On Weighted Bipartite Edge Coloring

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Abstract

We study WEIGHTED BIPARTITE EDGE COLORING problem, which is a generalization of two classical problems: BIN PACKING and EDGE COLORING. This problem has been inspired from the study of Clos networks in multirate switching environment in communication networks. In WEIGHTED BIPARTITE EDGE COLORING problem, we are given an edge-weighted bipartite multigraph $G = (V, E)$ with weights $w : E \to [0, 1]$. The goal is to find a *proper weighted coloring* of the edges with as few colors as possible. An edge coloring of the weighted graph is called a *proper weighted coloring* if the sum of the weights of the edges incident to a vertex of any color is at most one. Chung and Ross conjectured $2m - 1$ colors are sufficient for a proper weighted coloring, where m denotes the minimum number of unit sized bins needed to pack the weights of all edges incident at any vertex. We give an algorithm that returns a coloring with at most $\lceil 2.2223m \rceil$ colors improving on the previous result of $\frac{9m}{4}$ by Feige and Singh. Our algorithm is purely combinatorial and combines the König's theorem for edge coloring bipartite graphs and first-fit decreasing heuristic for bin packing. However, our analysis uses *configuration linear program* for the bin packing problem to give the improved result.

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1 Introduction

Clos networks were introduced by Clos [3] in the context of designing interconnection networks with small number of links to route multiple simultaneous connection requests such as telephone calls. Since then it has found various applications in data communications and parallel computing systems [1, 11]. The symmetric 3-stage Clos network is generally considered to be the most basic multistage interconnection network. Let $C(m, \mu, r)$ denote a symmetric 3-stage Clos network, where the input (first) stage consists of r crossbars (switches) of size $m \times \mu$, the center (second) stage consists of μ crossbars of size $r \times r$ and the output (third) stage consists of r crossbars of size $\mu \times m$. Moreover, there exists one link between every center switch and each of the r input or output switch. No link exists between other pair of switches.

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A request frame is a collection of connection requests between inlets and outlets in the network such that each inlet or outlet is associated with at most one request. A request frame is *routable* if all requests are routed through a middle switch such that no two requests share same link. An interconnection network is called to be *rearrangeably nonblocking* if all request frames are routable. In the classic switching environment all connection requests fully use a link and all have same bandwidth. However in modern networks, different requests might have different bandwidths (due to wide range of traffic such as voice, video, facsimile etc.) and may be combined in a given link if the combined request does not exceed the link capacity. In this *multirate* setting, a connection request is a triple (i, j, w) where i, j, w are inlet, outlet and demand of the connection respectively, and all links have capacity one. Here a request frame is a collection of connection requests between inlets and outlets in the network such that the total weight of all requests in the frame for any particular inlet or outlet is at most one. The central question in 3-stage Clos networks is finding the minimum number of crossbars μ $(= \mu(m, r))$ in the middle stage such that all request frames are routable. It is more interesting to obtain bounds independent of r.

Nonblocking rearrangeable properties of 3-stage Clos network $C(m, \mu, r)$ can be translated to the following graph theoretic problem. Formally, in WEIGHTED BIPARTITE EDGE COLORING problem, we are given an edge-weighted bipartite (multi)-graph $G := (V, E)$ with bipartitions A, B $(|A| = |B| = r)$ and edge weights $w : E \to [0, 1]$. Let w_e denote the weight of edge $e \in E$. The goal is to obtain a *proper weighted coloring* of all edges with minimum number of colors. An edge coloring of the weighted bipartite graph is called a *proper weighted coloring* if the sum of the same color edges incident to a vertex is at most one for any color and any vertex. Here the sets A and B correspond to the input and output switches, edge (u, v) corresponds to a request between input switch u and output switch v. A routable request frame translates into the condition that weights of all incident edges to any vertex can be proper weighted colored using m colors (or packed into m unit sized bins) and the switches in the middle stage correspond to the colors (or bins). We refer the reader to Correa and Goemans [5] for detailed discussion of this reduction.

