Strongly Monotone Drawings of Planar Graphs*

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Abstract

A straight-line drawing of a graph is a *monotone drawing* if for each pair of vertices there is a path which is monotonically increasing in some direction, and it is called a *strongly monotone drawing* if the direction of monotonicity is given by the direction of the line segment connecting the two vertices.

We present algorithms to compute crossing-free strongly monotone drawings for some classes of planar graphs; namely, 3-connected planar graphs, outerplanar graphs, and 2-trees. The drawings of 3-connected planar graphs are based on primal-dual circle packings. Our drawings of outerplanar graphs depend on a new algorithm that constructs strongly monotone drawings of trees which are also convex. For irreducible trees, these drawings are strictly convex.

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1 Introduction

Finding a path between a source vertex and a target vertex is one of the most important tasks when data are given by a graph, c.f. Lee et al. [16]. This task may serve as criterion for rating the quality of a drawing of a graph. Consequently researchers addressed the question of how to visualize a graph such that finding a path between any pair of nodes is easy. A user study of Huang et al. [13] showed that, in performing path-finding tasks, the eyes follow edges that go in the direction of the target vertex. This empirical study triggered the research topic of finding drawings with presence of some kind of geodesic paths. Several formalizations for the notion of geodesic paths have been proposed, most notably the notion of strongly monotone paths. Related drawing requirements are studied under the titles of self-approaching drawings and greedy drawings [1, 20].

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Let $G = (V, E)$ be a graph. We say that a path P is *monotone with respect to* a direction (or vector) d if the orthogonal projections of the vertices of P on a line with direction d appear in the same order as in P. A straight-line drawing of G is called *monotone* if for each pair of vertices $u, v \in V$ there is a connecting path that is monotone with respect to some direction. To support the path-finding tasks it is useful to restrict the monotone direction for each path to the direction of the line segment connecting the source and the target vertex: a path $v_1 v_2 \dots v_k$ is called *strongly monotone* if it is monotone with respect to the vector $\overrightarrow{v_1 v_k}$. A straight-line drawing of G is called *strongly monotone* if each pair of vertices $u, v \in V$ is connected by a strongly monotone path.

If crossings are allowed, then any strongly monotone drawing of a spanning tree of G yields a strongly monotone drawing of G, this has been observed by Angelini et al. [2]. For this reason, strongly monotone drawings referred to in this paper are always crossing-free.

Related Work

In addition to (strongly) monotone drawings, there are several other drawing styles that support the path-finding task. The earliest studied is the concept of *greedy drawings*, introduced by Rao et al. [20]. In a greedy drawing, one can find a source–target path by iteratively selecting a neighbor that is closer to the target. Triangulations admit crossing free greedy drawings [7], and more generally 3-connected planar graphs have greedy drawings [17]. Trees with a vertex of degree at least 6 have no greedy drawing. Nöllenburg and Prutkin [18] gave a complete characterization of trees that admit a greedy drawing.

Greedy drawings can have some undesirable properties, e.g., a greedy path can look like a spiral around the target vertex. To get rid of this effect, Alamdari et al. [1] introduced a subclass of greedy drawings, so-called *self-approaching drawings* which require the existence of a source–target path such that for any point p on the path the distance to another point q is decreasing along the path. In greedy drawings this is only required for q being the target-vertex. These drawings are related to the concept of self-approaching curves [14]. Alamdari et al. provide a complete characterization of trees that admit a self-approaching drawing.

Even more restricted are *increasing-chord drawings*, which require that there always is a source–target path which is self-approaching in both directions. Nöllenburg et al. [19] proved that every triangulation has a (not necessarily planar) increasing-chord drawing and every planar 3-tree admits a planar increasing-chord drawing. Dehkordi et al. [6] studied the problem of connecting a given point set in the plane with an increasing-chord graph.

Monotone drawings were introduced by Angelini et al. [2] They showed that any n-vertex tree admits a monotone drawing on a grid of size $O(n^{1.6}) \times O(n^{1.6})$ or $O(n) \times O(n^2)$. They also showed that any 2-connected planar graph has a monotone drawing having exponential area. Kindermann et al. [15] improved the area bound for trees to $O(n^{1.5}) \times O(n^{1.5})$ even with the property that the drawings are convex. The area bound was further lowered to $O(n^{1.205}) \times O(n^{1.205})$ by He and He [10]. Recently, the same authors [9] further reduced the area bound to $O(n \log n) \times O(n \log n)$.

Hossain and Rahman [12] showed that every connected planar graph admits a monotone drawing on a grid of size $O(n) \times O(n^2)$. For 3-connected planar graphs, He and He [11] proved that the convex drawings on a grid of size $O(n) \times O(n)$, produced by the algorithm of Felsner [8], are monotone. For the fixed embedding setting, Angelini et al. [3] showed that every plane graph admits a monotone drawing with at most two bends per edge, and all 2-connected plane graphs and all outerplane graphs admit a straight-line monotone drawing.

Angelini et al. [2] also introduced the concept of *strong monotonicity* and gave an example of a drawing of a planar triangulation that is not strongly monotone. Kindermann et al. [15]

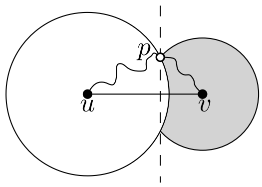


Figure 1 Any increasing-chord path is also strongly monotone.

showed that every tree admits a strongly monotone drawing. However, their drawing is not necessarily strictly convex and requires more than exponential area. Further, they presented an infinite class of 1-connected graphs that do not admit strongly monotone drawings. Nöllenburg et al. [19] have recently shown that exponential area is required for strongly monotone drawings of trees and binary cacti.

