Degree Four Plane Spanners: Simpler and Better

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Abstract

Let $\mathcal{P}$ be a set of n points embedded in the plane, and let $\mathcal{C}$ be the complete Euclidean graph whose point-set is $\mathcal{P}$. Each edge in $\mathcal{C}$ between two points p, q is realized as the line segment $[pq]$, and is assigned a weight equal to the Euclidean distance $|pq|$. In this paper, we show how to construct in $\mathcal{O}(n \lg n)$ time a plane spanner of $\mathcal{C}$ of maximum degree at most 4 and of stretch factor at most 20. This improves a long sequence of results on the construction of bounded degree plane spanners of $\mathcal{C}$. Our result matches the smallest known upper bound of 4 by Bonichon et al. on the maximum degree while significantly improving their stretch factor upper bound from 156.82 to 20. The construction of our spanner is based on Delaunay triangulations defined with respect to the equilateral-triangle distance, and uses a different approach than that used by Bonichon et al. Our approach leads to a simple and intuitive construction of a well-structured spanner, and reveals useful structural properties of the Delaunay triangulations defined with respect to the equilateral-triangle distance.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems, I.3.5 Computational Geometry and Object Modeling

Keywords and phrases geometric spanners; plane spanners; bounded degree spanners; Delaunay triangulations

Digital Object Identifier 10.4230/LIPIcs.SoCG.2016.45

1 Introduction

Let $\mathcal{P}$ be a set of n points embedded in the plane, and let $\mathcal{C}$ be the complete Euclidean graph whose point-set is $\mathcal{P}$. Each edge in $\mathcal{C}$ between two points p, q is realized as the line segment $[pq]$, and is assigned a weight equal to the Euclidean distance $|pq|$. In this paper, we consider the problem of constructing a plane spanner of $\mathcal{C}$ of small degree and small stretch factor. This problem has received considerable attention, and there is a long list of results on the construction of plane spanners of $\mathcal{C}$ that achieve various trade-offs between the degree and the stretch factor of the spanner.

The problem of constructing a plane spanner of $\mathcal{C}$ was considered as early as the 1980's, if not earlier. Chew [10] proved that the L_1-Delaunay triangulation of $\mathcal{P}$, which is the Delaunay triangulation of $\mathcal{P}$ defined with respect to the L_1-distance, is a spanner of $\mathcal{C}$ of stretch factor at most $\sqrt{10}$. Chew's result was followed by a series of papers showing that other Delaunay triangulations are plane spanners (of $\mathcal{C}$) as well. In 1987, Dobkin *et al.* [13] showed that the classical L_2-Delaunay triangulation of $\mathcal{P}$ is a plane spanner of stretch factor at most $\frac{\pi(1+\sqrt{5})}{2}$. This bound was subsequently improved by Keil and Gutwin [16] to $\frac{4\pi}{3\sqrt{3}}$. In the meantime, Chew [11] showed that the TD-Delaunay triangulation defined using a distance

32nd International Symposium on Computational Geometry (SoCG 2016).
Editors: Sándor Fekete and Anna Lubiw; Article No. 45; pp. 45:1–45:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

function based on an equilateral triangle – rather than a square (L_1-distance) or a circle (L_2-distance) – is a spanner of stretch factor 2. This result was generalized by Bose et al. [6], who showed that the Delaunay triangulation defined with respect to any convex distance function (*i.e.*, based on a convex shape) is a plane spanner. The bound on the stretch factor of the L_2-Delaunay triangulation by Keil and Gutwin stood unchallenged for many years until Xia recently improved the bound to below 2 [20]. Recently, Bonichon *et al.* [3] improved Chew's original bound on the stretch factor of the L_1-Delaunay triangulation to $\sqrt{4+2\sqrt{2}}$, and showed this bound to be tight.

All the Delaunay triangulations mentioned above can have unbounded degree. In recent years, bounded degree plane spanners have been used as the building block of wireless network topologies. Wireless distributed system technologies, such as wireless ad-hoc and sensor networks, are often modeled as proximity graphs in the Euclidean plane. Spanners of proximity graphs represent topologies that can be used for efficient communication. For these applications, in addition to having low stretch factor, spanners are typically required to be plane and have bounded degree, where both requirements are useful for efficient routing [8, 19].

The wireless network applications motivated researchers to turn their attention to minimizing the maximum degree of the plane spanner as well as its stretch factor. It can be readily seen that 3 is a lower bound on the maximum degree of a spanner of $\mathcal{C}$, because a Hamiltonian path/cycle through a set of points arranged in a grid has unbounded stretch factor. Work on bounded degree but not necessarily plane spanners of $\mathcal{C}$ closely followed the above-mentioned work on plane spanners. In a 1992 breakthrough, Salowe [18] proved the existence of spanners of maximum degree 4. The question was then resolved by Das and Heffernan [12] who showed that spanners of maximum degree 3 always exist. The focus in this line of research was to prove the existence of low degree spanners and the techniques developed to do so were not tuned towards constructing spanners that had both low degree and low stretch factor. For example, the bound on the stretch factor of the degree-4 spanner by Salowe [18] is greater than 10^9, which is far from practical. Furthermore, these bounded-degree spanners are not guaranteed to be plane.

Bose *et al.* [7] were the first to show how to extract a subgraph of the L_2-Delaunay triangulation that is a bounded-degree, plane spanner of $\mathcal{C}$. The maximum degree and stretch factor they obtained were subsequently improved by Li and Wang [17], by Bose *et al.* [9], and by Kanj and Perković [14] (see Table 1). The approach used in all these results was to extract a bounded degree spanning subgraph of the L_2-Delaunay triangulation and the main goal was to obtain a bounded-degree plane spanner of $\mathcal{C}$ with the smallest possible stretch factor. In a breakthrough result, Bonichon *et al.* [2] lowered the bound on the maximum degree of a plane spanner from 14 to 6. Instead of using the L_2-Delaunay triangulation as the starting point of the spanner construction, they used the Delaunay triangulation based on the equilateral-triangle distance, defined originally by Chew [11], and exploited a connection between these Delaunay triangulations and $\frac{1}{2}$-θ graphs. The plane spanner they constructed also has a small stretch factor of 6. Independently, Bose *et al.* [5] were also able to obtain a plane spanner of maximum degree at most 6, by starting from the L_2-Delaunay triangulation. Recently, Bonichon et al. [4] were able to construct a plane spanner of degree at most 4 and of stretch factor at most 156.82. Their construction is based on the L_1-Delaunay triangulation. Most of the above spanner constructions can be performed in time $\mathcal{O}(n \lg n)$, where n is the number of points in $\mathcal{P}$.

In this paper, we present a construction of a plane spanner $\mathcal{S}$ of $\mathcal{C}$ of degree at most 4 and of stretch factor at most 20. This result matches the smallest known upper bound of 4 on the

Table 1 Results on plane spanners with maximum degree bounded by Δ. The constant $C_0 = 1.998$ is the best known upper bound on the stretch factor of the L_2-Delaunay triangulation [20].

Paper	Δ	**Stretch factor bound**
Bose *et al.* [7]	27	$(\pi + 1)C_0 \approx 8.27$
Li and Wang [17]	23	$(1 + \pi \sin \frac{\pi}{4})C_0 \approx 6.43$
Bose *et al.* [9]	17	$(2 + 2\sqrt{3} + \frac{3\pi}{2} + 2\pi \sin(\frac{\pi}{12}))C_0 \approx 23.56$
Kanj and Perković [14]	14	$(1 + \frac{2\pi}{14\cos(\frac{\pi}{14})})C_0 \approx 2.91$
Bonichon *et al.* [2]	6	6
Bose *et al.* [5]	6	$1/(1 - \tan(\pi/7)(1 + 1/\cos(\pi/14)))C_0 \approx 81.66$
Bonichon *et al.* [4]	4	156.82
This paper	**4**	**20**

maximum degree of the spanner by Bonichon et al. [4], while significantly improving their stretch factor bound from 156.82 to 20. Our construction is also simpler and more intuitive. It is based on Delaunay triangulations defined with respect to the equilateral-triangle distance, similar to the degree 6 spanner construction used by Bonichon et al. [2], which could be viewed as the starting point of our construction. To get down to maximum degree 4, our approach introduces fresh techniques in both the construction and the analysis of the spanner. Unlike the approach in [2], our approach has a bias – from the beginning – towards certain edges of the Delaunay triangulation; this bias ensures that the constructed spanner is well structured. To make up for edges not in the spanner, we make use of recursion which, unlike the construction in [4], may have depth not bounded by a constant. To ensure that the recursion is controlled and yields short paths, we aggressively add shortcut edges to the spanner to ensure the existence of paths with specific properties, which we refer to as monotone weak paths. Finally, in our analysis we use a new type of distance metric and we also take the extra step of analyzing the stretch factor of our spanner with respect to $\mathcal{C}$ directly, rather than with respect to the underlying Delaunay triangulation.