The weighted bipartite edge coloring problem naturally generalizes two classically studied problems, the EDGE COLORING PROBLEM and the BIN PACKING PROBLEM. If all edge weights are one, this problem reduces to the classical edge coloring problem. On the other hand, if there is only one vertex in each partition in the bipartite graph, this problem reduces to the bin packing problem.

Now let us introduce some notation. Let $\chi'_w(G)$ denote the minimum number of colors needed to obtain a proper weighted coloring of G. Let $m, r \in \mathbb{Z}^+$, and $\mu(m, r) = max_G \chi'_w(G)$ where the maximum is taken over all bipartite graphs $G = (A \cup B, E)$ with $|A| = |B| = r$ and where m is the maximum over all the vertices of the number of unit-sized bins needed to pack the weights of incident edges. Chung and Ross [2] made the following conjecture.

▶ **Conjecture 1.** *Given an instance of the* WEIGHTED BIPARTITE EDGE COLORING *problem, there is a proper weighted coloring using at most* $2m - 1$ *colors where* m *denotes the maximum over all the vertices of the number of unit-sized bins needed to pack the weights of edges incident at the vertex. In other words,* $\mu(m, r) \leq 2m - 1$.

There has been a series of results achieving weaker bounds on $\mu(m, r)$ (see related works for details), the current best bound by Feige and Singh [7] shows that $\mu(m, r) \leq 2.25m$.

1.1 Our results and techniques.

Our main result is to make progress towards resolution of Conjecture 1 by showing $\mu(m,r) \leq \frac{20m}{9} + o(m)$.

▶ **Theorem 2.** *There is a polynomial time algorithm for the* WEIGHTED BIPARTITE EDGE COLORING *problem which returns a proper weighted coloring using at most* $\lceil 2.2223m \rceil$ *colors where* m *denotes the maximum over all the vertices of the number of unit-sized bins needed to pack the weights of incident edges.*

In our algorithm and analysis, we exploit that WEIGHTED BIPARTITE EDGE COLORING problem displays features of the classical edge coloring problem as well as the bin packing problem. Our algorithm starts by decomposing the *heavy* weight edges into matchings by applying König's theorem to find an edge coloring of the subgraph induced by these edges. For the light weight edges, we employ the *first-fit decreasing* heuristic where we consider the remaining edges in decreasing order of weight and give them the first available color. The detailed algorithm is given in Figure 1 and builds on the algorithm by Feige and Singh [7].

Our work diverges from previous results on this problem in the analysis of this simple combinatorial algorithm. We employ strong mathematical formulations for the bin packing problem; in particular, we use the *configuration linear program (LP)* for the bin packing problem. This linear program has been used to design the best approximation algorithm for the bin packing problem [12, 17, 9]. In our work, we use it as follows. We show that if the algorithm is not able to a color an edge (u,v), then the edges incident at u or v cannot be packed in m bins as promised. To show this, we formulate the configuration linear program for the two bin packing problems, one induced by edges incident at u and the other induced by edges incident at v. We then construct feasible dual solutions to one of these linear programs of value more than m. Appealing to linear programming duality, it implies that the optimal primal value, and therefore the optimal bin packing number, is more than m for at least one of the programs, giving us the desired contradiction. While the weights on the edges incident at u (or v) can be arbitrary reals between 0 and 1, we group the items according to weight classes and how our algorithm colors these edges. This allows us to reduce the number of item types, reducing the complexity of the configuration LP and makes it easier to analyze. While the grouping according to weight classes is natural in bin packing algorithms; the grouping based on the output of our algorithm helps us relate the fact that the edge (u,v) could not be colored by our algorithm to the bin packing bound at u and v. We mention two additional extensions of our techniques to the problem. Firstly, a more careful and detailed analysis (based on computer search) can improve the bounds slightly showing the current bound is not tight. Secondly, our analysis can also be extended to show $\lceil 2.2m \rceil$ colors are sufficient when all edge weights are more than 1/4.

