A Lower Bound on Opaque Sets*

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Abstract

It is proved that the total length of any set of countably many rectifiable curves, whose union meets all straight lines that intersect the unit square U, is at least 2.00002. This is the first improvement on the lower bound of 2 by Jones in 1964. A similar bound is proved for all convex sets U other than a triangle.

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1 Introduction

A *barrier* or an *opaque set* for $U \subseteq \mathbb{R}^2$ is a set $B \subseteq \mathbb{R}^2$ that intersects every line that intersects U. For example, when U is a square, any of the four sets depicted in thick lines in Figure 1 is a barrier. The question of finding small barriers for polygons was first considered by Mazurkiewicz a century ago [12]. Note that some part of the barrier may lie outside U in our setting (Figure 2), and the barrier need not be connected.

We are interested in "short" barriers B for a given object U, and hence we restrict attention to those barriers that can be written as a union $B = \bigcup_{b \in \mathcal{B}} b$ of some countable set $\mathcal{B}$ of curves[1] b that each have finite length $|b|$ and the sum of these lengths $|\mathcal{B}| = \sum_{b \in \mathcal{B}} |b|$ is finite. We call such a set $\mathcal{B}$ (and not the union B, strictly speaking) a *rectifiable barrier*, and $|\mathcal{B}|$ its *length*.

Finding the shortest barrier is difficult, even for simple shapes U, such as the square, the equilateral triangle, and the disk [6, 10]. The shortest known barrier for the unit square is the rightmost one in Figure 1, with length $2.638\ldots$. This problem and its relatives have been considered by many authors. See [6, 11] and the introduction of [5] for more history, background, and related problems.

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[1] In this paper, a *curve* or a *line segment* can by definition include both, one or none of the endpoints.

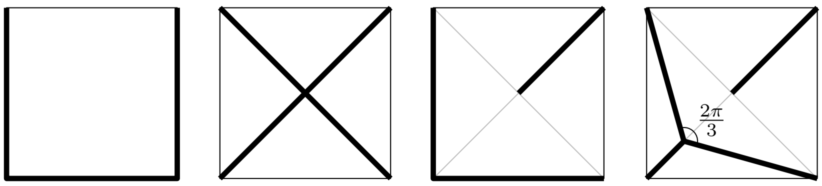

Figure 1 Barriers (in thick lines) for the unit square. The first one (three sides) and the second one (diagonals) have lengths 3 and $2\sqrt{2} = 2.828\ldots$, respectively. The third barrier consists of two sides and half of a diagonal, and has length $2 + 1/\sqrt{2} = 2.707\ldots$. The last one is the shortest known barrier for the unit square, with length $\sqrt{2} + \sqrt{6}/2 = 2.638\ldots$, consisting of half a diagonal and the Steiner tree of the lower left triangle.

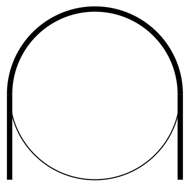

Figure 2 A barrier (in thick lines) for a disk that is shorter than the perimeter. This is not the shortest one; see [6].

The best known lower bound for the unit square has been 2, established by Jones in 1964 [9]. In general, for convex U, a barrier needs to have length at least half the perimeter of U (we review a proof in Section 2):

▶ **Lemma 1.** *$|\mathcal{B}| \geq p$ for any rectifiable barrier $\mathcal{B}$ of a convex set $U \subseteq \mathbb{R}^2$ with perimeter $2p$.*

Thus, from the point of view of finding short barriers, the trivial strategy of enclosing the entire perimeter (or the perimeter of the convex hull if U is a non-convex connected set) gives a 2-approximation. See [4] and references therein for algorithms that find shorter barriers. The current best approximation ratio is $1.58\ldots$ [5].

Proving a better lower bound has been elusive (again, even for specific shapes U). There has been some partial progress under additional assumptions about the shape (single arc, connected, etc.) and location (inside U, near U, etc.) of the barrier [1, 3, 7, 11, 14], but establishing an unconditional lower bound strictly greater than 2 for the unit square has been open (see [4, Open Problem 5] or [3, Footnote 1]). We prove such a lower bound in Section 4:

▶ **Theorem 2.** *$|\mathcal{B}| \geq 2.00002$ for any rectifiable barrier $\mathcal{B}$ of the unit square $\square$.*

Dumitrescu and Jiang [3] recently obtained a lower bound of $2 + 10^{-12}$ under the assumption that the barrier lies in the square obtained by magnifying $\square$ by 2 about its centre. Their proof, conceived independently of ours and at about the same time, is based on quite different ideas, most notably the line-sweeping technique. It will be worth exploring whether their techniques can be combined with ours.