The structure of our spanner guarantees that if the given point-set is in convex position then the constructed spanner has maximum degree at most 3. Therefore, for any point-set in convex position, there exists a plane spanner of $\mathcal{C}$ of maximum degree at most 3. We also show that 3 is a lower bound on the maximum degree of plane spanners of $\mathcal{C}$ for point-sets in convex position. This completely and satisfactorily resolves the question about the maximum degree of plane spanners of $\mathcal{C}$ for point-sets in convex position. Due to the lack of space, the formal statement and the proof of the aforementioned result, as well as some other proofs in the paper are omitted and can be found in a full version of the paper [15].

2 Preliminaries

Given a set of points $\mathcal{P}$ embedded in the Euclidean plane, we consider the complete weighted graph $\mathcal{C}(\mathcal{P})$, or simply $\mathcal{C}$, where each edge between any two points $p, q \in \mathcal{P}$ is associated with the line segment $[pq]$, and is assigned a weight equal to the Euclidean distance $|pq|$.

Given a subgraph G of $\mathcal{C}$, G is said to be *plane* if the edges of G do not cross each other, *i.e.*, the line segments associated with the edges of G intersect only at their endpoints. The *maximum degree* of G is the maximum degree (in G) over all points in $\mathcal{P}$; we say that a family of graphs has *bounded degree* if there is an integer constant $c \geq 0$ such that every graph in the family has a maximum degree at most c.

If graph G is connected, we define the *distance* between any two points $p, q \in \mathcal{P}$, denoted $d_G(p, q)$, to be the weight of a minimum-weight path between p and q in G, where the weight of a path is the sum of the weights of its edges.

Given a constant $\rho \geq 1$, we say that G is a *ρ-spanner* of $\mathcal{C}$ if for any two points $p, q \in \mathcal{P}$, $d_G(p, q) \leq \rho \cdot |pq|$; we refer to the minimum such constant ρ as the *stretch factor* of G. We also say that a family of geometric graphs, one for every finite set $\mathcal{P}$ of points in the plane, is a *spanner* if there is a constant $\rho \geq 1$ such that every graph $G(\mathcal{P})$ in the family is a ρ-spanner of $\mathcal{C}(\mathcal{P})$; we refer to the minimum such constant ρ as the *stretch factor* of the family. In this paper, the family we construct consists of the set of spanners $G(\mathcal{P})$, where $G(P)$ is the spanner obtained by applying our algorithm to a point-set $\mathcal{P}$.

In this paper, we rely on a metric that is different from the Euclidean metric. In order to define this metric, we fix an equilateral triangle with two of its points lying on the x-axis at coordinates $(0, 0)$ and $(1, 0)$, and the third point lying below the x-axis; we use the symbol $\bigtriangledown$ to refer to this equilateral triangle. We define a triangle to be a *$\bigtriangledown$-homothet* if it can be obtained through a translation of $\bigtriangledown$ followed by a scaling. We define the *triangular metric*, $d_{\bigtriangledown}$, as follows:

▶ **Definition 1.** For any two points $p, q \in \mathcal{P}$, define $d_{\bigtriangledown}(p, q)$ to be the side-length of the smallest $\bigtriangledown$-homothet that contains p and q on its boundary; we denote this triangle $\bigtriangledown(p, q)$.

It is easy to verify that $d_{\bigtriangledown}$ is indeed a metric. In particular, for any two points p, q, we have $d_{\bigtriangledown}(p, q) = 0 \Leftrightarrow p = q$, we have symmetry as in $d_{\bigtriangledown}(p, q) = d_{\bigtriangledown}(q, p)$, and for any third point r, we have the triangle inequality $d_{\bigtriangledown}(p, q) \leq d_{\bigtriangledown}(p, r) + d_{\bigtriangledown}(r, q)$. It is also easy to see that p or q must be a vertex of the triangle $\bigtriangledown(p, q)$ and that $|pq| \leq d_{\bigtriangledown}(p, q)$.

Using the triangular metric $d_{\bigtriangledown}$, we define a subgraph $\mathcal{D}$ of $\mathcal{C}$ as follows. For every point $w \in \mathcal{P}$, we partition the space around w into six equiangular cones whose common apex is w, three above and three below the horizontal line passing through w, as illustrated in Figure 1(a). We denote the middle cone above the horizontal line and the two outer cones below the horizontal line as the *positive* cones of w, and the remaining three cones as the *negative* cones of w. Each point w chooses an edge in each of its three positive cones by selecting the point $v \neq w$ in the cone such that $d_{\bigtriangledown}(w, v)$ is minimum. Assuming that $\mathcal{P}$ is in general position[1], for any two distinct points v, v' in a positive cone of w, we obtain $d_{\bigtriangledown}(w, v) \neq d_{\bigtriangledown}(w, v')$. We define $\mathcal{D}$ to be the graph whose vertex-set is $\mathcal{P}$ and whose edge-set is the set of edges selected as described.

We make the following observation regarding the graph $\mathcal{D}$. The *$\frac{1}{2}$-θ graph* of $\mathcal{P}$ is the graph whose point-set is $\mathcal{P}$, and whose edges are obtained as follows: at each point w, and for each of the three positive cones of apex w, select the edge wv in the cone where v is the point whose projection distance to the angular bisector of the cone is minimum. Bonichon et al. [1] showed that the $\frac{1}{2}$-θ graph of $\mathcal{P}$ is the same as the *TD-Delaunay triangulation* of $\mathcal{P}$ [11] defined based on the empty triangle property: there is an edge between two points $v, w \in P$ if there exists a homothet of $\bigtriangledown$ containing v and w on its boundary whose interior is empty of points of $\mathcal{P}$. It is easy to see that the $\frac{1}{2}$-θ graph of $\mathcal{P}$ coincides with the graph $\mathcal{D}$ defined above, and hence with the TD-Delaunay triangulation[2].

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[1] $\mathcal{P}$ is in general position if no pair of points $v, w \in \mathcal{P}$ lie on a line parallel to any of the boundary lines defining the six cones. We note that it is always possible to rotate the equilateral triangle that defines the metric $d_{\bigtriangledown}$ to ensure that the finite set $\mathcal{P}$ is in general position and so the results in this paper hold for all sets of points and not just for points in general position.

[2] A TD-Delaunay triangulation of $\mathcal{P}$ is not necessarily a triangulation of $\mathcal{P}$ as defined traditionally (a triangulation of the convex hull of the set of points). Just as Chew [11] did, we abuse the term *triangulation* because TD-Delaunay triangulations are closely related to classical L_2-Delaunay triangulations.

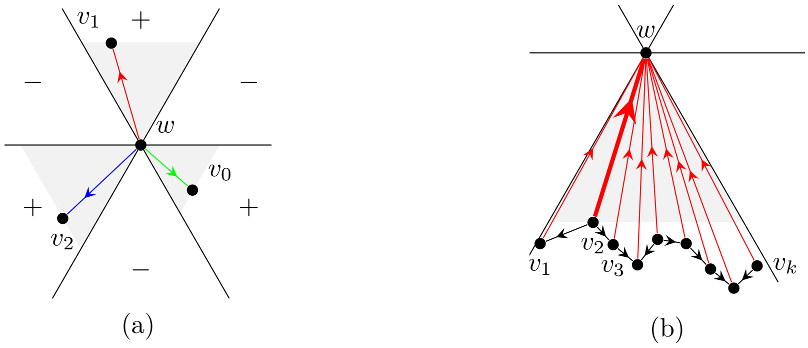

Figure 1 (a) To construct graph $\mathcal{D}$, every point w chooses the shortest edge, according to the $d_\triangledown$ distance, in every positive cone. (b) Edge (v_2, w) is the anchor of w in the negative cone shown because it is the shortest edge according to the $d_\triangledown$ distance, among edges incoming to w in the cone; the path $v_1, \ldots, v_k$ is the canonical path of anchor (v_2, w).