1.2 Related Works

Edge-coloring problem has been one of the central problems in graph theory and discrete mathematics since its appearance in 1880 [21] in relation with the *four-color problem.* The chromatic index of a graph is the number of colors required to color the edges of the graph such that no two adjacent edges have the same color. Three classical results on edge coloring are König's theorem [14] for coloring a bipartite graph with Δ colors, Vizing's theorem [22] for coloring any simple graph with $\Delta + 1$ colors and Shanon's theorem [18] for coloring any multigraph with at most $3\Delta/2$ colors where Δ is the maximum degree of the graph. Though one can find optimal edge coloring for a bipartite graph in polynomial time using König's

1. $F \leftarrow \emptyset$, $t \leftarrow 2.2223$.
2. Include edges with weight $> \gamma = \frac{1}{10}$ in F in nonincreasing order of weight maintaining the property that $deg_F(v) \leq \lceil tm \rceil$ for all $v \in V$.
3. Decompose F into $r = \lceil tm \rceil$ matchings $M_1, \ldots, M_r$ and color them using colors $1, \ldots, r$. Let $F_i \leftarrow M_i$ for each $1 \leq i \leq r$.
4. Add remaining edges in nonincreasing order of weight to any of the F_i's maintaining that weighted degree of each color at each vertex is at most one, i.e., $\sum_{e \in \delta(v) \cap F_i} w_e \leq 1$ for each $v \in V$ and $1 \leq i \leq r$.

Figure 1 Algorithm for Edge Coloring Weighted Bipartite Graphs.

theorem, Holyer [10] showed that it is NP-hard even to decide whether the chromatic index of a cubic graph is 3 or 4. We refer the readers to [20] for a survey on edge coloring.

On the other hand classical bin packing problem is NP-hard and has been studied extensively from the classical work of Garey et al. [8]. The problem finds numerous applications in scheduling, logistics, layout design and other resource allocation problems. The present best polynomial time algorithm is due to Hoberg and Rothvoß [9] based on rounding of configuration LP using connections to discrepancy [17] and achieves a logarithmic additive error. However only known hardness bound is Opt + 1 assuming P $\neq$ NP. We refer the readers to [4] for a survey on the current literature for bin packing.

Now we review the literature related to weighted bipartite edge coloring. First, we introduce some more notation. When the weight function $w : E \to I$ is restricted to a subinterval $I \subseteq [0, 1]$, then we denote the minimum number of colors by $\mu_I(m, r)$. Slepian [19] showed that $\mu_{[1,1]}(m, r) = m$ using König's theorem. Melen and Turner [15] showed that $\mu_{[0,B]}(m, r) \leq \frac{m}{1-B}$ for $B \leq 1$. In particular, $\mu_{[0,1/2]}(m, r) \leq 2m - 1$. There has been a series of work improving the bounds for $\mu(m, r)$ [2, 6, 16, 5]. The best known lower bound for $\mu(m, r)$ is 5/4 due to Ngo and Vu [16]. Correa and Goemans introduced a novel graph decomposition result and perfect packing of an associated continuous bin packing instance to show $\mu(m, r) \leq 2.5480m + o(m)$. The present best algorithm is due to Feige and Singh [7] who showed $\mu(m, r) \leq \frac{9m}{4}$. Their result holds even if m is the maximum over all the vertices of the total weight of edges incident at the vertex. For other related results, see [7].

2 Edge Coloring Weighted Bipartite Graphs

In this section we present our main result and prove Theorem 2.

▶ **Theorem 3** (Restatement of Theorem 2). *There is a polynomial time algorithm for the* WEIGHTED BIPARTITE EDGE COLORING *problem which returns a proper weighted coloring using at most* $\lceil 2.2223m \rceil$ *colors where* m *denotes the maximum over all the vertices of the number of unit-sized bins needed to pack the weights of incident edges.*

Our complete algorithm for edge-coloring weighted bipartite graphs is given in Figure 1. In the algorithm, we set a threshold $\gamma = \frac{1}{10}$ and consider the subgraph induced by edges with weights more than γ and apply a combination of König's Theorem and a greedy algorithm with $\lceil tm \rceil$ colors where $t = 2.2223 > 20/9$. The remaining edges of weights at most γ are then added greedily.