Our proof can be generalized (Section 5):

▶ **Theorem 3.** *For any closed convex set U with perimeter $2p$ that is not a triangle, there is $\varepsilon > 0$ such that every rectifiable barrier of U has length at least $p + \varepsilon$.*

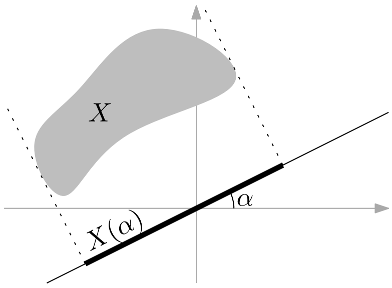

Figure 3 $X(\alpha)$ is the projection of X onto the angle-α (directed) line identified with $\mathbb{R}$.

Thus, the only convex objects for which we fail to establish a lower bound better than Lemma 1 are triangles[2].

The rest of this paper is structured as follows. In Section 2, we present a (known) proof for Lemma 1. We also prove that instead of rectifiable barriers, it is sufficient to restrict our attention to barriers comprised of line segments. In Section 3, we present three preliminary lemmas, analyzing some important special cases in which we can expect to improve on the bound from Lemma 1. The proof of one of these lemmas is postponed to Section 6. These lemmas are combined in Section 4 to obtain our lower bound for the length of a barrier for the square (Theorem 2). In Section 5, we show how to generalize these arguments to other convex sets (Theorem 3). In the last section, we discuss a closely related question.

2 Preliminaries: A general lower bound

For a set $X \subseteq \mathbb{R}^2$ and an angle $\alpha \in [0, 2\pi)$ (all angle calculation in this paper will be performed modulo 2π), we write

$$X(\alpha) = \left\{ x \cos \alpha + y \sin \alpha : (x, y) \in X \right\} \tag{1}$$

for the projection of X at angle α (Figure 3). To say that a set $B \subseteq \mathbb{R}^2$ is a barrier of $U \subseteq \mathbb{R}^2$ means that $B(\alpha) \supseteq U(\alpha)$ for all α. We are interested in lower bounds on the length $|\mathcal{B}|$ of a rectifiable barrier $\mathcal{B}$ such that the union $B = \bigcup_{b \in \mathcal{B}} b$ satisfies this. For this purpose, it is no loss of generality to assume that $\mathcal{B}$ consists of line segments, as the following lemma shows. We call such $\mathcal{B}$ a *straight barrier*.

▶ **Lemma 4** ([5, Lemma 1]). *Let $\mathcal{B}$ be a rectifiable barrier for $U \subseteq \mathbb{R}^2$. Then, for any $\varepsilon > 0$, there exists a straight barrier $\mathcal{B}_\varepsilon$ for U such that $|B_\varepsilon| \le (1 + \varepsilon)|B|$.*

Proof. Since the proof in [5] has a gap, we provide another proof. We will show that for any $\varepsilon > 0$ and any rectifiable curve b, there is a straight barrier $\mathcal{B}^b_\varepsilon$ of b of length $\le (1 + \varepsilon)|b|$. This then gives a straight barrier $\mathcal{B}_\varepsilon = \bigcup_{b \in \mathcal{B}} \mathcal{B}^b_\varepsilon$ of U.

If b is already a line segment, we are done by setting $\mathcal{B}^b_\varepsilon = \{b\}$. Otherwise, the convex hull H of b has an interior point. Let b' be the curve obtained by magnifying b by $1 + \varepsilon$ about this point. Since the convex hull of b' contains the compact set H in its interior, so does the convex hull of some finitely many points on b'. The set $\mathcal{B}^b_\varepsilon$ of line segments connecting these points along b' is a barrier of b of length at most $|b'| = (1 + \varepsilon)|b|$. ◀

[2] During the preparation of this manuscript, Izumi [8] announced such a nontrivial lower bound for the equilateral triangle.

By Lemma 4, we may focus attention on straight barriers: U has a rectifiable barrier of length $< l$ if and only if it has a straight barrier of length $< l$.

As mentioned in the introduction (Lemma 1), it has been known that any barrier of a convex set must be at least half the perimeter. We include a short proof of this bound here, for completeness and further reference. See [2] for another elegant proof.

