widetilde{\Gamma_2};\mathbb{E}(\widetilde{\Gamma_1};p))]$. But note that it only takes a finite number of (backward) EA_2 applications (*Cf.* Proposition 14) on $\widetilde{\Gamma_2};\mathbb{E}(\widetilde{\Gamma_1};p) \vdash p$ to upward derive $\widetilde{\Gamma_2};\widetilde{\Gamma_1};p \vdash p$. The implication is that, since $\widetilde{\Gamma_2};\mathbb{E}(\widetilde{\Gamma_1};p) \vdash p$ results upward from $\mathbb{E}'(\widetilde{\Gamma_2};\mathbb{E}(\widetilde{\Gamma_1};p)) \vdash p$ also in a finite number of backward EA_2 applications, the antecedent structure must be in the form: $\mathbb{E}''(\widetilde{\Gamma_2};\widetilde{\Gamma_1};p)$.

2.

$$\dfrac{\dfrac{}{\mathbb{E}(\widetilde{\Gamma_1};p) \vdash p}\; id \qquad \dfrac{}{\mathbb{E}'(\Gamma_2(p);q) \vdash q}\; id}{\mathbb{E}'(\Gamma_2(\mathbb{E}(\widetilde{\Gamma_1};p));q) \vdash q}\; \texttt{Cut}$$

This and the other patterns for which one of the premises is an axiom sequent are straightforward.

For the remaining cases, if the cut formula is principal only for one of the premise sequents, then we follow the routine [25] to permute up the other premise sequent for which it is the principal. For example, in case we have the derivation pattern below:

$$\dfrac{\dfrac{D_1 \qquad D_2}{D_5 : \Gamma_1(H_1 \vee H_2) \vdash F_1 \supset F_2}\; \vee L \qquad \dfrac{D_3 : \mathbb{E}(\widetilde{\Gamma_3};F_1 \supset F_2) \vdash F_1 \qquad D_4 : \Gamma_2(F_2;\mathbb{E}(\widetilde{\Gamma_3};F_1 \supset F_2)) \vdash H}{D_6 : \Gamma_2(\mathbb{E}(\widetilde{\Gamma_3};F_1 \supset F_2)) \vdash H}\; \supset L}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_3};\Gamma_1(H_1 \vee H_2))) \vdash H}\; \texttt{Cut}$$

for $D_1 : \Gamma_1(H_1) \vdash F_1 \supset F_2$ and $D_2 : \Gamma_1(H_2) \vdash F_1 \supset F_2$, the cut formula $F_1 \supset F_2$ is not the principal on the left premise. In this case, we simply apply `Cut` on the pairs: (D_1, D_6) and (D_2, D_6), to conclude:

$$\dfrac{\dfrac{D_1 \qquad D_6}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_3};\Gamma_1(H_1))) \vdash H}\; \texttt{Cut} \qquad \dfrac{D_2 \qquad D_6}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_3};\Gamma_1(H_2))) \vdash H}\; \texttt{Cut}}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_3};\Gamma_1(H_1 \vee H_2))) \vdash H}\; \vee L$$

Of course, for this particular permutation to be correct, we must be able to demonstrate, in the permuted derivation tree, that $\mathbb{E}(\widetilde{\Gamma_3};\Gamma_1(H_1 \vee H_2)) = \mathbb{E}'(\widetilde{\Gamma_3}) \star \Gamma_1(H_1 \vee H_2)$ with $\star$ either a semi-colon or a comma, that $\mathbb{E}(\widetilde{\Gamma_3};\Gamma_1(H_1)) = \mathbb{E}'(\widetilde{\Gamma_3}) \star \Gamma_1(H_1)$, and that $\mathbb{E}(\widetilde{\Gamma_3};\Gamma_1(H_2)) = \mathbb{E}'(\widetilde{\Gamma_3}) \star \Gamma_1(H_2)$. But this is vacuous since the cut formula which is replaced by the structure $\Gamma_1(H_1)$ or $\Gamma_1(H_2)$ is a formula.

The cases that remain are those for which both premises of the cut instance have the cut formula as the principal. We go through each of them to conclude the proof.

($\wedge L, \wedge R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1 \vdash F_1 \qquad D_2 : \Gamma_1 \vdash F_2}{\Gamma_1 \vdash F_1 \wedge F_2}\; \wedge R \qquad \dfrac{D_3 : \Gamma_2(F_1;F_2) \vdash H}{\Gamma_2(F_1 \wedge F_2) \vdash H}\; \wedge L}{\Gamma_2(\Gamma_1) \vdash H}\; \texttt{Cut}$$

$\Rightarrow$

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\Gamma_2(\Gamma_1; F_2) \vdash H}\,\texttt{Cut}}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}_{CS}$$

($\vee L, \vee R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1 \vdash F_i \quad (i \in \{1,2\})}{\Gamma_1 \vdash F_1 \vee F_2}\,\vee R \qquad \dfrac{D_2 : \Gamma_2(F_1) \vdash H \qquad D_3 : \Gamma_2(F_2) \vdash H}{\Gamma_2(F_1 \vee F_2) \vdash H}\,\vee L}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{D_1 \qquad D_{(2\ or\ 3)}}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}$$

The value of i decides which of D_2 or D_3 is the right premise sequent.