There are some relations among the aforementioned drawing styles. Plane increasing-chord drawings are self-approaching by definition, but they are also strongly monotone: consider an increasing-chord path from u to v and any point p on this path. By definition, by walking along this path, the distance to u increases and the distance to v decreases; hence, any point q on the walk from p to v lies outside the circle around u through p but inside the circle around v through p. Since these points all lie on the same side of the line through p with direction $\overrightarrow{uv}^{\perp}$ as v, the path is strongly monotone; see Figure 1. Self-approaching drawings are greedy by definition. On the other hand, (plane) self-approaching drawings are not necessarily monotone, and vice-versa.

Our Contribution

After giving some basic definitions used throughout the paper in Section 2, we present four results. First, we show that any 3-connected planar graph admits a strongly monotone drawing induced by primal-dual circle packings (Section 3). Then, we answer in the affirmative the open question of Kindermann et al. [15] on whether every tree has a strongly monotone drawing which is strictly convex. We use this result to show that every outerplanar graph admits a strongly monotone drawing (Section 4). Finally, we prove that 2-trees can be drawn strongly monotone (Section 5). All our proofs are constructive and admit efficient drawing algorithms. Our main open question is whether every planar 2-connected graph admits a strongly monotone drawing (Section 6). It would also be interesting to understand which graphs admit strongly monotone drawings on a grid of polynomial size.

2 Definitions

Let $G = (V, E)$ be a graph. A *drawing* Γ of G maps the vertices of G to distinct points in the plane and the edges of G to simple Jordan curves between their end-points. In a *straight-line* drawing, each edge is mapped to a straight line segment. A planar drawing induces a *combinatorial embedding* which is the class of topologically equivalent drawings. In particular, an embedding specifies the connected regions of the plane, called *faces*, whose boundary consists of a cyclic sequence of edges. The unbounded face is called the *outer face*, the other faces are called *internal faces*. For connected graphs, an embedding can also be defined by a *rotation system*, that is, the circular order of the incident edges around each vertex. Note that both definitions are equivalent for planar graphs.

A (straight-line) drawing of a planar graph is a *convex drawing* if it is crossing free and internal faces are realized as convex non-overlapping (polygonal) regions. The *augmentation* of a straight-line drawn tree is obtained by substituting each edge incident to a leaf by a ray which begins with the edge and extends across the leaf. A drawing of a tree is a *(strictly) convex drawing* if the augmented drawing is crossing free and has (strictly) convex faces, i.e., all the angles of the unbounded polygonal regions are less or equal to (strictly less than) π. Note that strict convexity forbids vertices of degree 2. We call a tree *irreducible* if it contains no vertices of degree 2. It has been observed before that a convex drawing of a tree is also monotone but a monotone drawing is not necessarily convex, see [2, 4].

A *k-tree* is a graph which can be produced from a complete graph K_{k+1} and then repeatedly adding vertices in such a way that the neighbors of the added vertex form a k-clique. We say that the new vertex is *stacked* on the clique. By construction k-trees are chordal graphs. They can also be characterized as maximal graphs with treewidth k, that is, no edges can be added without increasing the treewidth. Note that 1-trees are equivalent to trees and 2-trees are equivalent to maximal series-parallel graphs.

We denote an undirected edge between two vertices $a, b \in V$ by (a, b). In a drawing of G, we may identify each vertex with the point in the plane it is mapped to. For two vectors x and y, we define the angle $\measuredangle(x, y)$ as the smallest angle between the two vectors, that is, $\measuredangle(x, y) = \arccos\left(\frac{\langle x,y\rangle}{|x||y|}\right)$, and for three points p, q, r, we define $\measuredangle pqr = \measuredangle(\overrightarrow{qp}, \overrightarrow{qr})$. We say that a vector x is monotone with respect to y if $\measuredangle(x, y) < \pi/2$. This yields an alternative definition of a strongly monotone path: A path $v_1 v_2 \dots v_k$ is strongly monotone if $\measuredangle(\overrightarrow{v_i v_{i+1}}, \overrightarrow{v_1 v_k}) < \pi/2$, for $1 \le i \le k-1$. Note that we interpret monotonicity as strict monotonicity, i.e., we do not allow edges on the path that are orthogonal to the segment between the endpoints.

3 3-Connected Planar Graphs

In this section, we prove the following theorem.

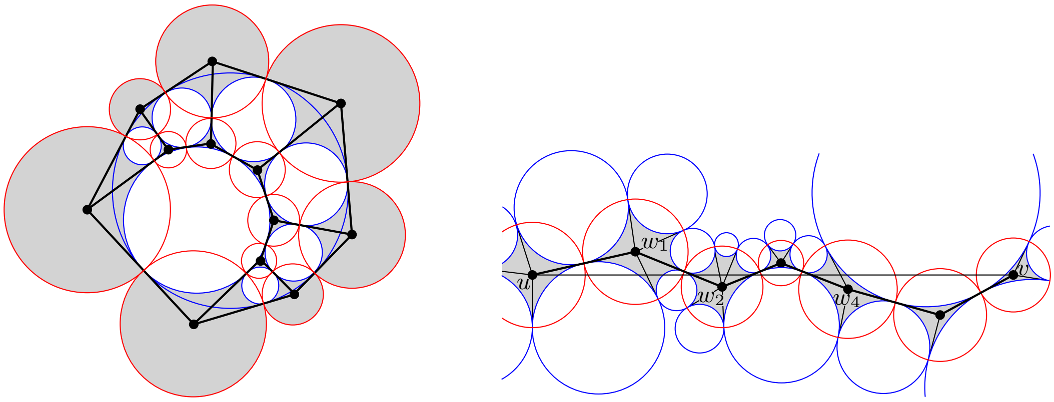
▶ **Theorem 1.** *Every 3-connected planar graph has a strongly monotone drawing.*

Proof. We show that the straight-line drawing corresponding to a primal-dual circle packing of a graph G is already strongly monotone. The theorem then follows from the fact that any 3-connected planar graph $G = (V, E)$ admits a primal-dual circle packing. This was shown by Brightwell and Scheinerman [5]; for a comprehensive treatment of circle packings we refer to Stephenson's book [21].