Proof of Lemma 1. By Lemma 4, we may assume that $\mathcal{B}$ is straight. We have

$$|U(\alpha)| \leq \Big|\bigcup_{b\in\mathcal{B}} b(\alpha)\Big| \leq \sum_{b\in\mathcal{B}}|b(\alpha)| = \sum_{b\in\mathcal{B}}|b|\cdot|\cos(\alpha-\theta_b)| \tag{2}$$

for each $\alpha \in [0, 2\pi)$, where θ_b is the angle of a line segment b. Integrating over $[0, 2\pi)$, we obtain

$$\int_{\alpha=0}^{2\pi}|U(\alpha)|\,\mathrm{d}\alpha \leq \sum_{b\in\mathcal{B}}\Big(|b|\cdot\int_{\alpha=0}^{2\pi}|\cos(\alpha-\theta_b)|\,\mathrm{d}\alpha\Big) = 4\sum_{b\in\mathcal{B}}|b| = 4|\mathcal{B}|. \tag{3}$$

When U is a convex set, the left-hand side equals twice the perimeter (cf. the Cauchy–Crofton formula [13, Theorem 16.15]). ◀

3 Preliminary lemmas

Note that Theorems 2 and 3 do not merely state the non-existence of a straight barrier $\mathcal{B}$ of length exactly half the perimeter of U. Such a claim can be proved easily as follows: If $\mathcal{B}$ is such a barrier, the inequality (3) must hold with equality, and so must (2) for almost every α. Thus, the second inequality in (2) must hold with equality, which means that segments in $\mathcal{B}$ never overlaps one another when projected onto the line with angle α. Since this must be the case for almost every α, the entire $\mathcal{B}$ must lie on a line, which is clearly impossible.

The theorems claim more strongly that a barrier must be longer by an absolute constant. The following lemma says that in order to obtain such a bound, it suffices to find a part $\mathcal{B}' \subseteq \mathcal{B}$ of the barrier whose contribution to covering U is less than the optimal by at least a fixed positive constant (the proof is not hard and will be presented in the journal version).

▶ **Lemma 5.** *Let $\mathcal{B}$ be a rectifiable barrier of a closed convex set U of perimeter $2p$. Then $|\mathcal{B}| \geq p + \delta$ if there is a subset $\mathcal{B}' \subseteq \mathcal{B}$ with*

$$\int_{\alpha=0}^{2\pi}\Big|\Big(\bigcup_{b\in\mathcal{B}'} b\Big)(\alpha)\cap U(\alpha)\Big|\,\mathrm{d}\alpha \leq 4|\mathcal{B}'| - 4\delta. \tag{4}$$

There are several ways in which such a “waste” can occur, and we make use of two of them (Figure 4). The first one is when there is a significant part of the barrier that lies far outside U, as described in the following lemma (the proof is again not hard and will be included in the journal version):

▶ **Lemma 6.** *Let b be a line segment that lies outside a convex region U. Suppose that the set $A := \{\,\alpha \in [0, 2\pi) : U(\alpha) \cap b(\alpha) \neq \emptyset\,\}$ (of angles of all lines through U and b) has measure $\leq 2\pi - 4\varepsilon$. Then*

$$\int_{\alpha=0}^{2\pi}|b(\alpha)\cap U(\alpha)|\,\mathrm{d}\alpha \leq 4|b|\cos\varepsilon. \tag{5}$$

The second situation where we have a significant waste required in Lemma 5 is when there are two sets of barrier segments that roughly face each other:

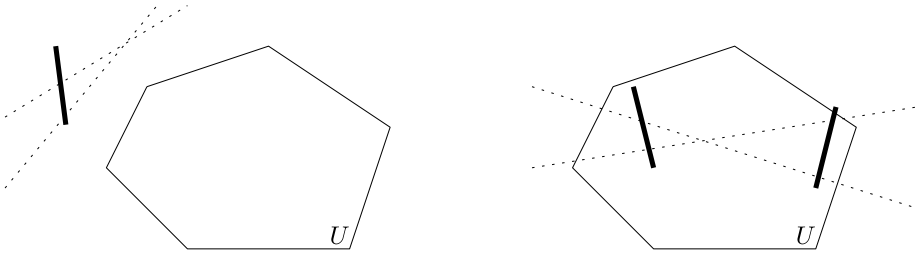

Figure 4 Two wasteful situations. In the left figure, a barrier segment (thick) lies far outside the object U, which leads to significant waste because this segment covers in vain some lines (dotted) that do not pass through U; this is discussed in Lemma 6. In the right figure, there are two parts of the barrier (thick) that face each other, which also results in significant waste because they cover some lines (dotted) doubly; this is roughly the situation discussed in Lemma 7.