Analysis

Now we prove Theorem 2. Though the algorithm is purely combinatorial, the analysis uses configuration LP and other techniques from bin packing to prove the correctness of the algorithm. First, we state the König's Theorem since we use it as a subroutine in our algorithm to ensure a decomposition of F into $\lceil tm \rceil$ matchings.

▶ **Theorem 4** ([14]). *Given a bipartite graph $G = (V, E)$, there exists a coloring of edges with $\Delta = max_{v \in V} deg_E(v)$ colors such that all edges incident at a common vertex receive a distinct color. Moreover, such a coloring can be found in polynomial time.*

The following lemma from Correa and Goemans [5] (it was also implicit in [6]) ensures that if the algorithm succeeds in coloring all edges of weight at least γ, the greedy coloring will be able to color the remaining edges of weight at most γ.

▶ **Lemma 5** ([5, 6]). *Consider a bipartite weighted graph $G = (V, E)$ with a coloring of all edges of weight $> \gamma$ using at least $\frac{2m}{1-\gamma}$ colors for some $\gamma > 0$. Then the greedy coloring will succeed in coloring the edges with weight at most γ without any additional colors.*

In our setting, we have $\gamma = \frac{1}{10}$ and the number of colors is $\leq \frac{20}{9}m = \frac{2m}{1-\frac{1}{10}}$ and thus Lemma 5 applies. Hence, it suffices to show the algorithm is able to color all edges with weight $> \frac{1}{10}$ using $\lceil tm \rceil$ colors as the remaining smaller edges can be colored greedily. Thus, w.l.o.g, we assume that the graph has no edges of weight $\leq \frac{1}{10}$ and prove the following lemma.

▶ **Lemma 6.** *If all edges have weight more than $\frac{1}{10}$ and $t = 2.2223$ $(> 20/9)$ then the algorithm in Figure 1 returns a coloring of all edges using $\lceil tm \rceil$ colors such that the weighted degree of each color at each vertex is at most one, i.e., $\sum_{e \in \delta(v) \cap F_i} w_e \leq 1$.*

Proof. Suppose for the sake of contradiction, the algorithm is not able to color all edges. Let $e := (u, v)$ be the first edge that cannot be colored by any color in Step (3) or Step (4) of the algorithm. Let the weight of edge e, w_e, be α. Moreover, when e is considered in Step (2), degree of either u or v is already $\lceil tm \rceil$ else we would have included e in F. Without loss of generality let that vertex be u, i.e., $deg_F(u) = \lceil tm \rceil$. Now we characterize the colors F_i of edges incident at u and consider the edges incident at u and v to get a series of inequalities. Thereafter, we show that $\alpha \leq 1/3$ and use these inequalities to arrive at a contradiction.

For each color $1 \leq i \leq \lceil tm \rceil$, either $\sum_{f \in \delta(v) \cap F_i} w_f > 1 - \alpha$ or $\sum_{f \in \delta(u) \cap F_i} w_f > 1 - \alpha$, else we can color e in Step (4). Let $H_v = \{i | \sum_{f \in \delta(v) \cap F_i} w_f > 1 - \alpha\}$, $\beta m = |H_v|$. Now for each color $i \notin H_v$, we have $\sum_{f \in \delta(u) \cap F_i} w_f > 1 - \alpha$. Moreover, $deg_F(u) = \lceil tm \rceil$ and for all edge $f \in \delta(u)$, we have $w_f \geq \alpha$ as the edges were considered in nonincreasing order of weight. Hence, for each color $1 \leq i \leq \lceil tm \rceil$, there is an edge incident at u colored with color i with weight at least α. Let us call a color i *tight* at u if $\sum_{f \in \delta(u) \cap F_i} w_f > (1 - \alpha)$ and a color i *open* at u if $\sum_{f \in \delta(u) \cap F_i} w_f \in [\alpha, 1 - \alpha]$. Let τ be the number of tight colors at u and θ be the number of open colors at u. Thus we have,