A *primal-dual circle packing* of a plane graph G consists of two families $\mathcal{C}_V$ and $\mathcal{C}_F$ of circles such that, there is a bijection $v \leftrightarrow C_v$ between the set V of vertices of G and circles of $\mathcal{C}_V$ and a bijection $f \leftrightarrow C_f$ between the set F of faces of G and circles of $\mathcal{C}_F$. Moreover, the following properties hold:

1. The circles in the family $\mathcal{C}_V$ are interiorly disjoint and their contact graph is G, i.e., $C_u \cap C_v \neq \emptyset$ if and only if $(u, v) \in E(G)$.
2. If $C_o \in \mathcal{C}_F$ is the circle of the outer face o, then the circles of $\mathcal{C}_F \setminus \{C_o\}$ are interiorly disjoint while C_o contains all of them. The contact graph of $\mathcal{C}_F$ is the dual G^* of G, i.e., $C_f \cap C_g \neq \emptyset$ if and only if $(f, g) \in E(G^*)$.
3. The circle packings C_V and C_F are orthogonal, i.e., if $e = (u, v)$ and the dual of e is $e^* = (f, g)$, then there is a point $p_e = C_u \cap C_v = C_f \cap C_g$; moreover, the common tangents t_{e^*} of C_u, C_v and t_e of C_f, C_g cross perpendicularly in p_e.

Let a primal-dual circle packing of a graph G be given. For each vertex v, let p_v be the center of the corresponding circle C_v. By placing each vertex v at p_v, we obtain a planar straight-line drawing Γ of G. In this drawing, the edge $e = (u, v)$ is represented by the



(a) **(b)**

Figure 2 (a) Drawing Γ of 3-connected graph $G = (V, E)$. Red circles are vertex circles $\mathcal{C}_V$, Blue circles are face circles $\mathcal{C}_F$. Regions of faces in white, regions of vertices in gray. (b) A strongly monotone path (thick edges) from u to v.

segment with end-points p_u and p_v on t_e. The face circles are inscribed circles of the faces of Γ; moreover, C_f is touching each boundary edge of the face f; see Figure 2a.

A straight-line drawing Γ^* of the dual G^* of G with the dual vertex of the outer face o at infinity can be obtained similarly by placing the dual vertex of each bounded face f at the center of the corresponding circle C_f. In this drawing, a dual edge $e^* = (f, o)$ is represented by the ray supported by t_{e^*} that starts at p_f and contains p_e.

In the following, we will make use of a specific partition Π of the plane. The regions of Π correspond to the vertices and the faces of G. For a vertex or face x, let D_x be the interior disk of C_x.

- The region R_f of a bounded face f is D_f.
- The region R_v of a vertex v is obtained from the disk D_v by removing the intersections with the disks of bounded faces, i.e., $R_v = D_v \setminus \bigcup_{f \neq o} R_f = D_v \setminus \bigcup_{f \neq o} D_f$; see Figure 2a.

To get a partition of the whole plane, we assign the complement of the already defined regions to the outer face, i.e, $R_o = \mathbb{R}^2 \setminus (\bigcup_{f \neq o} R_f \cup \bigcup_v R_v) = \mathbb{R}^2 \setminus (\bigcup_{f \neq o} D_f \cup \bigcup_v D_v)$.

Note that the edge-points p_e are part of the boundary of four regions of Π and if two regions of Π share more than one point on the boundary, then one of them is a vertex region R_v, the other is a face-region D_f, and (v, f) is an incident pair of G.

We are now prepared to prove the strong monotonicity of Γ. Consider two vertices u and v and let ℓ be the line spanned by p_u and p_v. W.l.o.g., assume that ℓ is horizontal and p_u lies left of p_v. Let ℓ_s be the directed segment from p_u to p_v. Since $p_u \in R_u$ and $p_v \in R_v$, the segment ℓ_s starts and ends in these regions. In between, the segment will traverse some other regions of Π. This is true unless (u, v) is an edge of G whence the strong monotonicity for the pair is trivial. We assume non-degeneracy in the following sense.

Non-degeneracy: The interior of the segment ℓ_s contains no vertex-point p_w, edge-point p_e, or face-point p_f.

Möbius transformations of the plane map circle packings to circle packings. In fact, the primal-dual circle packing of G is unique up to Möbius transformation; see Stephenson [21]. Now, any degenerate primal-dual circle packing of G can be mapped to a non-degenerate one by a Möbius transformation. This justifies the non-degeneracy assumption. Later we will give a more direct handling of degenerate situations.

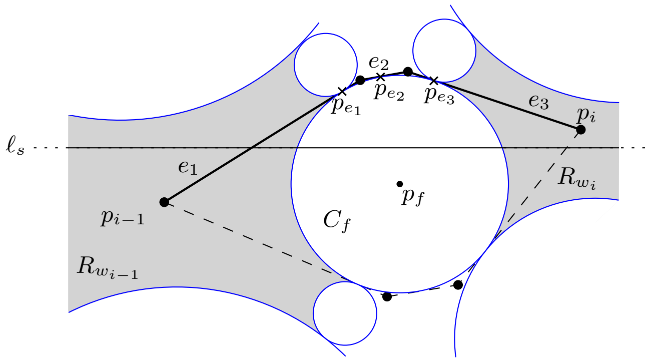


Figure 3 The path P_i connecting p_{i-1} and p_i.