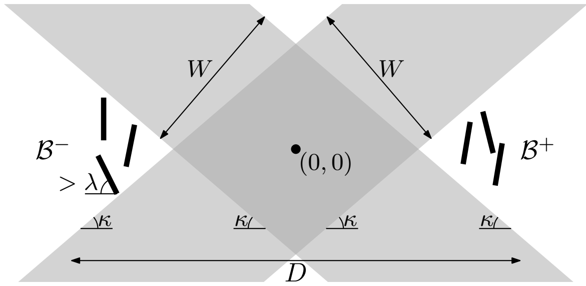

Figure 5 Sets $\mathcal{B}^-$ and $\mathcal{B}^+$ (Lemma 7).

▶ **Lemma 7.** *Let $\lambda \in (0, \frac{\pi}{2})$, $\kappa \in (0, \lambda)$ and l, $D > 0$. Let $\mathcal{B}^-$ and $\mathcal{B}^+$ be sets of n line segments of length l (Figure 5) such that*

1. *every segment of $\mathcal{B}^- \cup \mathcal{B}^+$ makes angle $> \lambda$ with the horizontal axis, and lies entirely in the disk of diameter D centred at the origin;*
2. *the segments in $\mathcal{B}^-$ and the segments in $\mathcal{B}^+$ are separated by bands of angle κ and width $W := nl \sin(\lambda - \kappa)$ centred at the origin, as depicted in Figure 5—that is, each point (x, y) on each segment in $\mathcal{B}^\pm$ satisfies $\pm(x \sin \kappa + y \cos \kappa) \geq W/2$ and $\pm(x \sin \kappa - y \cos \kappa) \geq W/2$ (where $\pm$ should be read consistently as $+$ and $-$).*

Then

$$\int_{\alpha=0}^{2\pi} \Big| \bigcup_{b \in \mathcal{B}^- \cup \mathcal{B}^+} b(\alpha) \Big| \, \mathrm{d}\alpha \leq 8nl - \frac{2W^2}{D}. \tag{6}$$

Note that $8nl = 4|\mathcal{B}^- \cup \mathcal{B}^+|$, so (6) is of the form (4) in Lemma 5.

The proof of Lemma 7 requires a more involved argument than Lemma 6, and will be given in Section 6. Before that, we prove Theorems 2 and 3 using Lemmas 6 and 7.

4 Proof of Theorem 2

We prove Theorem 2 using Lemmas 5, 6 and 7. The proof roughly goes as follows. Consider a barrier whose length is very close to 2.

1. There cannot be too much of the barrier far outside □, because that would be too wasteful by Lemma 6.

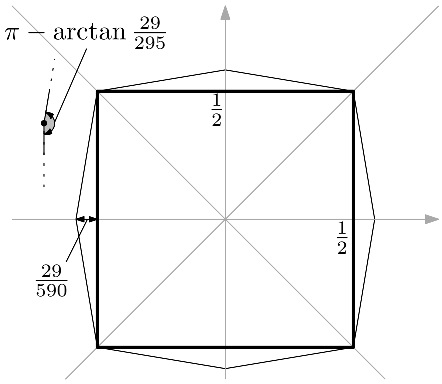

Figure 6 Viewed from any point outside the octagon ⬡, the square □ lies inside an angle that is smaller than π by the constant $\arctan \frac{29}{295}$.

2. This implies that there must be a significant part of the barrier near each corner of □, because this is the only place to put barrier segments that block those lines that clip this corner closely.
3. Among the parts of the barrier that lie near the four corners, there are parts that face each other and thus lead to waste by Lemma 7.

Proof of Theorem 2. Let □ be the unit square (including the boundary), which we assume to be axis-aligned and centred at the origin. Let $\mathcal{B}$ be a rectifiable barrier of □. By Lemma 4, we may assume that $\mathcal{B}$ is a straight barrier. Let ⬡ be the octagon (Figure 6) obtained by attaching to each edge of □ an isosceles triangle of height $\frac{29}{590}$ (and thus whose identical angles are $\arctan \frac{29}{295}$). By splitting some of the segments in $\mathcal{B}$ into several pieces, we may assume that $\mathcal{B} = \mathcal{B}_{\text{out}} \cup \mathcal{B}_{\text{in}}$, where each segment in $\mathcal{B}_{\text{in}}$ lies entirely in ⬡, and each segment in $\mathcal{B}_{\text{out}}$ lies entirely outside ⬡ and inside one of the eight regions delimited by the two axes and the two bisectors of the axes.