($\supset L, \supset R$):

$$\dfrac{\dfrac{D_1 : \Gamma_3; F_1 \vdash F_2}{D_4 : \Gamma_3 \vdash F_1 \supset F_2}\,\supset R \qquad \dfrac{D_2 : \mathbb{E}(\widetilde{\Gamma_1}; F_1 \supset F_2) \vdash F_1 \qquad D_3 : \Gamma_2(F_2; \mathbb{E}(\widetilde{\Gamma_1}; F_1 \supset F_2)) \vdash H}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_1}; F_1 \supset F_2)) \vdash H}\,\supset L}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{\dfrac{D_4 \qquad D_2}{\mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3) \vdash F_1}\,\texttt{Cut} \qquad D_1}{\Gamma_3; \mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3) \vdash F_2}\,\texttt{Cut} \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2(F_2; \mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H}\,\texttt{Cut}}{\Gamma_2(\Gamma_3; \mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H}\,\texttt{Cut}_{CS}$$

$$\cdots\cdots\cdots \text{ Proposition 13}$$

$$\Gamma_2(\widetilde{\Gamma_1}; \Gamma_3; \mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H$$

$$\cdots\cdots\cdots \text{ Proposition 14}$$

$$\Gamma_2(\mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3); \mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H$$

$$\cdots\cdots\cdots \text{ Theorem 19}$$

$$\Gamma_2(\mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H$$

The derivation steps with a dotted line are depth-preserving.

($*L, *R$):

$$\dfrac{\dfrac{D_1 : Re_i \vdash F_1 \qquad D_2 : Re_j \vdash F_2}{\Gamma_1 \vdash F_1 * F_2}\,*R \qquad \dfrac{D_3 : \Gamma_2(F_1, F_2) \vdash H}{\Gamma_2(F_1 * F_2) \vdash H}\,*L}{\Gamma_2(\Gamma_1) \vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\Gamma_2(Re_i, F_2) \vdash H}\,\texttt{Cut}}{\Gamma_2(Re_i, Re_j) \vdash H}\,\texttt{Cut}$$

$$\cdots\cdots\cdots \text{ Proposition 13}$$

$$\Gamma_2(\Gamma_1) \vdash H$$

($-\!\!* L, -\!\!* R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1, F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 -\!\!* F_2}\,-\!\!* R \qquad \dfrac{D_2 : Re_i \vdash F_1 \qquad D_3 : \Gamma_2((\widetilde{Re_j}, F_2); (\widetilde{\Gamma}', \mathbb{E}(\widetilde{\Gamma_3}; F_1 -\!\!* F_2))) \vdash H}{\Gamma_2(\widetilde{\Gamma}', \mathbb{E}(\widetilde{\Gamma_3}; F_1 -\!\!* F_2)) \vdash H}\,-\!\!* L_1}{\Gamma_2(\widetilde{\Gamma}', \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1)) \vdash H}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2((\widetilde{Re_j}, F_2); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H}\,\texttt{Cut}}{\Gamma_2((\widetilde{Re_j}, \Gamma_1, F_1); (\Gamma', \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H}\,\texttt{Cut}}{\Gamma_2((\widetilde{Re_j}, \Gamma_1, Re_i); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H}\,\texttt{Cut}$$

$$\begin{array}{cl} \Gamma_2((\widetilde{Re_j}, \Gamma_1, Re_i); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Proposition 13} \\ \Gamma_2((\widetilde{\Gamma'}, (\widetilde{\Gamma_3}; \Gamma_1)); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Proposition 14} \\ \Gamma_2((\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1)); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \text{Theorem 19} \\ \Gamma_2(\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1)) \vdash H & \end{array}$$

◀

5 Conclusion

We addressed the problem of structural rule absorption in BI sequent calculus. This problem was around for a while. As far back as we can see, the first proximate attempt was made in [21]. References to the problem were subsequently made [9, 21, 4] in a discussion. The work that came closest to ours is one by Donnelly *et al.* [7]. They consider weakening absorption in the context of forward theorem proving (where weakening rather than contraction is a source of non-termination). One inconvenience in their approach, however, is that the effect of weakening is not totally isolated from that of contraction: it is absorbed into contraction as well as into logical rules. But then structural weakening is still possible through the new structural contraction. Also, the coupling of the two structural rules amplifies the difficulty of analysis on the behaviour of contraction. Further, their work is on a subset of BI without units. In comparison, our solution covers the whole BI. Techniques we used in this work should be useful in the derivation of contraction-free sequent calculi of other non-classical logics that come with a non-formula structural contraction rule. For instance, nested sequent calculi [15, 12, 17] of some constructive modal logics (those only with k1 and k2 axioms) [23], when they are extended with additional modal axioms including 5 axiom, are known to truly require non-formula contractions, in the presence of which cut-elimination proof becomes demanding. As is always the case, there are fewer cases to cover in cut-elimination proof when there are no structural contraction. There are also more recent BI extensions in sequent calculus such as [14], to which this work has relevance. Seeing the complexity of LBIZ, one may also consider development of another formalism that may represent BI and other similar non-classical logics more informatively.

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