$$\tau \geq (t - \beta)m \quad (1)$$

$$\theta = (tm - \tau) \leq \beta m \quad (2)$$

Now consider all edges incident on v. We get, $m > \beta m(1 - \alpha)$

$$\Rightarrow \quad 1 > \beta(1 - \alpha) \quad (3)$$

Similarly considering all edges incident on u, we get: $m > (tm - \beta m)(1 - \alpha) + (\beta m)\alpha$

$$\Rightarrow \quad 1 > t(1 - \alpha) + \beta(2\alpha - 1) \quad (4)$$

Now a unit sized bin can contain at most two items with weight $> 1/3$. As all edges incident to a vertex can be packed into m unit sized bins, there can be at most $2m$ edges incident to a vertex with weight $> 1/3$. Since $t > 2$, we get that all edges with weight more than $\frac{1}{3}$ must have been included in F in Step (2). Thus $\alpha \leq 1/3$.

Thus we get from (3):
$$\beta < 1/(1-\alpha) \leq 3/2 \tag{5}$$

Now there are two cases:

Case A: $\alpha \leq 1/4$**.** Consider the RHS of (4): $t(1-\alpha) + \beta(2\alpha - 1)$. Now,

$$\begin{aligned} t(1-\alpha) + \beta(2\alpha-1) - 1 &> t(1-\alpha) - \frac{(1-2\alpha)}{(1-\alpha)} - 1, \quad \text{[From (3)]} \\ &\geq \frac{20(1-\alpha)^2 - 9(1-2\alpha) - 9(1-\alpha)}{9(1-\alpha)} \quad [\because t > 20/9] \\ &\geq \frac{(20\alpha^2 - 13\alpha + 2)}{9(1-\alpha)} = \frac{(4\alpha-1)(5\alpha-2)}{9(1-\alpha)} \geq 0, \text{as } \alpha \leq 1/4 \end{aligned}$$

Thus $t(1-\alpha) + \beta(2\alpha - 1) > 1$, which contradicts (4).

Case B: $1/4 < \alpha \leq 1/3$**.** In this case, we will show in Lemma 7 that if $\beta \leq 13/9$, then all edges incident at u can not be packed in m bins. On the other hand, in Lemma 14 we show that if $\beta > 13/9$, then all edges incident at v can not be packed in m bins.

This two facts together give us the desired contradiction.

▶ **Lemma 7.** *If $\beta \leq 13/9$, then edges incident at u can not be packed in m bins.*

Proof. To give a lower bound on the number of bins required, we will consider a relaxation to the bin packing problem for edges incident at vertex u and show that the optimal value of the relaxation, and thus the optimal number of bins required, is greater than m. The lower bound will be exhibited by constructing a feasible dual solution to the relaxation to the bin packing problem.

Since $deg_F(u) = \lceil tm \rceil$ when edge e was considered in Step (2) of the algorithm and not included in F, we have that all edges incident at u in F have weight at least the weight of e. Moreover, edges are considered in the decreasing order of weight in Step (4), the weight of all edges incident at u when e is considered in Step (4) is $\geq w_e$. We restrict our attention to these edges incident at u with weight $\geq \alpha$ and show that they cannot be packed in m unit sized bins. Let us divide these edges incident at u into three size classes.

- Large $L := \{f \in \delta(u) : w_f \in (1/2, 1]\}$.
- Medium $M := \{f \in \delta(u) : w_f \in (1/3, 1/2]\}$.
- Small $S := \{f \in \delta(u) : w_f \in [\alpha, 1/3]\}$.

First we have the following observation.

▶ **Observation 8.** *In any bin packing solution, in any bin there can be at most one item from L, two items from $L \cup M$ and three items from $L \cup M \cup S$.*

Now consider the following two simple claims.