Let $u = w_0, w_1, \ldots, w_k = v$ be the sequence of vertices whose region is intersected by ℓ_s, in the order of intersection from left to right, and let $p_i = p_{w_i}$; see Figure 2b. We will construct a strongly monotone path P from p_u to p_v in Γ that contains $p_u = p_0, p_1, \ldots, p_k = p_v$ in this order. Let P_i be the subpath of P from p_{i-1} to p_i. Since ℓ_s may revisit a vertex-region, it is possible that $p_{i-1} = p_i$; in this case we set $P_i = p_i$. Now suppose that $p_{i-1} \neq p_i$. Non-degeneracy implies that the segment ℓ_s alternates between vertex-regions and face-regions; hence, a unique disk D_f is intersected by ℓ_s between the regions of w_{i-1} and w_i. It follows that w_{i-1} and w_i are vertices on the boundary of f. The boundary of f contains two paths from w_{i-1} to w_i. In Γ, one of these two paths from p_{i-1} to p_i lies above D_f, we call it the *upper path*; the other one lies below D_f, we call it the *lower path*. If the center p_f of D_f lies below ℓ, then we choose the upper path from p_{i-1} to p_i as P_i; otherwise, we choose the lower path.

Suppose that this rule led to the choice of the upper path; see Figure 3. The case that the lower path was chosen works analogously. We have to show that P_i is monotone with respect to ℓ, i.e., to the x-axis. Let $e_1, \ldots, e_r$ be the edges of this path and let $e_j = (q_{j-1}, q_j)$; in particular, $q_0 = p_{i-1}$ and $q_r = p_i$. Since $R_{w_{i-1}}$ is star-shaped with center p_{i-1}, the segment connecting p_{i-1} with the first intersection point of ℓ with C_f belongs to $R_{w_{i-1}}$. Therefore, the point p_{e_1} of tangency of edge e_1 at C_f lies above ℓ. Similarly, p_{e_r} and, hence, all the points p_{e_j} lie above ℓ. Since the points $p_{e_1}, \ldots, p_{e_r}$ appear in this order on C_f and the center of C_f lies below ℓ, we obtain that their x-coordinates are increasing in this order. This sequence is interleaved with the x-coordinates of $q_0, q_1, \ldots, q_r$; hence, this is also monotone. This proves that the chosen path P_i is monotone with respect to ℓ. Monotonicity also holds for the concatenation $P = P_1 + P_2 + \ldots + P_k$; see Figure 2b.

We have shown strong monotonicity under the non-degeneracy assumption. Next, we consider degenerate cases and show how to find strongly monotone paths in these cases.

If ℓ_s contains a vertex-point p_w with $w \neq u, v$, the path P between u and v is just the concatenation of monotone paths between the pairs u, w and w, v; hence, it is strongly monotone. Next suppose that ℓ_s contains an edge-point p_e. If the edge e in Γ is horizontal, then we also have two vertex-points on ℓ_s and are in the case described above; otherwise, we consider the region which is touching ℓ from above as intersecting and the region which is touching ℓ from below as non-intersecting. This recovers the property that there is an alternation between vertex-regions and face-regions intersected by ℓ_s. Hence, the definition of the path for u and v gives a strongly monotone path unless it contains a vertical edge. The use of a vertical edge can be excluded by properly adjusting degeneracies of the form $p_f \in \ell$. For faces f with $p_f \in \ell$, we use the upper path, i.e., we consider p_f to be below ℓ. Thus, even in degenerate situations the drawing corresponding to a primal-dual circle packing is strongly monotone. This concludes the proof. ◀

4 Trees and Outerplanar Graphs

Kindermann et al. [15] have shown that any tree has a strongly monotone drawing and that any irreducible binary tree has a strictly convex strongly monotone drawing. They left as an open question whether every tree admits a convex strongly monotone drawing; noticing that, in the positive case, this would imply that every Halin graph has a convex strongly monotone drawing.

In this section, we show that every tree has a convex strongly monotone drawing. Moreover, if the tree is irreducible, then the drawing is strictly convex. We use the result on trees to prove that every outerplanar graphs admits a strongly monotone drawing.

▶ **Theorem 2.** *Every tree has a convex strongly monotone drawing. If the tree is irreducible, then the drawing is strictly convex.*

Proof. We actually prove something stronger, namely, that any tree T has a drawing Γ with the following properties:

I1. Every leaf of T is placed on a corner of the convex hull of the vertices in Γ.

I2. If $a_1, \dots, a_\ell$ is the counterclockwise order of the leaves on the convex hull, then for $i = 1, \dots, \ell$ the vectors $(\overrightarrow{a_i a_{i-1}})^\perp$, $\overrightarrow{p_i, a_i}$, $(\overrightarrow{a_{i+1} a_i})^\perp$ appear in counterclockwise radial order, where p_i denotes the unique vertex adjacent to a_i (see Figure 4a).

I3. The angle between two consecutive edges incident to a vertex $v \in V(T)$ is at most π and is equal to π only when v has degree two.

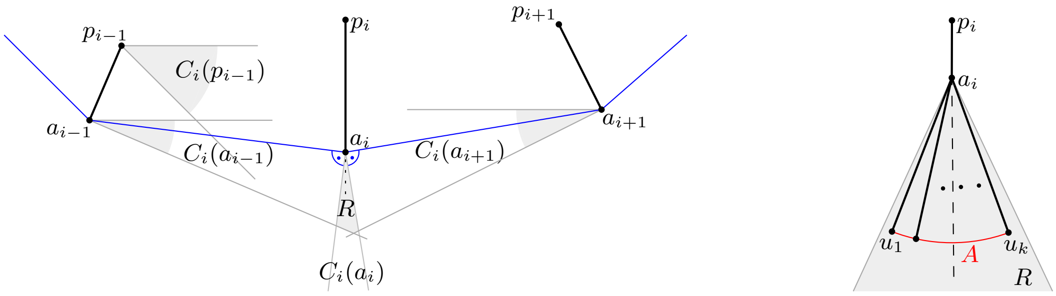
I4. Γ is strongly monotone.