For convenience, we label the positive cones at each point of $\mathcal{P}$, in clockwise order and starting with the positive cone above the horizontal line, *red*, *green*, and *blue*; we also label the negative cones, in clockwise order and starting with the negative cone below the horizontal line, red, green, and blue. We assign an orientation and a color to the edges of $\mathcal{D}$ by orienting each edge outwards from the point w that selects it and by coloring it red, blue, or green depending on whether the edge lies in the positive red, blue, or green cone of point w, as illustrated in Figure 1(a). We emphasize that the edge orientations are only used for the purpose of constructing the spanner and proving its desired properties; the final spanner in our construction is an undirected graph obtained by removing edge orientations. In fact, we abuse terminology and, throughout the paper, use the term *path* to refer to weak paths in $\mathcal{D}$; we always use the term *directed path* when edge orientations are relevant.

We observe that for any point $w \in \mathcal{P}$ there is at most one edge outgoing from w in a positive cone of w, but there can be an unbounded number of edges incoming to w in a negative cone of w, and that in such cases all these edges have the same color as the cone itself (e.g., see Figure 1(b)). We follow the same approach as Bonichon et al. [2], and identify in each negative cone of point w an edge that plays a key role in the spanner construction:

▶ **Definition 2.** For any point $w \in \mathcal{P}$, and for each negative cone of w that contains at least one edge incoming to w, let (directed) edge $(v, w) \in \mathcal{D}$ be the edge in the cone such that $d_\triangledown(v, w)$ is minimum. We define (v, w) to be the *anchor* of w in the cone.

We say that anchors incident to the same point w are *adjacent* if their cones are adjacent. Note that for any two adjacent anchors incident to w, one of the two adjacent anchors must lie in a positive cone of w and must be an anchor of a point other than w.

We introduce next more terminology that we will need. Consider a negative cone of a point $w \in \mathcal{P}$ containing at least one incoming edge to w in $\mathcal{D}$. Let $(v_1, w), \ldots, (v_k, w) \in \mathcal{D}$, where $k \geq 1$, be all the incoming edges to w that lie in the cone, listed in counterclockwise order, as illustrated in Figure 1(b), and let (v_j, w), for some j such that $1 \leq j \leq k$, be the anchor of w in the cone. We call $(v_1, w), \ldots, (v_k, w)$ the *fan* of the anchor (v_j, w). We identify (v_j, w) as the anchor of each edge in the fan. Note that every edge in $\mathcal{D}$ has an anchor which could be itself. We call the first edge (v_1, w) and the last edge (v_k, w) of the fan the *boundary edges* of the anchor (v_j, w). Note that either one (possibly both) of the boundary edges of an anchor could be the anchor itself. If $k \geq 2$, since $\mathcal{D}$ is a triangulation, it follows that (v_i, v_{i+1}, w) is a triangle in $\mathcal{D}$, for $i = 1, \ldots, k - 1$. Hence, $v_1, \ldots, v_k$ is a (weak)

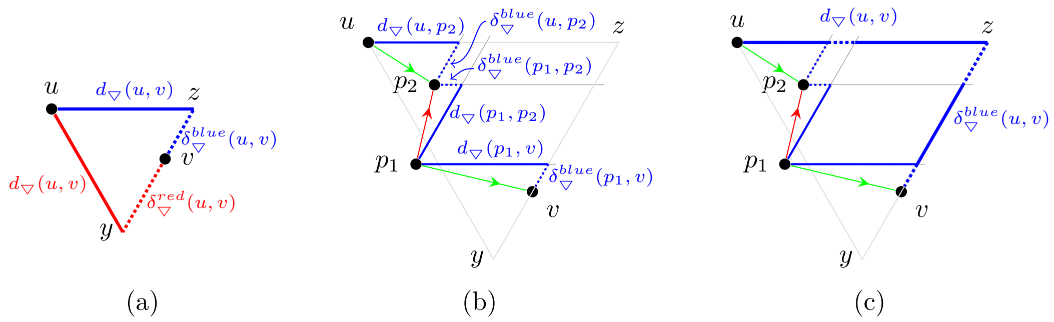

Figure 2 (a) If point v lies in the positive green cone of point u, and the vertices of $\triangledown(u,v)$, u, y, z, are colored green, red, and blue, respectively, then $d_\triangledown(u,v) = |zu|$ and $\delta_\triangledown^{blue}(u,v) = |zv|$. (b) and (c) P is a monotone path between u and v with edges colored green or red. The projection onto zu (resp. zy) of (u,p_2), (p_1,p_2), and (p_1,v) do not overlap and are contained within $[zu]$ (resp. $[zv]$).

path in $\mathcal{D}$ between the endpoints v_1 and v_k. We call this path the *canonical path* of w in the designated cone; we also call each edge on this path a *canonical edge* of w. Finally, we refer to the (weak) subpath $v_r, \ldots, v_s$ of the canonical path $v_1, \ldots, v_k$ of w as the canonical path between v_r and v_s of w. The two *sides* of an edge are the two half-planes defined by the line obtained by extending the edge. We say that a canonical edge e is canonical on a side of e if it is a canonical edge of a point that lies on that side of e. Note that a canonical edge can be canonical on both sides.

We state the following easy to verify facts without proof:

▶ **Lemma 3.** *Let (s,t) be a canonical edge of a point w, and let (s',t) be the anchor of (s,t).*
(a) *The edges (s,w) and (t,w) are in $\mathcal{D}$.*
(b) *The edge (s,w) cannot be a canonical edge on the side containing t.*
(c) *The edge (t,w) is not an anchor.*
(d) *The edge (s,t) is a boundary edge of its anchor (s',t).*

3 Monotone (weak) paths

We define next a type of path in $\mathcal{D}$ that generalizes canonical paths and that will be a key tool in our construction. We give two equivalent definitions of such a path; we leave the proof of the equivalence between these two definitions to the reader.

▶ **Definition 4.** Let v be a point lying in the positive cone of u whose color is c. A (weak) path in $\mathcal{D}$ between u and v is *monotone* if the path is bi-colored, with c being one of the colors, and the path satisfies the two equivalent properties:

- After reversing the direction of all edges not colored c, the path is directed from u to v.
- No two consecutive edges of the path lie in neighboring cones of the shared endpoint.

The key property of a monotone path between u and v is that its length can be bounded by twice the side-length of $\triangledown(u,v)$, *i.e.*, by $2d_\triangledown(u,v)$. This follows from a stronger insight which we develop next. To facilitate our discussion, we label the vertices of a $\triangledown$-homothet *green*, *blue*, and *red*, in clockwise order starting from the upper left vertex.