▶ **Claim 9.** *Edges in $L \cup M$ are included in Step (2) of Algorithm 1 and are a subset of F.*

Proof. If we are unable to add an edge f in Step (2), it means one of its endpoints have $\lceil tm \rceil > 2m$ edges with weight $\geq w_f$. Since all edges incident at any vertex $v \in V$ can be packed in m bins, there are at most $2m$ edges incident at it with weight more than $\frac{1}{3}$. Thus all edges of weight more than $\frac{1}{3}$, i.e., all edges in $L \cup M$ must be included in F in Step (2) of the algorithm. ◀

▶ **Claim 10.** *For any color i, there is at most one edge in $L \cup M$ with color i.*

Proof. As all edges in $L \cup M$ must be included in F in Step (2) of the algorithm. In Step (3) of the algorithm, we include at most one edge of F incident at any vertex in each F_i. Thus each color class obtains at most one edge incident at each vertex from F and therefore, from $L \cup M$. ◀

Using the observation and the above claim, we itemize the configuration of each of the tight colors depending on the size of edges with that color. Note that tight colors must have weight $> 1 - \alpha \geq 1 - 1/3 = 2/3$.

1. If the tight color has a single edge f. Then we have that $w_f > \frac{2}{3}$ and only possibility is $i)(L)$; Here by (L), we denote that the bin contains only one item and that item is an item from the set L.
2. If the tight color contains exactly two edges. Here (S, S) is not tight as the total weight of edges in such a bin is $\leq 2/3$. So, the bin can contain at most one item from S. On the other hand, the bin can contain at most one item from $L \cup M$ from Claim 10. Thus the possible size types of these edges are $ii)(L, S); iii)(M, S)$; As above, by (L, S) we denote that the bin contains only two items: exactly one item from set L and exactly one item from S.
3. If the tight color contains three edges. The bin can contain at most one item from $L \cup M$ from Claim 10. However if it contains one L item, sum of weights of three items exceeds one. Thus the only possible size types of these edges are $iv)(M, S, S); v)(S, S, S)$.

Now consider the following LP: $LP_{bin}(u)$:

$$min \sum_{i=1}^{5} y_i$$

$$x_1 + x_2 + x_3 + x_4 + x_5 \geq \tau \tag{6}$$

$$y_1 + y_2 \geq x_1 + x_2 + z_1 \tag{7}$$

$$y_1 + 2y_3 + y_4 \geq x_3 + x_4 + z_2 \tag{8}$$

$$y_2 + y_3 + 2y_4 + 3y_5 \geq x_2 + x_3 + 2x_4 + 3x_5 + z_3 \tag{9}$$

$$z_1 + z_2 + z_3 \geq \theta \tag{10}$$

$$x_j, y_k, z_l \geq 0 \quad \forall j \in [5], k \in [5], l \in [3] \tag{11}$$

▶ **Lemma 11.** *The optimal number of unit sized bins needed to pack all edges incident at u is at least the optimum value of $LP_{bin}(u)$.*

Proof. Given a feasible packing of edges incident at u in at most m unit sized bins, we construct a feasible solution $(\bar{x}, \bar{y}, \bar{z})$ to the linear programming relaxation whose objective is at most the number of unit sized bins needed in the packing. In the feasible solution $(\bar{x}, \bar{y}, \bar{z})$, the variables $\bar{x}$ and $\bar{z}$ are constructed using the coloring given by the algorithm. The variables $\bar{y}$ are constructed using the optimal bin packing.

We first define the variables $\bar{x}$. Let $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5$ be the number of tight colors at u of type $(L), (L, S), (M, S), (M, S, S), (S, S, S)$, respectively. Since the coloring of the edges incident at u is one of the five types described above and there are at least τ tight colors, we have that $\sum_{i=1}^{5} \bar{x}_i \geq \tau$ and thus the solution satisfies constraint (6). Now we define the variables $\bar{z}_1, \bar{z}_2, \bar{z}_3$ to be the number of items in open colors from L, M and S respectively. There are θ open colors. Each open color contains at least one item $L \cup M \cup S$. Thus, $\bar{z}_1 + \bar{z}_2 + \bar{z}_3 \geq \theta$ and thus the solution satisfies constraint (10).

To construct the solution $\bar{y}$, we will group the bins in the optimal bin packing solutions depending on the subset of items present in them into five classes and the number of bins in each class will define the variables $\bar{y}$. The constraints (7)-(9) will correspond to making sure that the optimal bin packing solution has appropriate number of items of each size type.