Suppose that $|\mathcal{B}_{\text{out}}| > \frac{1}{60}$. For each $b \in \mathcal{B}_{\text{out}}$, observe that, viewed from each point on b, the square □ lies entirely in an angle of size $\pi - \arctan \frac{29}{295}$ (Figure 6). This allows us to apply Lemma 6 and obtain

$$\int_{\alpha=0}^{2\pi} |b(\alpha) \cap \square(\alpha)| \, \mathrm{d}\alpha \le 4|b| \cos\left(\frac{1}{2} \arctan \frac{29}{295}\right) < (4 - 0.0048)|b|. \tag{7}$$

Summing up for all $b \in \mathcal{B}_{\text{out}}$ (and using the triangle inequality), we have

$$\int_{\alpha=0}^{2\pi} \Big| \bigcup_{b \in \mathcal{B}_{\text{out}}} b(\alpha) \cap \square(\alpha) \Big| \, \mathrm{d}\alpha < (4 - 0.0048)|\mathcal{B}_{\text{out}}| \le 4|\mathcal{B}_{\text{out}}| - 0.0048 \cdot \frac{1}{60}, \tag{8}$$

which yields $|\mathcal{B}| \ge 2.00002$ by Lemma 5. From now on, we can and will assume that $|\mathcal{B}_{\text{out}}| \le \frac{1}{60}$.

Let $I_0 := \{ (x, y) \in \mathbb{R}^2 : \frac{7}{8} \le x + y \le 1 \}$ and $R_0 := I_0 \cap$ ⬡ (Figure 7). Again, by splitting some of the segments in $\mathcal{B}$ into several pieces (which may or may not include endpoints), we may assume that each segment in $\mathcal{B}$ lies entirely in R_0 or entirely outside R_0. Let $\mathcal{B}_0 \subseteq \mathcal{B}$ consist of those that lie in R_0. Since $\bigcup_{b \in \mathcal{B}} b(\frac{\pi}{4}) \supseteq \square(\frac{\pi}{4}) = [-\sqrt{2}/2, \sqrt{2}/2] \supseteq [\frac{7}{8}\sqrt{2}/2, \sqrt{2}/2]$, and since the only segments $b \in \mathcal{B}$ for which $b(\frac{\pi}{4})$ can intersect this interval $[\frac{7}{8}\sqrt{2}/2, \sqrt{2}/2]$ of length $\sqrt{2}/16$ are those in $\mathcal{B}_0 \cup \mathcal{B}_{\text{out}}$, we have $|\mathcal{B}_0 \cup \mathcal{B}_{\text{out}}| \ge \sqrt{2}/16$, and hence

$$|\mathcal{B}_0| \ge \frac{\sqrt{2}}{16} - \frac{1}{60} > 0.07172 =: 2\eta. \tag{9}$$

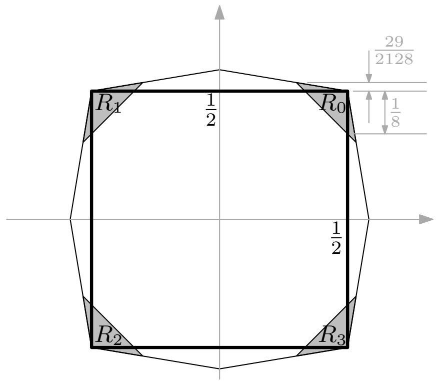

Figure 7 The regions R_0, R_1, R_2, R_3.

Likewise, let R_1, R_2, R_3 be the upper left, lower left, and lower right corners of ⬡, respectively, and define $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ analogously to $\mathcal{B}_0$, so that $|\mathcal{B}_1|$, $|\mathcal{B}_2|$, $|\mathcal{B}_3| > 2\eta$. Observe that the interval $R_0(\frac{\pi}{2} - 0.1813)$ lies above $R_1(\frac{\pi}{2} - 0.1813)$, with a gap of size

$$\frac{7}{8}\sin 0.1813 - \left(\frac{1}{8} + 2\cdot\frac{29}{2128}\right)\cos 0.1813 > 0.008; \tag{10}$$

and $R_0(\frac{3\pi}{4} - 0.1813)$ lies above $R_2(\frac{3\pi}{4} - 0.1813)$, with an even bigger gap.