▶ **Definition 5.** Let v be a point lying in a positive cone of u of color c_1. With u being the vertex of $\triangledown(u,v)$ of color c_1, let y and z be the vertices of $\triangledown(u,v)$ of colors c_2 and c_3

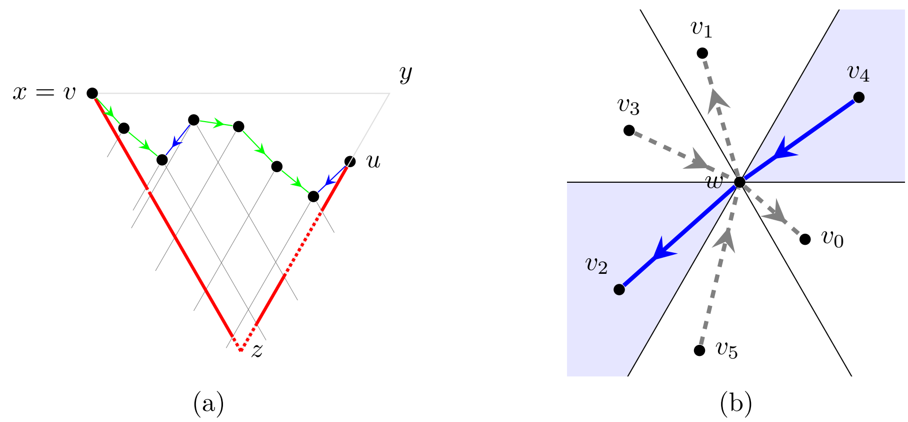

Figure 3 (a) Illustration of Lemma 7. (b) w is incident to an anchor in every cone. In step 1, both of the blue anchors are added. In step 2, on the other hand, no more than one white anchor is added below the horizontal line through w and no more than one white anchor is added above.

respectively (refer to Figure 2 where $c_1 = green$, $c_2 = red$, and $c_3 = blue$). We define the following distance functions $\delta^{c_2}_{\triangledown}, \delta^{c_3}_{\triangledown}$, and $\delta^{\min}_{\triangledown}$:

1. $\delta^{c_2}_{\triangledown}(u, v) = \delta^{c_2}_{\triangledown}(v, u) = |yv|$.
2. $\delta^{c_3}_{\triangledown}(u, v) = \delta^{c_3}_{\triangledown}(v, u) = |zv|$.
3. $\delta^{\min}_{\triangledown}(u, v) = \delta^{\min}_{\triangledown}(v, u) = \min\{\delta^{c_2}_{\triangledown}(u, v), \delta^{c_3}_{\triangledown}(u, v)\}$.

Given the assumptions of Definition 5, let P be a monotone path in $\mathcal{D}$ between u and v whose edges are colored c_1 or c_2. We define, using the lines zv and zu as axes of a coordinate system of the Euclidean plane, *the projection onto zv and onto zu*. In the following lemma, we use this projection to map the edges of P and derive an upper bound on the length of P:

▶ **Lemma 6.** *Let v be a point lying in a positive cone of u of color c_1. Let P be a monotone path between u and v whose edges are colored c_1 or c_2 (refer to Figure 2 where $c_1 = green$, $c_2 = red$, and $c_3 = blue$). With u being the vertex of $\triangledown(u, v)$ of color c_1, let z be the vertex of $\triangledown(u, v)$ of color c_3. Then, the monotone path P satisfies the following:*

(a) *The projections of all edges of P onto zu (resp., zv) do not overlap and are contained within the segment $[zu]$ (resp., $[zv]$); see Figures 2(b) and 2(c).*
(b) *If (p, q) is an edge of P colored c_1 (resp., c_2) then the projection onto zu (resp., zv) of (p, q) has length $d_{\triangledown}(p, q) \geq |pq|$.*
(c) *The sum of the lengths of the edges of P colored c_1 is at most $d_{\triangledown}(u, v) = |zu|$.*
(d) *The sum of the lengths of the edges of P colored c_2 is at most $\delta^{c_3}_{\triangledown}(u, v) = |zv|$.*
(e) *The length of P is at most $d_{\triangledown}(u, v) + \delta^{c_3}_{\triangledown}(u, v) \leq 2d_{\triangledown}(u, v)$.*

Proof. For part (a), we consider the coordinates of the points of P in the coordinate system of the Euclidean plane defined by using the lines zv and zu as axes. When visiting the points of P in the order in which they appear on P, the coordinates of the points along the zu (resp., zv) axis form a monotonic sequence (decreasing or increasing) between the coordinates of u and z (resp. z and v), and part (a) follows. Since zu is parallel to an edge of $\triangledown(p, q)$, and hence the projection of (p, q) onto zu has length $d_{\triangledown}(p, q)$, part (b) follows. Parts (c) and (d) follow from parts (a) and (b), and part (e) follows from parts (c) and (d). ◀

Implied by Lemma 6, the following lemma makes explicit an insight implicit in Lemma 2 of [2] on canonical paths (see Figure 3(a)).

▶ **Lemma 7.** *For any two edges (v, w) and (u, w) that lie in the same fan:*
1. *The canonical path P between v and u is monotone.*
2. *The sum of the lengths of all monochromatic edges on P is at most $d_{\bigtriangledown}(v, u)$.*
3. *The length of the canonical path P between v to u is at most $2d_{\bigtriangledown}(v, u)$.*

Proof. For part *(a)*, we assume, without loss of generality, that w lies in the red positive cones of v and of u. We then observe that for every point p on P, (p, w) is an edge in $\mathcal{D}$. Therefore, every edge of P must lie in the blue or green positive cones of its tail, and thus path P is bi-colored. Furthermore, since $\mathcal{D}$ is planar, at every intermediate point p of P, the two edges of P incident to p must lie in non-adjacent cones. The canonical path P between v and u is thus monotone. Hence, parts (b) and (c) follow by Lemma 6. ◀

4 The Spanner

In this section, we describe the construction of a plane spanner of $\mathcal{C}$ of maximum degree at most 4 and stretch factor at most 20. In our construction, we will bias blue – positive and negative – cones and edges. This bias results in a spanner satisfying structural properties that allow us to prove the desired upper bounds on the spanner degree and stretch factor. These structural properties also ensure that the spanner has maximum degree at most 3 when the point-set $\mathcal{P}$ is in convex position. In the algorithm description and the remainder of the paper, we find it convenient to refer to the four non-blue cones, as well as all the red and green edges, as *white* (see Figure 3(b)). We also use some new terminology which we define next.

If e is a canonical edge of a point w that lies in a white (resp., blue) cone of w, we say that e is a canonical edge *in a white* (resp., *blue*) *cone*. We note that a canonical edge could be in a white cone of one point and in a blue cone of another. Given a white anchor (v, w), the ray starting from w extending (v, w) partitions the (white) negative cone of w containing v into two *sides*: we refer to the side of the cone that is adjacent to a blue cone as the *blue side*, and we refer to the other side that is adjacent to a white cone as the *white side*. We say that an edge (u, w) in the fan of (v, w) is on the *white side* (resp. *blue side*) if it is on the white side (resp. blue side) of (v, w).

The following describes the construction of the spanner $\mathcal{S}$ of $\mathcal{C}$. The construction is based on the underlying triangulation $\mathcal{D}$ of $\mathcal{C}$. We start by constructing a degree-4 anchor subgraph $\mathcal{A}$ of $\mathcal{S}$ that includes all blue anchors. We then augment $\mathcal{S}$ by adding *some* white canonical edges and shortcut edges.

1. We add to $\mathcal{A}$ (which is initially empty) every blue anchor.
2. In increasing order of length with respect to the metric $d_{\bigtriangledown}$, for every white anchor a, we add a to $\mathcal{A}$ if no white anchor adjacent to a is already in $\mathcal{A}$.
3. We set $\mathcal{S}$ to $\mathcal{A}$ and then add to $\mathcal{S}$ every (white) canonical edge in a blue cone if the edge is not in $\mathcal{A}$.
4. For every pair of canonical edges $(p, q), (r, q)$ in a blue cone such that $(p, q), (r, q) \in \mathcal{S} \setminus \mathcal{A}$, we add to $\mathcal{S}$ the shortcut edge (p, r), color it white, and remove (p, q) and (r, q) from $\mathcal{S}$.
5. We add to $\mathcal{S}$ every white canonical edge that is on the white side of its (white) anchor, but only if its anchor is not in $\mathcal{A}$.
6. For every white anchor (v, w) and its boundary edge $(u, w) \neq (v, w)$ on the white side, let P be the canonical path $(u = p_0, p_1, \ldots, p_k = v)$. We apply the following procedure at a current point p_i starting with $i = 0$ and stopping when $i = k$:
   a. If the canonical edge (p_{i+1}, p_i) is white, we skip this edge and set i to $i + 1$;