Now let us characterize the possible bin configurations to explain constraints (7)-(9).

▶ **Claim 12.** *Consider any feasible bin packing of edges incident at u restricted to edges in $L \cup M \cup S$. Then each bin must contain items which correspond to a subset of one of the following 5 configurations or subsets of these configurations.*

$C_1 : (L, M)$ $\quad C_2 : (L, S)$ $\quad C_3 : (M, M, S)$
$C_4 : (M, S, S)$ $\quad C_5 : (S, S, S)$

Proof. Observe that in any bin there can be at most one item from L, two items from $L \cup M$ and three items from $L \cup M \cup S$. Now let us consider two cases.
1. If the bin contains an item from L. In this case, the bin can not contain three items as the sum of their weights exceeds one. So, it can contain at most one item from L and one item from $M \cup S$. Thus C_1 and C_2 cover such two cases.
2. If the bin does not contain any item from L. In this case, the bin can contain three items from $M \cup S$ and at most two of these items can be from M. Thus C_3, C_4 and C_5 cover such possibilities. ◀

We map each configuration in the optimal bin packing solution to one of types C_i where the configuration is either C_i or its subset. Let $\bar{y}_i$ denote the number of bins mapped to type C_i for each $1 \leq i \leq 5$. We now count the number of items of each type to show feasibility of the constraints of the linear program.

Constraint (7). Items of type L equal $\bar{x}_1 + \bar{x}_2 + \bar{z}_1$ and can only be contained in configuration C_1 and C_2. Thus we have $\bar{y}_1 + \bar{y}_2 \geq \bar{x}_1 + \bar{x}_2 + \bar{z}_1$ satisfying constraint (7).
Constraint (8). Items of type M equal $\bar{x}_3+\bar{x}_4+\bar{z}_2$ and are contained once in configurations C_1, C_4 and twice in configuration C_3. Thus we have $\bar{y}_1+2\bar{y}_3+\bar{y}_4 \geq \bar{x}_3+\bar{x}_4+\bar{z}_2$ satisfying constraint (8).
Constraint (9). Items of type S equal $\bar{x}_2 + \bar{x}_3 + 2\bar{x}_4 + 3\bar{x}_5 + \bar{z}_3$ and occur once in configurations C_2, C_3, twice in configurations C_4 and thrice in C_5. Thus, we have $\bar{y}_2 + \bar{y}_3 + 2\bar{y}_4 + 3\bar{y}_5 \geq \bar{x}_2 + \bar{x}_3 + 2\bar{x}_4 + 3\bar{x}_5 + \bar{z}_3$ showing feasibility of constraint (9).
This implies that $(\bar{x}, \bar{y}, \bar{z})$ is a feasible solution to LP_{bin} and its objective equals the number of bins needed to pack the edges incident at u in unit sized bins. Thus we have the lemma. ◀

We now show a contradiction by showing the optimal value of the $LP_{bin}(u)$ is more than m.

▶ **Lemma 13.** *The optimal solution to $LP_{bin}(u)$ is strictly more than m.*

Proof. We prove this by considering the dual linear program of the $LP_{bin}(u)$. Since every feasible solution to the dual LP gives a lower bound on the objective of the primal $LP_{bin}(u)$, it is enough to exhibit a feasible dual solution of objective strictly more than m to prove the lemma. Now the dual of the LP_{bin} is given in the next page.

A feasible dual solution is: $v_1 = \frac{2}{3}, v_2 = \frac{2}{3}, v_3 = \frac{1}{3}, v_4 = \frac{1}{3}, v_5 = \frac{1}{3}$.