Let T be a tree on at least 3 vertices, rooted at some vertex v_0 with degree at least 2. We inductively produce a drawing of T. We begin with placing the root v_0 at any point in the plane and the children $u_1, \dots, u_k$ of v_0 at the corners of a regular k-gon with center v_0. The resulting drawing clearly fulfills the four desired properties.

Let T^- be a subtree of T such that T^- has at least one leaf a_i that is not a leaf in T. Let Γ^- be a drawing of T^- that fulfills the properties I1–I4. Let $u_1, \dots, u_k$ be the children of a_i in T. Let T^+ denote the subtree of T induced by $V(T^-) \cup \{u_1, \dots, u_k\}$. In the inductive step, we explain how to extend the drawing Γ^- of T^- to a drawing Γ^+ of T^+ such that it fulfills the properties I1–I4.

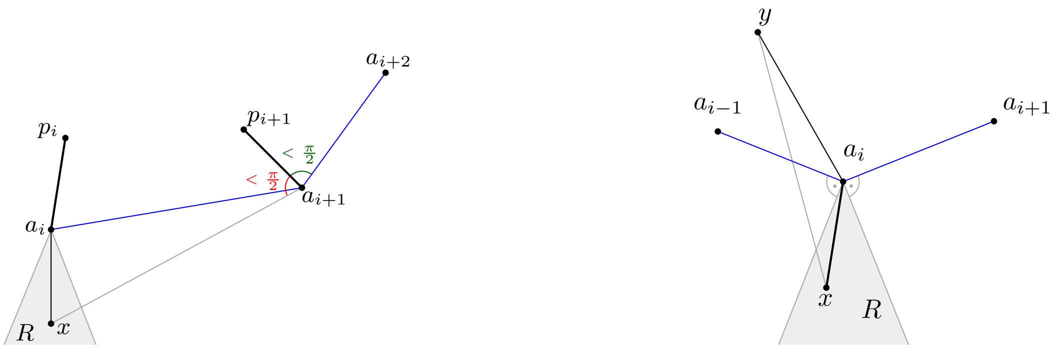
We first define a region R which is appropriate for the placement of $u_1, \dots, u_k$; see Figure 4a for an illustration. Let $C_i(a_i)$ be the open cone containing all points x such that the vectors $(\overrightarrow{a_i a_{i-1}})^\perp$, $\overrightarrow{a_i x}$, and $(\overrightarrow{a_{i+1} a_i})^\perp$ are ordered counterclockwise. From property I2, it follows that $C_i(a_i)$ contains the *prolongation* h_{p_i, a_i} of $\overrightarrow{p_i a_i}$, i.e., the ray that starts with $\overrightarrow{p_i a_i}$ and extends across a_i. For every vertex $y \neq a_i$ of T^-, let $C_i(y)$ be the open cone consisting of all points p such that the path from y to a_i in T^- is monotone with respect to $\overrightarrow{yp}$. Since the drawing Γ^- is strongly monotone in a strict sense, $C_i(y)$ contains an open disk centered at a_i. Therefore, the intersection $\cap_{y \in V(T^-) \setminus \{a_i\}} C_i(y)$ of the all these cones contains an open disk D centered at a_i. We define the region R to be the intersection of all cones $C_i(y)$ and the cone $C_i(a_i)$, i.e., $R = \cap_{v \in V(T^-)} C_i(v)$. The intersection of disk D with $C_i(a_i)$ yields an open 'pizza slice' contained in R. In particular, R is non-empty.

Since R is an open convex set, we can construct a circular arc A in R with center a_i that contains points on both sides of the prolongation h_{p_i, a_i} of $\overrightarrow{p_i a_i}$; see Figure 4b. We place the vertices $u_1, \dots, u_k$ on the arc A between the intersection of the tangent to A



(a) **(b)**

Figure 4 (a) The region R which is used for placing all the children of vertex a_i. The boundary of the convex hull is drawn blue. (b) Placement of the children $u_1, \ldots, u_k$ on the arc $A \subset R$. The prolongation h_{p_i,a_i} is drawn dashed, the arc A is drawn red.



(a) **(b)**

Figure 5 (a) An illustration for the proof of property I1 and property I2. (b) An illustration of the case where $y \in V(T^-)$ and $x \in \{u_1, \ldots, u_k\}$.

through a_{i-1} (if it exists) and the intersection of the tangent to A through a_{i+1} (if it exists) such that $\measuredangle p_i a_i u_1 = \measuredangle u_k a_i p_i$. This placement implies that in case a_i has degree 2 (that is, $k = 1$), $\measuredangle p_i a_i u_1 = \measuredangle u_1 a_i p_i = \pi$, and otherwise all the angles $\measuredangle p_i a_i u_1$, $\measuredangle u_k a_i p_i$, $\measuredangle u_j a_i u_{j+1}$, for $j = 1, \ldots, k-1$, are all less than π. This ensures property I3.

Next, we prove that the drawing Γ^+ of T^+ fulfills property I1. We first show that a_{i-1} and a_{i+1} lie on the convex hull of Γ^+; see Figure 5a. Consider the path from a_{i+1} to a_i in T^-, and let x be a point in R. By definition of R, this path is monotone (in a strict sense) with respect to $\overrightarrow{a_i x}$; therefore, $\measuredangle p_{i+1} a_{i+1} x < \pi/2$. Considering the strictly monotone path from a_{i+2} to a_{i+1} in T^- we obtain that $\measuredangle a_{i+2} a_{i+1} p_{i+1} < \pi/2$. The two inequalities above sum up to $\measuredangle x a_{i+1} a_{i+2} < \pi$ which means that a_{i+1} lies on the convex hull of Γ^+. Analogously, we obtain that a_{i-1} lies on the convex hull of Γ^+.