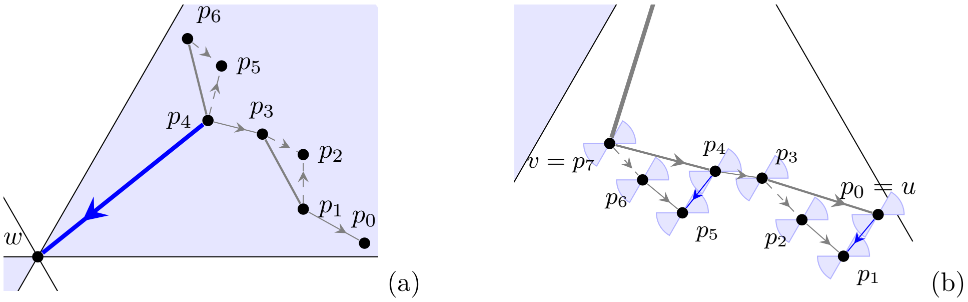

Figure 4 (a) In step 3, white canonical edges of w in the negative blue cone of w are added to $\mathcal{S}$ if not in $\mathcal{A}$ already; in step 4, any pair of canonical edges of w added in step 3 that are incoming at the same point are replaced by a shortcut between the outgoing endpoints ((p_6, p_5) and (p_4, p_5) replaced by shortcut (p_6, p_4) and ($p_{3,2}$) and (p_1, p_2) replaced by shortcut (p_3, p_1).) (b) Shortcut edges (p_3, p_0) and (p_7, p_4) are added to $\mathcal{S}$ in step 6; edges (p_7, p_6) and (p_3, p_2) are not in $\mathcal{S}$ unless they are anchors in $\mathcal{A}$.

b. Otherwise, (p_i, p_{i+1}) must be blue. Let $j > i$ be the largest index of a point on P such that the line segment $[p_i p_j]$ does not intersect the canonical path from p_i to p_j (except at p_i and p_j). We add the shortcut (p_j, p_i) to $\mathcal{S}$ and color it white; we remove the (white) canonical edge (p_j, p_{j-1}) from $\mathcal{S}$ if $(p_j, p_{j-1}) \in \mathcal{S} \setminus \mathcal{A}$; and we set i to j.

In the following section we prove that this algorithm yields a plane spanner of maximum degree at most 4 and stretch factor at most 20. We provide here a high-level overview of our arguments.

To show planarity, we note that the underlying graph $\mathcal{D}$ is planar and that the only edges of $\mathcal{S}$ not in $\mathcal{D}$ are the shortcut edges added in steps 4 and 6.b. We prove in Lemmas 10 and 11 that each such edge does not intersect any other edge of $\mathcal{S}$. For the degree upper bound, we note that the first two steps of the algorithm yield the subgraph $\mathcal{A}$ of maximum degree at most 4. In the remaining steps, we carefully add additional edges, whether canonical edges or shortcuts of canonical paths. To prove the degree bound, we develop a charging argument that assigns each edge of $\mathcal{S}$ to a cone at each endpoint and show, in Lemma 12, that no more than 4 cones are charged at every point.

To prove that $\mathcal{S}$ is a spanner, we show that every edge (u, w) in $\mathcal{D}$ but not in $\mathcal{S}$ can be *reconstructed*, by which we mean that there is a short path between u and w in $\mathcal{S}$. To do this, we consider the path between u and w in $\mathcal{D}$ consisting of the anchor (v, w) of (u, w) and the canonical path from u to v of anchor (v, w), and we argue that every edge on that path can be reconstructed. Because canonical edges are boundary edges, it is sufficient to show that all anchors and boundary edges are reconstructible.

In step 1, we add all blue anchors to $\mathcal{S}$ and in steps 3 and 4 we add to $\mathcal{S}$ all white canonical edges in blue cones, except for some consecutive pairs of canonical edges that are replaced with shortcut edges. Together, these steps ensure that (almost) all blue edges are reconstructible as we show in Lemma 13; in particular, all blue boundary edges are reconstructible.

If (u, w) is a white boundary edge, the edges on the canonical path between u and v are blue boundary edges (which are reconstructible, as discussed above) or white boundary edges on the white side of their anchor. Steps 5 and 6 ensure that white boundary edges on the white side of their anchor are reconstructible with a monotone path that is constructed recursively using shortcuts added in step 6. Therefore, if white anchor (v, w) is in $\mathcal{S}$, edge (u, w) is reconstructible as we show in Lemma 14. If (v, w) is not in $\mathcal{S}$ then (v, w) must be a white anchor (since step 1 added all blue anchors to $\mathcal{S}$) and there must exist a shorter anchor adjacent to (v, w) in $\mathcal{S}$ (by step 2.) In Lemma 17, we show that this shorter anchor can be used to reconstruct anchor (v, w) implying that (u, w) is reconstructible as well.

5 Properties of the Spanner

In this section, we prove the three properties of the spanner $\mathcal{S}$ obtained using our algorithm: planarity, the maximum degree upper bound of 4, and the stretch factor bound of 20. We start with the following justification for coloring white the shortcut edges added in step 6:

▶ **Lemma 8.** *For every shortcut edge (p_j, p_i) added to $\mathcal{S}$ in step 6, p_j and $v = p_k$ both lie in the same negative white cone of p_i, and they both lie in the same negative white cone of p_{j-1}.*

Proof. Because $d_{\bigtriangledown}(w, p_k) < d_{\bigtriangledown}(w, p_i)$ and $(p_i, w) \in \mathcal{D}$, p_k must lie in a negative white cone of p_i. The lemma thus holds if $j = k$. Otherwise, by the choice of p_j, p_j must lie on the same side of line $p_i p_k$ as point w; again, because $(p_i, w) \in \mathcal{D}$, p_j must lie in a (negative) white cone of p_i that also contains p_k. Similar arguments apply to p_{j-1}. ◀

Next, we show that $\mathcal{S}$ is plane. We first need the following definition and lemma.

▶ **Definition 9.** An edge $(u, w) \in \mathcal{D}$ is *uncrossed* if no shortcut in $\mathcal{S}$ crosses (u, w).

▶ **Lemma 10.** *All anchors, all canonical edges, and all boundary edges are uncrossed.*

Proof. Let (p, r) be a shortcut that was added in step 4 of the spanner construction, let (p, q) and (r, q) be the pair of canonical edges in the blue cone as described in step 4, and let w be the apex of this blue cone. It is easy to verify that $(q, w) \in \mathcal{D}$ is the only edge in $\mathcal{D}$ that (p, r) crosses, and that (q, w) is not a boundary edge, a canonical edge, or an anchor. Next, consider a shortcut (p_j, p_i) that was added in step 6 of the spanner construction, and let (v, w) be the white anchor and $p_{i+1}, p_{i+2}, \ldots, p_{j-1}$ be the points on the canonical path between p_i and p_j as described in step 6. Again, it is easy to verify that $(p_{i+1}, w), (p_{i+2}, w), \ldots, (p_{j-1}, w)$ are the only edges in $\mathcal{D}$ that the shortcut (p_j, p_i) crosses, and that none of them is a boundary edge, a canonical edge, or an anchor. ◀

▶ **Lemma 11.** *The subgraph $\mathcal{S}$ is a plane subgraph of $\mathcal{C}$.*

Proof. Let $\mathcal{S}_1 = \mathcal{D} \cap \mathcal{S}$ and $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$. Note that $\mathcal{S}_1$ consists of $\mathcal{A}$ plus those canonical edges that are added in steps 3 or 5 and kept after steps 4 and 6. Note also that $\mathcal{S}_2$ consists only of the shortcuts which are added in steps 4 and 6. Since $\mathcal{S}_1$ is a subgraph of $\mathcal{D}$, $\mathcal{S}_1$ is plane. By Lemma 10, all the edges in $\mathcal{S}_1$ are uncrossed, *i.e.*, no shortcut (edge in $\mathcal{S}_2$) crosses an edge in $\mathcal{S}_1$. To conclude the proof, we show that no two edges in $\mathcal{S}_2$ cross either. Observe that any two shortcuts connect pairs of endpoints of canonical paths that either belong to different fans or that belong to the same fan. In the former case, the shortcuts do not cross because they belong to different fans. In the latter case, the shortcuts do not cross because shortcuts always connect the endpoints of non-overlapping canonical paths. ◀

To facilitate the discussion in the proof of the degree upper bound, we refer to the two adjacent white cones above (resp., below) the horizontal line through a point $p \in \mathcal{P}$ as the *upper* (resp., *lower*) *white sector* of p; we also refer to the two blue cones at p as the *left* and *right blue sectors* of p. We develop a charging scheme to show that, for each point p, each edge incident to p in $\mathcal{S}$ can be mapped in a one-to-one fashion to one of the four sectors at p. To describe the charging scheme for every edge $e \in \mathcal{S}$ and for every endpoint p of e, we define $\sigma(e,p)$ to be the sector of p that contains e. Also for a point p, we denote by LB_p, RB_p, UW_p, and LW_p, the left blue, the right blue, the upper white, and the lower white sectors of p respectively. We describe in the table below the charging scheme for every edge $e=(x,y)\in\mathcal{S}$ based on which step of the construction e is added to $\mathcal{S}$.