For each i, we partition $\mathcal{B}_i$ into three parts $\mathcal{B}_{i,i+1}$, $\mathcal{B}_{i,i+2}$, $\mathcal{B}_{i,i+3}$ (the subscripts are modulo 4), consisting respectively of segments whose angles are in $[\frac{\pi}{2}i - \frac{\pi}{4}, \frac{\pi}{2}i + \frac{\pi}{8})$, $[\frac{\pi}{2}i + \frac{\pi}{8}, \frac{\pi}{2}i + \frac{3\pi}{8})$ and $[\frac{\pi}{2}i + \frac{3\pi}{8}, \frac{\pi}{2}i + \frac{3\pi}{4})$. Thus, $\mathcal{B}_{i,j}$ consists of segments in $\mathcal{B}_i$ that "roughly point towards R_j." Since $|\mathcal{B}_i| > 2\eta$, we have $|\mathcal{B}_i \setminus \mathcal{B}_{i,j}| > \eta$ for at least two of the three j for each i, and thus, for at least eight of the twelve pairs (i, j). Hence, there is (i, j) such that $|\mathcal{B}_i \setminus \mathcal{B}_{i,j}| > \eta$ and $|\mathcal{B}_j \setminus \mathcal{B}_{j,i}| > \eta$.

Let $\mathcal{B}^-$ and $\mathcal{B}^+$ be finite sets of line segments of the same length such that $|\mathcal{B}^-| = |\mathcal{B}^+| = \eta$ and $\bigcup_{b\in\mathcal{B}^-} b \subseteq \bigcup_{b\in\mathcal{B}_i\setminus\mathcal{B}_{i,j}} b$, $\bigcup_{b\in\mathcal{B}^+} b \subseteq \bigcup_{b\in\mathcal{B}_j\setminus\mathcal{B}_{j,i}} b$. Apply Lemma 7 to these $\mathcal{B}^-$ and $\mathcal{B}^+$, rotated and translated appropriately, and the constants $\kappa = 0.1813$, $\lambda = \frac{\pi}{8}$, $D = \sqrt{2}$. Note that the last assumption of Lemma 7 is satisfied because $W := \eta\sin(\lambda - \kappa) = 0.03586\sin(\frac{\pi}{8} - 0.1813) = 0.007524\ldots < 0.008$. This gives

$$\int_{\alpha=0}^{2\pi} \Big|\bigcup_{b\in\mathcal{B}^-\cup\mathcal{B}^+} b(\alpha)\Big|\,\mathrm{d}\alpha \le 8\eta - \frac{2W^2}{D} < 8\eta - 0.00008, \tag{11}$$

whence $|\mathcal{B}| \ge 2.00002$ by Lemma 5. ◀

We did not attempt to seriously optimize the specific numbers in the above proof (such as $\frac{29}{590}$, $\frac{1}{60}$, $\frac{1}{8}$, ...), but they are somewhat carefully chosen, for the following intuition. What we needed to do is to cut out the four small regions R_i from the corners, and argue that some significant part of the barrier segments in these regions are in the relative position described in the assumption of Lemma 7 (Figure 5). If these regions are too small, we would obtain only a small amount of such segments (i.e., η would be smaller). If they are too big, we would know less about the relative position of the segments (and thus have to use worse values of λ and κ). The number $\frac{1}{8}$ for the size of the corners R_i was (roughly) chosen for the right balance between these factors. The number $\frac{29}{590}$ was then chosen big enough to imply (via Lemma 6) that $|\mathcal{B}_{\text{out}}|$ is quite small (specifically $\frac{1}{60}$, which is so small that η defined in (9) is still a significant positive value), but not so big that the regions R_i stretch too much and deteriorate κ.

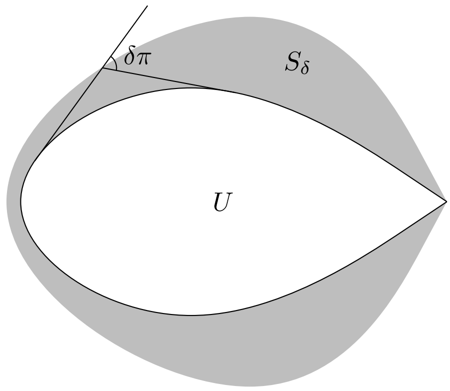

Figure 8 S_δ is the set of points from which U looks big. Putting too much of the barrier outside S_δ is wasteful.