Notice that at least one of $u_1, \ldots, u_k$ lies on the convex hull of Γ^+ since they are placed outside of the convex hull of Γ^-. On the other hand, the construction of the circular arc A and the placement between the intersection points of the tangents through a_{i-1} and a_{i+1} ensures that $\measuredangle u_2 u_1 a_{i-1} < \pi$ and $\measuredangle a_{i+1} u_k u_{k-1} < \pi$. Hence, all of them lie on the convex hull of Γ^+. This ensures property I1.

For property I2, observe that $\measuredangle x a_i a_{i+1} > \pi/2$ holds for every $x \in C_i(a_i)$ (see Figure 5a), and therefore $\measuredangle a_{i+1} x a_i < \pi/2$, as these two angles lie in the triangle $\triangle x a_i a_{i+1}$. The last inequality implies property I2 for Γ^+.

Finally, we show that property I4 holds, i.e., that Γ^+ is a strongly monotone drawing. Consider $x, y \in V(T^+)$, let P_{xy} denote the path between x and y in T^+. We distinguish the following three cases:

1. If $x, y \in V(T^-)$, then the path P_{xy} is contained in T^-. Since Γ^- is a strongly monotone drawing by induction hypothesis, P_{xy} is strongly monotone.
2. If $y \in V(T^-)$ and $x \in \{u_1, \ldots, u_k\}$, then $P_{yx} = P_{ya_i} + (a_i, x)$; refer to Figure 5b. The path P_{ya_i} is monotone with respect to $\overrightarrow{yx}$ by construction because $x \in A \subset R \subset C_i(x)$. The definition of R also implies that $\measuredangle a_{i-1}a_i x$ and $\measuredangle a_{i+1}a_i x$ are greater than $\pi/2$. Since y lies inside the convex hull of Γ^-, the smallest angle $\measuredangle y a_i x$ is also greater than $\pi/2$. Thus, $\measuredangle a_i x y < \pi/2$ which implies that the vector $\overrightarrow{xa_i}$ is monotone with respect to $\overrightarrow{xy}$. We conclude that P_{xy} is strongly monotone.
3. If $x, y \in \{u_1, \ldots, u_k\}$, then the path $P_{xy} = (x, a_i) + (a_i, y)$ is strongly monotone since x and y are placed on the circular arc A centered at a_i.

We have proven that each tree has a drawing that fulfills the four properties I1–I4. Property I2 implies that the prolongations of the edges incident to the leaves do not intersect. This, together with property I3, implies the convexity of the drawing and strict convexity in case of an irreducible tree. This concludes the proof of the theorem. ◀

▶ **Theorem 3.** *Every outerplanar graph has a convex strongly monotone drawing.*

Proof. Let G be an outerplanar graph with at least 2 vertices. For every vertex $v \in V$, we add two dummy vertices v', v'' and edges (v, v'), (v, v''). By construction, the resulting graph H is outerplanar and does not contain vertices of degree 2. Let Γ_H be an outerplanar drawing of H. We will construct a convex strongly monotone drawing Γ'_H of H with the same combinatorial embedding as Γ_H.

Let T be an arbitrary spanning tree of H. By construction, no vertex in T has degree 2. Thus, according to Theorem 2, T admits a strongly monotone drawing Γ_T which is strictly convex and which also preserves the order of the children for every vertex, i.e., the rotation system coincides with the one in Γ_H.

Now, we insert all the missing edges. Recall that, by removing an edge from a planar drawing, the two adjacent faces are merged. Since the drawing Γ_T of T is strictly convex and since Γ_T preserves the rotation system of Γ_H, by inserting an edge $e = (u, v)$ of the graph H into Γ_T one strictly convex face is partitioned into two strictly convex faces. Note that vertices u, v have to be incident to the same strictly convex face. Otherwise let C be the cycle contained in $E(T) \cup \{e\}$. If u and v are not incident to the same strictly convex face of Γ_T, there exists a leaf of T that is contained in the interior of C in Γ_H contradicting to the outerplanarity of H. Furthermore, the insertion of edge e does not destroy strong monotonicity. We re-insert all edges of H iteratively. The resulting drawing Γ'_H of H is a strictly convex and strongly monotone.

Finally, we remove all the dummy vertices and obtain a strongly monotone drawing of G. Since Γ'_H has the same combinatorial embedding as Γ_H, every dummy vertex lies in the outer face. Hence, no internal face is affected by the removal of dummy vertices, and thus all internal faces remain strictly convex. ◀

5 2-Trees

In this section, we show how to construct a strongly monotone drawing for any 2-tree. We begin by introducing some notation. A *drawing with bubbles* of a graph $G = (V, E)$ is a straight-line drawing of G in the plane such that, for some $E' \subseteq E$, every edge $e \in E'$ is

(a) **(b)**

Figure 6 (a) A drawing of a 2-tree with bubbles (orange) and (b) an extension of the drawing.

associated with a circular region in the plane, called a *bubble* B_e; see Figure 6a. An *extension* of a drawing with bubbles is a straight-line drawing that is obtained by taking some subset of edges with bubbles $E'' \subseteq E'$ and stacking one vertex on top of each edge $e \in E''$ into the corresponding bubble B_e; see Figure 6b. (Since every bubble is associated with a unique edge we often simply say that a vertex is stacked into a bubble without mentioning the corresponding edge.) We call a drawing with bubbles Γ strongly monotone if *every* extension of Γ is strongly monotone. Note that this implies that if a vertex w is stacked on top of edge e into bubble B_e, then there exists a strongly monotone path from w to any other vertex in the drawing and, furthermore, there exists a strongly monotone path from w to any of the current bubbles, i.e., to any vertex that might be stacked into another bubble.