Step	**Classification of** $e=(x,y)$	**Charge at** x	**Charge at** y
1	Blue anchor in $\mathcal{A}$	$\sigma(e,x)=LB_x$	$\sigma(e,y)=RB_y$
2	White anchor in $\mathcal{A}$	$\sigma(e,x)=UW_x$ or LW_x	$\sigma(e,y)=LW_y$ or UW_y
3	White canonical edge in a blue cone	$\sigma(e,x)=UW_x$ or LW_x	LB_y
4	(White) shortcut in a blue cone	$\sigma(e,x)=UW_x$ or LW_x	$\sigma(e,y)=LW_y$ or UW_y
5	White canonical edge in a white cone	RB_x	$\sigma(e,y)=UW_y$ or LW_y
6	(White) shortcut in a white cone	RB_x	$\sigma(e,y)=UW_y$ or LW_y

▶ **Lemma 12.** *For any point $p\in\mathcal{P}$, each sector of p is charged with at most one edge in $\mathcal{S}$. Therefore, the maximum degree of $\mathcal{S}$ is at most 4.*

The remainder of this section is devoted to proving the upper bound of 20 on the stretch factor of $\mathcal{S}$. We do so by first proving a sequence of lemmas that derive upper bounds on the distance in $\mathcal{S}$ between the endpoints of different types of edges in $\mathcal{D}$; we then use these lemmas to derive the upper bound of 20 on the stretch factor of $\mathcal{S}$.

▶ **Lemma 13.** *For any uncrossed blue edge $(u,w)\in\mathcal{D}$, $d_{\mathcal{S}}(u,w)\le 3d_{\triangledown}(u,w)$.*

Proof. Let (v,w) be the blue anchor of the blue edge (u,w). In step 1 of the algorithm, we add all the blue anchors in $\mathcal{S}$, and thus $(v,w)\in\mathcal{S}$. Also, in step 3 of the algorithm, we add in $\mathcal{S}$ all the canonical edges in blue cones except that, in step 4, we substitute some pairs of these canonical edges with shortcuts. Since (u,w) is uncrossed, these canonical edges and shortcuts provide a path for connecting v and u. Using the triangle inequality, this path that includes the shortcuts is not longer than the canonical path between v and u. Hence, in the worst case, we may assume that the path connecting v and u consists only of canonical edges on the canonical path. This canonical path plus the anchor constitutes a path between u and w. By Lemma 7, the length of this canonical path is bounded by $2d_{\triangledown}(v,u)\le 2d_{\triangledown}(u,w)$. We also have that $|vw|\le d_{\triangledown}(v,w)\le d_{\triangledown}(u,w)$. Consequently, the length of this path is bounded by $d_{\triangledown}(u,w)$ (anchor) plus $2d_{\triangledown}(u,w)$ (canonical path). It follows that $d_{\mathcal{S}}(u,w)\le 3d_{\triangledown}(u,w)$. ◀

▶ **Lemma 14.** *For any white anchor (v,w) and any uncrossed white edge $(u,w)\in\mathcal{D}$ that lies on the white side of (v,w), $d_{\mathcal{S}}(v,u)\le d_{\triangledown}(v,u)+\delta^{blue}_{\triangledown}(v,u)\le 2d_{\triangledown}(v,u)$. Furthermore, if $(v,w)\in\mathcal{S}$, then $d_{\mathcal{S}}(u,w)\le d_{\triangledown}(u,w)+\delta^{blue}_{\triangledown}(u,w)\le 2d_{\triangledown}(u,w)$.*

Proof. We describe below how to construct a white monotone path in $\mathcal{S}$ between u and v. If $(v,w)\in\mathcal{S}$, we extend this path to a white monotone path between u and w. Then, we obtain the desired bounds using Lemma 6.

To describe the white monotone path between u and v, we consider the uncrossed edges of $\mathcal{D}$ on the fan of (v,w) whose endpoints lie on the canonical path between v and u. We observe that shortcuts and white canonical edges connect the (distinct) endpoints of those uncrossed edges, and that they form a white monotone path between u and v because at each point the white edges of the path incident to the point lie on opposite sides of the horizontal line through the point. We call this white monotone path the *white monotone connection* between v and u. We know that all of the shortcuts on this white monotone connection are in $\mathcal{S}$ and even though some of the white canonical edges may not be in $\mathcal{S}$, we know, for all such white canonical edges, that we have their anchors in $\mathcal{S}$. For each such white canonical edge (s,t), we recursively expand the current white monotone path by including the anchor

(r, t) of (s, t) and by including the white monotone connection between r and s. We point out that the path obtained after the expansion of (s, t) continues to be a white monotone path. This is because the anchor (r, t) already conforms to the existing white monotone path, and so does the white monotone connection between r and s. Therefore, by recursively expanding this path for white canonical edges that are not in $\mathcal{S}$, we obtain a white monotone path between v and u. Furthermore, if $(v, w) \in \mathcal{S}$, we expand this path to include the white anchor (v, w) while preserving its monotonicity. ◀

▶ **Lemma 15.** *For any white anchor (v, w) and any white edge $(u, w) \in \mathcal{D}$ that lies on the blue side of (v, w), $d_{\mathcal{S}}(v, u) \le 5d_{\bigtriangledown}(v, u)$. Furthermore, if $(v, w) \in \mathcal{S}$, then $d_{\mathcal{S}}(u, w) \le 6d_{\bigtriangledown}(u, w)$.*

Proof. The canonical path from v to u consists of blue and white canonical edges. The total length of the blue canonical edges does not exceed $d_{\bigtriangledown}(v, u)$, and the total length of the white canonical edges does not exceed $d_{\bigtriangledown}(v, u)$ by Lemma 7. By Lemma 10, we know that all of these canonical edges are uncrossed. By Lemma 13, the total length of the paths needed to reconstruct these blue canonical edges can be bounded by $3d_{\bigtriangledown}(v, u)$. Also, since either the white canonical edges themselves or their anchors are in $\mathcal{S}$, the total length of the white canonical edges can be bounded by $2d_{\bigtriangledown}(v, u)$ by Lemma 14. Therefore, $d_{\mathcal{S}}(v, u)$ can be bounded by $5d_{\bigtriangledown}(v, u)$ for the edge (v, u) as stated. Furthermore, if $(v, w) \in \mathcal{S}$, $d_{\mathcal{S}}(u, w)$ can be bounded by $5d_{\bigtriangledown}(v, u) + d_{\bigtriangledown}(v, w)$, which in turn is bounded by $6d_{\bigtriangledown}(u, w)$. ◀

▶ **Definition 16.** For any two points $p, q \in \mathcal{P}$ such that p lies in a white cone of q, we define $\delta^{white}_{\bigtriangledown}(p, q) = \delta^{white}_{\bigtriangledown}(q, p) = d_{\bigtriangledown}(p, q) - \delta^{blue}_{\bigtriangledown}(p, q)$.