5 Proof of Theorem 3

Theorem 3 is proved by modifying the proof of Theorem 2 (Section 4) as follows. Let x_i be distinct points ($i = 1, 2, 3, 4$) on the boundary of U at which U is strictly convex, i.e., there is a line that intersects U only at x_i; let α_i be the angle of this line. Note that such four points exist unless U is a triangle. Let R_i be a sufficiently small closed neighbourhood of x_i, so that no three of R_1, R_2, R_3, R_4 are stabbed by a line.

Instead of the octagon ⬡, we consider the set $S_\delta \supseteq U$ of points such that a random line through this point avoids U with probability less than a positive constant δ (Figure 8). By applying Lemma 6 in the same way (with some routine compactness argument), we know that $\mathcal{B}_{\text{out}}$ (the segments in the assumed straight barrier $\mathcal{B}$ that lie outside S_δ) must be small (under the assumption of $|\mathcal{B}| \le p + \varepsilon$, for an appropriately small ε). By taking δ sufficiently small, S_δ comes so close to U that the following happens for each $i = 1, 2, 3, 4$: there is a neighbourhood $N \subseteq U$ of x_i in U such that every angle-α_i line that intersects N intersects S_δ only in R_i. This guarantees that the part $\mathcal{B}_i$ of $\mathcal{B}$ that lies in R_i must have length at least some positive constant (just to block those angle-α_i lines that hit N). This allows us to define $\mathcal{B}_{i,j}$ in the way similar to Theorem 2 and apply Lemma 7 with appropriate κ, λ, D.

6 Proof of Lemma 7

It remains to prove Lemma 7. Let us first interpret what it roughly claims. By symmetry, we can halve the interval $[0, 2\pi]$ and replace (6) by

$$4nl - \int_{\alpha=0}^{\pi} \Big| \bigcup_{b \in \mathcal{B}^- \cup \mathcal{B}^+} b(\alpha) \Big| \, \mathrm{d}\alpha \ge \frac{W^2}{D}. \tag{12}$$

For each $b \in \mathcal{B}^- \cup \mathcal{B}^+$, consider the region

$$R_b := \{ (\alpha, v) \in [0, \pi] \times \mathbb{R} : v \in b(\alpha) \}, \tag{13}$$

whose area is $2l$. Note that the first term $4nl$ of (12) is the sum of this area for all $b \in \mathcal{B}^- \cup \mathcal{B}^+$, whereas the second term is the area of the union. Thus, (12) says that the area of the overlap (considering multiplicity) is at least W^2/D. To prove such a bound, we start with the following lemma, which provides a similar estimate on the size of potentially complicated overlaps, but of simpler objects, namely bands with fixed width.

▶ **Lemma 8.** *Let $I \subseteq \mathbb{R}$ be an interval and let W, $D \geq 0$. Let $\mathcal{U}$ be the set of functions f which take each $\alpha \in I$ to an interval $f(\alpha) = [\underline{f}(\alpha), \overline{f}(\alpha)]$ of length W/n and are $\frac{1}{2}D$-*Lipschitz*, that is, $|\underline{f}(\alpha_0) - \underline{f}(\alpha_1)| \leq \frac{1}{2}D \cdot |\alpha_0 - \alpha_1|$ for each α_0, $\alpha_1 \in I$. Suppose that $2n$ functions f_1, $\dots$, f_n, g_1, $\dots$, $g_n \in \mathcal{U}$ satisfy*

$$\underline{g_j}(\min I) - \overline{f_i}(\min I) \geq W, \qquad \underline{f_i}(\max I) - \overline{g_j}(\max I) \geq W \tag{14}$$

for each i, j (i.e., the functions f_i start far below g_j and end up far above). Then

$$\left|R_{f_1} \cup \dots \cup R_{f_n} \cup R_{g_1} \cup \dots \cup R_{g_n}\right| \leq 2W|I| - \frac{W^2}{D}, \tag{15}$$

where $R_f := \{\, (\alpha, v) \in I \times \mathbb{R} : v \in f(\alpha) \,\}$ denotes the graph of $f \in \mathcal{U}$.

The proof will be given in the full version.