Every 2-tree $T = (V, E)$ can be constructed through the following iterative procedure:

1. We start with one edge and tag it as *active*. During the entire procedure, every present edge is tagged either as active or *inactive*.
2. As an iterative step we pick one active edge e and stack vertices $w_1, \ldots, w_k$ on top of this edge for some $k \geq 0$ (we note that k might equal 0). Edge e is then tagged as inactive and all new edges incident to the stacked vertices $w_1, \ldots, w_k$ are tagged as active.
3. If there are active edges remaining, repeat Step 2.

Observe that Step 2 is performed exactly once per edge and that an according decomposition for T can always be found by the definition of 2-trees.

We construct a strongly monotone drawing of T by geometrically implementing the iterative procedure described above, so that after every step of the algorithm the present part of the graph is realized as a drawing with bubbles. We use the following additional geometrical condition:

(C) After each step of the algorithm every active edge comes with a bubble and the drawing with bubbles is strongly monotone. Additionaly, for an edge $e = (u, v)$ with bubble B_e for each point $w \in B_e$, the angle $\measuredangle(\overrightarrow{uw}, \overrightarrow{wv})$ is obtuse.

In Step 1, we arbitrarily draw the edge e_0 in the plane. Clearly, it is possible to define a bubble for e_0 that only allows obtuse angles. In Step 2, we place the vertices $w_1, \ldots, w_k$ over an edge $e = (u, v)$ as follows. The fact that stacking a vertex into B_e gives an obtuse angle allows us to place the to-be stacked vertices $w_1, \ldots w_k$ in B_e on a circular arc around u such that, for any $1 \leq i, j \leq k$, there exists a strongly monotone path between w_i and w_j; see Figure 7a. Due to condition 3, there also exists a strongly monotone path between any of the newly stacked vertices and any vertex of an extension of the previous drawing with bubbles. Hence, after removing the bubble B_e, the resulting drawing is a strongly monotone drawing with bubbles.

In order to maintain condition 3, it remains to describe how to define the bubbles for the new active edges incident to the stacked vertices. For this purpose, we state the following Lemma 4, which enables us to define the two bubbles for the edges incident to any degree-2 vertex with an obtuse angle. The Lemma is then iteratively applied to the vertices $w_1, \ldots, w_k$ and after every usage of the Lemma the produced drawing with bubbles is strongly monotone.

(a) Stacking vertices into a bubble

(b) The empty neighbourhood $\mathcal{N}$ (dotted)

Figure 7 Illustrations for the drawing approach for strongly monotone 2-trees.

This iterative approach is used to ensure that, when defining bubbles for some vertex w_i, the previously added bubbles for $w_1, \dots, w_{i-1}$ are taken into account.

▶ **Lemma 4.** *Let Γ be a strongly monotone drawing with bubbles and let w be a vertex of degree 2 with an obtuse angle such that the two incident edges $e_1 = (u, w)$ and $e_2 = (v, w)$ have no bubbles. Then, there exist bubbles B_{e_1} and B_{e_2} for edges e_1 and e_2 respectively that only allow obtuse angles such that Γ remains strongly monotone with bubbles if we add B_{e_1} and B_{e_2}.*

Proof. We begin by describing how we determine the size and location of the new bubbles. Since Γ is planar, there exists a neighborhood $\mathcal{N}$ of w, e_1 and e_2 that does not contain elements of any extension of Γ; see Figure 7b.

Furthermore, consider any extension Γ' of Γ. Since we consider monotonicity in a strict fashion, there exists a constant $\alpha > 0$ such that, for any pair of vertices s_0, s_t of Γ', there exists a strongly monotone path $P = (s_0, \dots, s_t)$ with $\measuredangle(\overrightarrow{s_0 s_t}, \overrightarrow{s_i s_{i+1}}) \leq \pi/2 - \alpha$ for $i = 0, \dots, t-1$. We refer to this property of P as being *α-safe* with respect to $\overrightarrow{s_o s_t}$. A simple compactness argument shows that this safety parameter can be chosen simultaneously for all the extensions of Γ: there exists $\alpha(\Gamma) > 0$ such that for every extension Γ' of Γ for every two vertices s_0, s_t of Γ' there exists a strongly monotone path connecting these vertices that is $\alpha(\Gamma)$-safe with respect to $\overrightarrow{s_o s_t}$. More precisely, for a fixed extension Γ', let $\alpha(\Gamma')$ denote the *maximal* α such that for any pair of vertices s_0, s_t of Γ' there exists a strongly monotone path connecting them which is α-safe with respect to $\overrightarrow{s_o s_t}$. Then we can chose the global safety parameter as $\alpha(\Gamma) := \inf_{\Gamma'} \alpha(\Gamma')$, where the infimum is taken over all the extensions Γ' of Γ. This infimum is strictly positive since the set of extensions is compact and the function $\alpha(\Gamma')$ is continuous and strictly positive.

For the edge e_1, we define the bubble B_{e_1} as the circle of radius r with center at the extension of the edge e_2 over w with distance ε to w as depicted in Figure 8a. In order to ensure the strong monotonicity, we choose r and ε such that the following properties hold (these properties clearly hold as soon as r, ε and r/ε are small enough):

(i) Bubble B_{e_1} is located inside the empty neighborhood $\mathcal{N}$. Moreover, to preserve obtusity, B_{e_1} needs to lie inside the semicircle with edge e_1 as diameter, as depicted in Figure 8a.

(ii) Consider angles β_1 and β_2 as illustrated in Figure 8b. We require that both angles are smaller than $\alpha(\Gamma)/4$.

(iii) For any vertex y of any extension of Γ, consider the angle β_y as illustrated in Figure 8c. We require that this angle is smaller than $\alpha(\Gamma)/4$. That guarantees that for any point $x \in B_{e_1}$ it holds that $\measuredangle(\overrightarrow{yw}, \overrightarrow{yx}) < \alpha(\Gamma)/4$.