▶ **Lemma 17.** *For any white anchor (v, w), $d_{\mathcal{S}}(v, w) \le 9d_{\bigtriangledown}(v, w)$. Furthermore, for any uncrossed white edge (u, w) in the fan of (v, w), we have $d_{\mathcal{S}}(u, w) \le 9d_{\bigtriangledown}(u, w) + \delta^{blue}_{\bigtriangledown}(u, w)$ if (u, w) lies on the white side of (v, w), and $d_{\mathcal{S}}(u, w) \le 9d_{\bigtriangledown}(u, w)$ otherwise.*

Proof. If $(v, w) \in \mathcal{S}$, then clearly $d_{\mathcal{S}}(v, w) \le d_{\bigtriangledown}(v, w)$. As for any uncrossed edge (u, w) in the fan, by Lemma 14, we get a bound of $2d_{\bigtriangledown}(u, w)$ on $d_{\mathcal{S}}(u, w)$ if (u, w) lies on the white side of (v, w), and by Lemma 15, we get a bound of $6d_{\bigtriangledown}(u, w)$ on $d_{\mathcal{S}}(u, w)$ if (u, w) lies on the blue side of (v, w). Then, we consider $(v, w) \notin \mathcal{S}$ and analyze two cases: (v, w) was not added in $\mathcal{A}$ because of an adjacent anchor at w, or because of an adjacent anchor at v.

If (v, w) was not added in $\mathcal{A}$ because of an adjacent (white) anchor at w, let (w, w') be that anchor (see Figure 5(a)). By our construction of $\mathcal{A}$, we know that (w, w') must be shorter than (v, w), *i.e.*, $d_{\bigtriangledown}(w, w') < d_{\bigtriangledown}(v, w)$. Therefore, v lies in the positive blue cone of w', and hence, there must be an outgoing blue edge at w'. Let (w', u') be that blue edge; then, (u', w) must be a white boundary edge of (v, w), and possibly $u' = v$. Using the fact that u' lies in the positive blue cone of w', it is easy to verify that $d_{\bigtriangledown}(v, u') \le \delta^{white}_{\bigtriangledown}(v, w) \le d_{\bigtriangledown}(v, w)$. Similarly, using the fact that u' lies in a positive white cone of v, and that $d_{\bigtriangledown}(w, w') < d_{\bigtriangledown}(v, w)$, it is easy to verify that $d_{\bigtriangledown}(w', u') \le d_{\bigtriangledown}(w, w') + \delta^{white}_{\bigtriangledown}(v, w) < 2d_{\bigtriangledown}(v, w)$. Following the path from w to w' to u' to v, we bound $d_{\mathcal{S}}(v, w)$ by $d_{\bigtriangledown}(w, w')$ for the edge (w, w'), by $3d_{\bigtriangledown}(w', u')$ for the edge (w', u') using Lemmas 10 and 13, and by $2d_{\bigtriangledown}(v, u')$ for the edge (v, u') using Lemmas 10 and 14. Also, using the above inequalities $d_{\bigtriangledown}(w, w') < d_{\bigtriangledown}(v, w)$, $d_{\bigtriangledown}(v, u') \le d_{\bigtriangledown}(v, w)$, and $d_{\bigtriangledown}(w', u') \le 2d_{\bigtriangledown}(v, w)$, we get the desired upper bound $d_{\mathcal{S}}(v, w) \le 9d_{\bigtriangledown}(v, w)$.

Next, we consider the uncrossed white edges. For any uncrossed edge (u, w) on the white side of the anchor, the bound on $d_{\mathcal{S}}(v, w)$ applies directly to $d_{\mathcal{S}}(u, w)$ because the path between v and w already connects u and w. Also, since $d_{\bigtriangledown}(v, w) \le d_{\bigtriangledown}(u, w)$, we immediately get $d_{\mathcal{S}}(u, w) \le 9d_{\bigtriangledown}(u, w)$. As for any white edge (u, w) on the blue side of the anchor, we start by observing that $\delta^{white}_{\bigtriangledown}(v, w) \le d_{\bigtriangledown}(u, w) - d_{\bigtriangledown}(v, u)$, and that $d_{\bigtriangledown}(v, w) \le d_{\bigtriangledown}(u, w)$. Also, using the above inequalities $d_{\bigtriangledown}(w, w') < d_{\bigtriangledown}(v, w)$, and $d_{\bigtriangledown}(v, u') \le \delta^{white}_{\bigtriangledown}(v, w)$, and

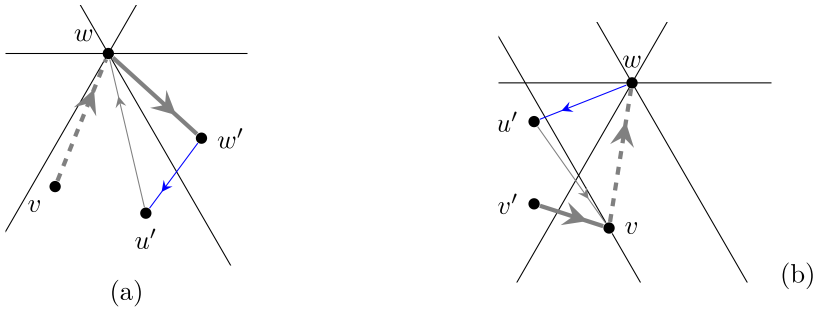

Figure 5 Illustrations of the proof of Lemma 17. (a) The case when $(v, w) \notin \mathcal{A}$ because a shorter adjacent anchor (w, w') was added first. Edge (w', u') is a blue boundary edge and there is a white monotone path from u' to v in $\mathcal{S}$. (b) The case when $(v, w) \notin \mathcal{A}$ because a shorter adjacent anchor (v', v) was added first. Edge (w, u') is a blue boundary edge and (u', v) is a white boundary edge on the white side of its cone and there is a white monotone path between u' and v' in $\mathcal{S}$.

$d_\triangledown(w', u') \leq d_\triangledown(w, w') + \delta_\triangledown^{white}(v, w)$, we obtain the inequalities $d_\triangledown(w, w') \leq d_\triangledown(u, w)$, and $d_\triangledown(v, u') \leq d_\triangledown(u, w) - d_\triangledown(v, u)$, and $d_\triangledown(w', u') \leq 2d_\triangledown(u, w) - d_\triangledown(v, u)$. Then, following the above path from w to v and extending it to u using the canonical edges, we get $d_\mathcal{S}(u, w) \leq d_\triangledown(w, w') + 3(d_\triangledown(w', u') + d_\triangledown(v, u)) + 2(d_\triangledown(v, u') + d_\triangledown(v, u))$, which is then bounded by $d_\mathcal{S}(u, w) \leq d_\triangledown(u, w) + 6d_\triangledown(u, w) + 2d_\triangledown(u, w) = 9d_\triangledown(u, w)$.

In the case when (v, w) was not added in $\mathcal{A}$ because of an adjacent (white) anchor at v, (see Figure 5(b)) one can obtain the desired bounds using similar analysis. ◀

▶ **Lemma 18.** *For any crossed blue edge $(u, w) \in \mathcal{D}$, $d_\mathcal{S}(u, w) \leq 3d_\triangledown(u, w) + 9\delta_\triangledown^{\min}(u, w)$.*

Proof. Let (p, q) be a shortcut that crosses (u, w). Since this shortcut is in the blue cone, it must have been added in $\mathcal{S}$ replacing two white canonical edges incoming at u, namely (p, u) and (q, u). As p and q are endpoints of the shortcut (p, q), both blue edges (p, w) and (q, w) are uncrossed. Consequently, we have $d_\mathcal{S}(p, w) \leq 3d_\triangledown(p, w)$ and $d_\mathcal{S}(q, w) \leq 3d_\triangledown(q, w)$ by Lemma 13. Furthermore, by Lemmas 10 and 17 we know that both of the canonical edges (p, u) and (q, u) satisfy the inequalities $d_\mathcal{S}(p, u) \leq 9d_\triangledown(p, u)$ and $d_\mathcal{S}(q, u) \leq 9d_\triangledown(q, u)$. Since both of these canonical edges are incoming at u, we also know that one of them, say (p, u), is not longer than $\delta_\triangledown^{\min}(u, w)$, i.e., $d_\triangledown(p, u) \leq \delta_\triangledown^{\min}(u, w)$. Following the path from w to p to u, we bound $d_\mathcal{S}(u, w)$ by $3d_\triangledown(p, w) + 9d_\triangledown(p, u)$, which in turn is bounded by $3d_\triangledown(u, w) + 9\delta_\triangledown^{\min}(u, w)$ as stated in the lemma. ◀

Due to space constraints, we omit the technical proof of the following lemma on crossed white edges; the interested reader can find the proof in the full version of the paper.