Proof of Lemma 7. Let R_b be as in (13). As explained there, our goal is to prove (12), which says that the area of the overlap among R_b for $b \in \mathcal{B}^- \cup \mathcal{B}^+$ is at least W^2/D. We claim that this is true *even if we replace each R_b by its subset $\tilde{R}_b \subseteq R_b$ defined below.*

Let $I := [\frac{\pi}{2} - \kappa, \frac{\pi}{2} + \kappa]$. Note that, because of the configuration of segments (Figure 5), we have $|b(\alpha)| \geq l \sin(\lambda - \kappa) = W/n$ for each $\alpha \in I$ and $b \in \mathcal{B}^- \cup \mathcal{B}^+$. We define $\tilde{R}_b$ from R_b by restricting α to I and replacing the interval $b(\alpha)$ by its subinterval $\tilde{b}(\alpha) := [\min(b(\alpha)), \min(b(\alpha)) + W/n]$. Thus,

$$\tilde{R}_b := \{\, (\alpha, v) \in I \times \mathbb{R} : v \in \tilde{b}(\alpha) \,\}. \tag{16}$$

Note that these functions $\tilde{b}$ are $\frac{1}{2}D$-Lipschitz (see Lemma 8), because the segments $b \in \mathcal{B}^- \cup \mathcal{B}^+$ lie within distance $\frac{1}{2}D$ of the origin. We can thus apply Lemma 8 to $\{f_1, \dots, f_n\} := \{\, \tilde{b} : b \in \mathcal{B}^- \,\}$ and $\{g_1, \dots, g_n\} := \{\, \tilde{b} : b \in \mathcal{B}^+ \,\}$, since (14) is satisfied because of the width-W separation (Figure 5) assumed in Lemma 7. We can thus apply Lemma 8 and obtain $\left|\bigcup_{b \in \mathcal{B}^- \cup \mathcal{B}^+} \tilde{R}_b\right| \leq 2W|I| - W^2/D$, as was desired. ◀

7 Half-line barriers

We propose an analogous question, obtained by replacing lines by half-lines in the definition of barriers: a set $B \subseteq \mathbb{R}^2$ is a *half-line barrier* of $U \subseteq \mathbb{R}^2$ if all half-lines intersecting U intersect B. This intuitively means "hiding the object U from outside," which we find perhaps as natural, if not more, than the notion of opaque sets. Similarly to Lemma 1, we have

▶ **Lemma 9.** *$|B| \geq p$ for any rectifiable half-line barrier B of a convex set $U \subseteq \mathbb{R}^2$ that is not a line segment and has perimeter p.*

Thus, unlike for line barriers, the question is completely answered when U is connected: the shortest half-line barrier is the boundary of the convex hull.

If U is disconnected, there can be shorter half-line barriers. For example, if U consists of two connected components that are enough far apart from each other, it is more efficient to cover them separately than together. One might hope that an optimal half-line barrier is always obtained by grouping the connected components of U in some way and taking convex hulls of each. This is not true, as the example in Figure 9 shows. We have not been able to find an algorithm that achieves a nontrivial approximation ratio for this problem.

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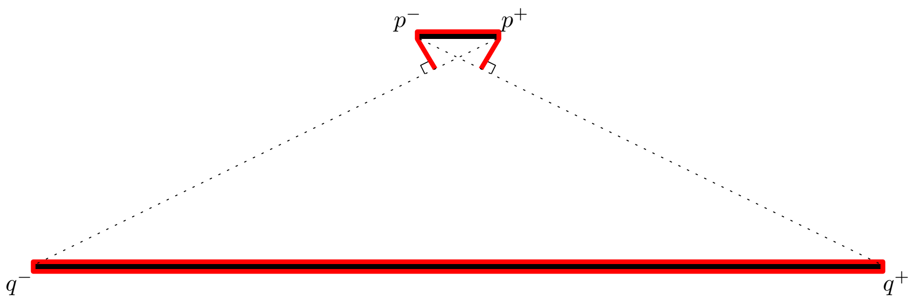

Figure 9 Consider the line segments p^-p^+ and q^-q^+, where $p^\pm = (\pm 1, 8)$ and $q^\pm = (\pm 15, 0)$, and let U be the union of these segments with small "thickness": U consists of a rectangle with vertices $(\pm 1, 8 \pm \varepsilon)$ and another with vertices $(\pm 15, \pm\varepsilon)$, for a small $\varepsilon > 0$. The boundaries of these thick line segments have total length 64 (plus a small amount due to the thickness). The boundary of the convex hull of all of U has length $2 + 30 + 2\sqrt{260} > 64.24$ (plus thickness). But we have another half-line barrier depicted above, whose total length is $2 + 60 + 2/\sqrt{5} + 2/\sqrt{5} < 63.79$ (plus thickness, which can be made arbitrarily small).

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