We define the bubble B_{e_2} for the edge e_2 analogously with B_{e_1}. Moreover, we can use the same pair of parameters r and ε for B_{e_1} and B_{e_2}.

For the strong monotonicity of the drawing Γ with two new bubbles B_{e_1} and B_{e_2} we have to show two conditions: (1) that from any vertex stacked into one of the new bubbles there exists a strongly monotone path to any vertex y of any extension of Γ and (2) that there exists a strongly monotone path between any vertex stacked into B_{e_1} and any vertex stacked into B_{e_2}.

Since we use the same pair of r and ε for defining B_{e_1} and B_{e_2}, the condition (2) clearly holds as soon as r/ε is small enough. Thus we are left with ensuring that the condition (1) holds.

Consider the new bubble B_{e_1}, a point $x \in B_{e_1}$ and any vertex y of any extension Γ' of Γ. Since the drawing Γ' is strongly monotone, there exists a strongly monotone path P_{yw} in Γ' between y and w, furthermore by definition of α-safety we can choose P_{yw} being $\alpha(\Gamma)$-safe. Since w has only two incident edges in Γ', the last edge of the path P_{yw} is either $e_1 = (u, w)$ or $e_2 = (v, w)$. We distinguish between these two cases: in the first case we construct a path P_{yx} from y to x by re-routing the last edge of P_{yw} from (u, w) to (u, x) as illustrated in Figure 9a; in the second case we construct a path P_{yx} by appending the edge (w, x) to the end of P_{yw} as illustrated in Figure 9b;

It remains to show that P_{yx} is strongly monotone. First, observe that P_{yw} is strongly monotone and $\alpha(\Gamma)$-safe. By property 2, the final edges e_w of P_{yw} and e_x of P_{yx} satisfy $\measuredangle(\overrightarrow{e_w}, \overrightarrow{e_x}) < \alpha(\Gamma)/4$ and all other edges of these paths are identical. Thus, P_{yx} is $(3\alpha(\Gamma)/4)$-safe with respect to $\overrightarrow{yw}$. By property 3 $\measuredangle(\overrightarrow{yw}, \overrightarrow{yx}) < \alpha(\Gamma)/4$ and, therefore, P_{yx} is $(\alpha(\Gamma)/2)$-safe with respect to $\overrightarrow{yx}$ and thus in particular it is strongly monotone.

The arguments for a vertex stacked on e_2 into B_{e_2} are identical. ◀

Thus, we obtain the main result of this section:

▶ **Theorem 5.** *Every 2-tree admits a strongly monotone drawing.*

6 Conclusion

We have shown that any 3-connected planar graph, tree, outerplanar graph, and 2-tree admits a strongly monotone drawing. All our drawings require exponential area. For trees, this area bound has been proven to be required; however, it remains open whether the other graph classes can be drawn in polynomial area. Further, the question whether any 2-connected planar graph admits a strongly monotone drawing remains open. Last but not least, we could observe (using a computer-assisted search) that 2-connected graphs with at most 9 vertices admit a strongly monotone drawing, while there is exactly one connected graph with 7 vertices that is the smallest graph not admitting a strongly monotone drawing; see Figure 10.

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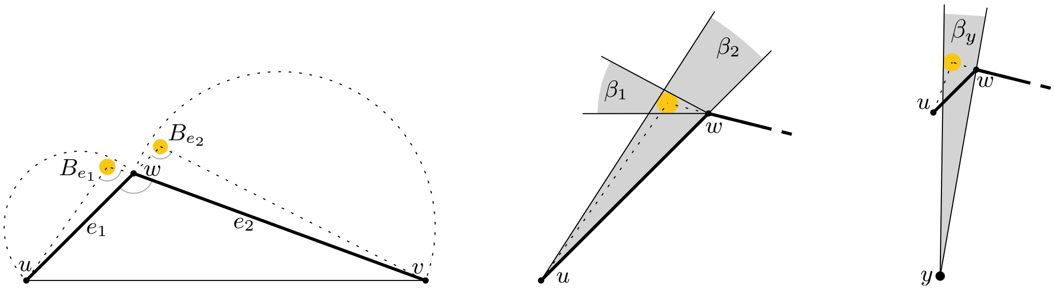
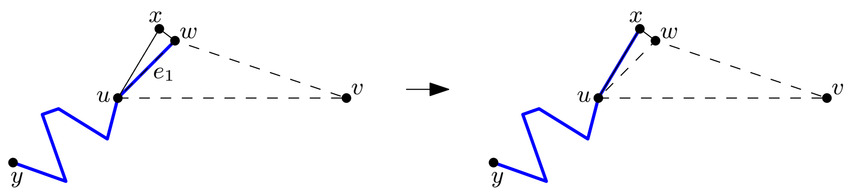
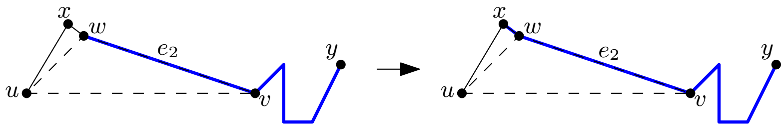


(a) Semi circles around e_1 and e_2 **(b)** Cones centered at u and w **(c)** Cone centered at y

Figure 8 Illustrations for the placement of the new bubbles B_{e_1} and B_{e_2}.



(a) Rerouting in case the last edge of P_{yw} is e_1.



(b) Rerouting in case the last edge of P_{yw} is e_2.

Figure 9 A strongly monotone path P_{yw} from y to w is re-routed to x. Two possible cases are distinguished: The last edge of P_{yw} is either e_1 or e_2.

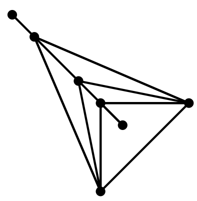


Figure 10 The unique connected 7-vertex graph without a strongly monotone drawing.

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