▶ **Lemma 19.** *For any crossed white edge $(u, w) \in \mathcal{D}$, $d_\mathcal{S}(u, w) \leq 10d_\triangledown(u, w) + 10\delta_\triangledown^{\min}(u, w)$.*

By Lemmas 13, 17, 18, and 19 we have that $\mathcal{S}$ is a 20-spanner of $\mathcal{D}$. Since $\mathcal{D}$ is a 2-spanner of $\mathcal{C}$ ([11]) it follows that $\mathcal{S}$ is a 40-spanner of $\mathcal{C}$. We prove, however, a much better stretch factor upper bound of 20 next.

▶ **Lemma 20.** *For any two points $p, q \in \mathcal{P}$, $d_\mathcal{S}(p, q)$ is bounded by $20|pq|$.*

Proof. We prove the lemma by first constructing, in $\mathcal{D}$, a monotone path π between p and q that lies inside $\triangledown(p, q)$. We then consider the path π' in $\mathcal{S}$ obtained by replacing every edge of π not in $\mathcal{S}$ with a short path in $\mathcal{S}$.

We define the path π between p and q consisting of k edges in $\mathcal{D}$ using a sequence of pairs of points $\{p,q\} = \{p_0,q_0\},\{p_1,q_1\},\ldots,\{p_k,q_k\}$ such that any two consecutive pairs of points $\{p_{i-1},q_{i-1}\}$ and $\{p_i,q_i\}$ satisfy exactly one of the equations $p_i = p_{i-1}$ and $q_i = q_{i-1}$ and the equation that is not satisfied describes the i^{th} edge. If $p_i \neq p_{i-1}$, then the i^{th} edge is $(p_{i-1},p_i) \in \mathcal{D}$, otherwise the i^{th} edge is $(q_{i-1},q_i) \in \mathcal{D}$. We define this sequence recursively for the next pair of points $\{p_{i+1},q_{i+1}\}$ by first identifying which of the points p_i and q_i lie in the other's positive cone. If q_i lies in the positive cone of p_i, then we define $q_{i+1} = q_i$ and $p_{i+1} = r$ such that $(p_i,r) \in \mathcal{D}$, noting that by definition of $\mathcal{D}$, r is the unique such point in the positive cone of p_i that contains q_i. Otherwise, if p_i lies in a positive cone of q_i, then we define $p_{i+1} = p_i$ and $q_{i+1} = r'$ such that $(q_i,r') \in \mathcal{D}$. We stop when $p_k = q_k$.

We prove inductively that the aforementioned path π lies within $\triangledown(p,q)$. For the base case, clearly the path consisting of the only edge in the sequence $\{p_{k-1},q_{k-1}\},\{p_k,q_k\}$ lies within $\triangledown(p_{k-1},q_{k-1})$. For the inductive step, assuming that the path for the sequence $\{p_i,q_i\},\ldots,\{p_k,q_k\}$ lies within $\triangledown(p_i,q_i)$, we show that the path for the sequence $\{p_{i-1},q_{i-1}\},\{p_i,q_i\},\ldots,\{p_k,q_k\}$ lies within $\triangledown(p_{i-1},q_{i-1})$. First, we observe that in either case that the first edge is (p_{i-1},p_i) or (q_{i-1},q_i), it lies within $\triangledown(p_{i-1},q_{i-1})$ by definition. Finally, we observe that $\triangledown(p_i,q_i)$, hence the rest of the path lies within $\triangledown(p_{i-1},q_{i-1})$ as well. Therefore, we prove the inductive step.

We then prove that all of the edges of the form (p_i,p_{i+1}) lie in the same corresponding positive cones of their respective points p_i. More specifically, we prove by induction on the sequence of such edges $e_0 = (p = p_{i_0-1}, p_{i_0}), e_1 = (p_{i_0} = p_{i_1-1}, p_{i_1}), \ldots, e_\ell = (p_{i_{\ell-1}} = p_{i_\ell-1}, p_{i_\ell} = p_k)$, that $p_{i_0}, p_{i_1}, \ldots, p_{i_\ell}$ lie in the same corresponding cones of $p_{i_0-1}, p_{i_1-1}, \ldots, p_{i_\ell-1}$ respectively.

The base case follows trivially, and for the inductive step we assume for edges $e_0, e_1, \ldots, e_\lambda$ that $p_{i_0}, p_{i_1}, \ldots, p_{i_\lambda}$ lie in the same corresponding cones of $p_{i_0-1}, p_{i_1-1}, \ldots, p_{i_\lambda-1}$ respectively. For the inductive step we need to prove for the edge $e_{\lambda+1}$ that $p_{i_{\lambda+1}}$ lies in the same corresponding cone of $p_{i_{\lambda+1}-1} = p_{i_\lambda}$. We have already proven that the edge $e_{\lambda+1}$ lies within $\triangledown(p_{i_\lambda}, q_{i_\lambda})$. Also, by definition of $\mathcal{D}$, we know that $\triangledown(p_{i_\lambda-1}, p_{i_\lambda})$ is empty of points of $\mathcal{P}$ in its interior. Then we conclude the inductive proof by observing that the only positive cone that can possibly include $e_{\lambda+1}$ at p_{i_λ} is part of the same corresponding cone of p_{i_λ}. Having proven this critical property about this path, we denote it by π_p and refer to it as one of the two *branches*, where the other *branch* π_q is defined analogously using points $q_0, q_1, \ldots, q_k$. We conclude that the path π consisting of these two branches π_p and π_q is monotone.

Finally, we prove the claimed bound on the length of the path π' between p and q. Because the constants in Lemma 19 are the largest among Lemmas 13, 17, 18, and 19, the worst case happens when q lies in the white positive cone of p and π is white monotone. Letting z be the blue vertex of $\triangledown(p,q)$, by Lemma 6, the projections of all edges of π onto zp (resp. zq) do not overlap, are contained within $[zp]$ (resp. zq), $|zp| = d_\triangledown(p,q)$, and $|zq| = \delta_\triangledown^{blue}(p,q)$. In the worst case, each edge of π is crossed, and Lemma 19 applies to reconstruct each edge (s,t) of π. Therefore, the length of the path π' and thus $d_\mathcal{S}(p,q)$ can be upper bounded by $\sum_{(s,t)\in\pi} 10 d_\triangledown(s,t) + 10\delta_\triangledown^{\min}(s,t) \leq 10 \sum_{(s,t)\in\pi} d_\triangledown(s,t) + \delta_\triangledown^{blue}(s,t)$. It follows that $d_\mathcal{S}(p,q) \leq 10(|zp| + |zq|)$ and therefore that $d_\mathcal{S}(p,q) \leq 20|pq|$ as desired. ◀

▶ **Theorem 21.** *$\mathcal{S}$ is a plane spanner of $\mathcal{C}$ with maximum degree at most 4 and stretch factor at most 20 and $\mathcal{S}$ can be constructed in $\mathcal{O}(n \log n)$ time.*

Proof. The planarity, maximum degree, and stretch factor properties of $\mathcal{S}$ were proven in Lemmas 11, 12, and 20, respectively. The TD-Delaunay triangulation $\mathcal{D}$ can be constructed in $\mathcal{O}(n \log n)$ time [11]. Given $\mathcal{D}$ and the fact that it is plane, $\mathcal{S}$ can be constructed in $\mathcal{O}(n \log n)$ time: sorting the anchors takes $\mathcal{O}(n \log n)$ time, and adding edges to $\mathcal{S}$ can be done in $\mathcal{O}(n)$ time. ◀

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