$$\begin{array}{ll} max \quad \tau \cdot v_1 + \theta \cdot v_5 & \\ \text{Subject to:} & \\ v_1 - v_2 \le 0, & v_1 - v_2 - v_4 \le 0, \\ v_1 - v_3 - v_4 \le 0, & v_1 - v_3 - 2v_4 \le 0, \\ v_1 - 3v_4 \le 0, & v_2 + v_3 \le 1, \\ v_2 + v_4 \le 1, & 2v_3 + v_4 \le 1, \\ v_3 + 2v_4 \le 1, & 3v_4 \le 1, \\ v_5 - v_2 \le 0, & v_5 - v_3 \le 0, \\ v_5 - v_4 \le 0, & v_i \ge 0 \quad \forall i \in [5] \end{array}$$

Thus dual optima $\geq \frac{2\tau}{3} + \frac{\theta}{3}$ and we need at least these many colors to color items in τ tight colors and θ open colors. Using the fact that $\theta = tm - \tau, \tau \geq (t-\beta)m$ and $t > \frac{20}{9}m, \beta \leq \frac{13}{9}$, we obtain that the number bins required to pack all items incident on u is:

$$\begin{array}{lllllll} \geq & \tau \cdot v_1 + \theta \cdot v_4 & \geq & \frac{2}{3}\tau + \frac{1}{3}(tm - \tau) & = & \frac{1}{3}\tau + \frac{1}{3}(tm) & \\ \geq & \frac{1}{3}(t-\beta)m + \frac{1}{3}(tm) & \geq & \frac{2t}{3}m - \frac{\beta}{3}m & > & m(\frac{2}{3}\cdot\frac{20}{9} - \frac{1}{3}\cdot\frac{13}{9}) & = \quad m \end{array}$$

Thus the number bins required to pack all items incident on u is strictly greater than m. This is a contradiction. ◀

This concludes the proof of Lemma 7. ◀

▶ **Lemma 14.** *If $\beta > 13/9$, then edges incident at v can not be packed in m bins.*

Proof. Similar to the previous lemma, to give a lower bound on the number of bins required, we will consider a relaxation to the bin packing problem for edges incident at vertex v and show that the optimal value of the relaxation, and thus the optimal number of bins required, is greater than m. Again, the lower bound will be exhibited by constructing a feasible dual solution to the relaxation to the bin packing problem.

As $\beta(1-\alpha) < 1$, we get,

$$\alpha > 1 - 1/\beta \geq 4/13 > 0.3. \tag{12}$$

Let us call a color i *tight* at v if $\sum_{f \in \delta(v) \cap F_i} w_f > (1-\alpha)$. Now consider any tight color $\mathcal{B}$ at v. At most one edge f in $\mathcal{B}$ was colored in Step (3) of the algorithm and remaining edges (if any) in $\mathcal{B}$ were colored in Step (4) of the algorithm. Now, w_f can be smaller than w_e as it might be the case that when e was considered in Step (2) then already degree of other endpoint u was $\lceil tm \rceil$. However, edges are considered in the nonincreasing order of weight in Step (4), thus the weight of all edges incident at v when e is considered in Step (4) is also $\geq w_e$. Thus, all the remaining edges (if any) in $\mathcal{B}$ that were colored in Step (4) of the algorithm have weight more than α.

We restrict our attention to the edges at tight colors at v and show that if $\beta > \frac{13}{9}$ they cannot be packed in m unit sized bins. Let us divide these edges incident at u into four size classes.

- Large $L := \{f \in \delta(v) : w_f \in (1/2, 1]\}$.
- Medium $M := \{f \in \delta(v) : w_f \in (1/3, 1/2]\}$.
- Small $S := \{f \in \delta(v) : w_f \in [\alpha, 1/3]\}$.
- Tiny $T := \{f \in \delta(v) : w_f \in (1/10, \alpha)\}$.

First we have the following observation.

▶ **Observation 15.** *In any bin packing solution, in any bin there can be at most one item from L, two items from $L \cup M$, three items from $L \cup M \cup S$ and nine items from $L \cup M \cup S \cup T$.*

Now let us claim the following.

▶ **Claim 16.** *For any tight color i at v, all edges added in Step (4) of the algorithm are in S. As a corollary, there is at most one edge incident on v with color i that is in $L \cup M \cup T$ and it can only be added in Step (2) of the algorithm.*

Proof. From Claim 9, it follows that all edges in $L \cup M$ must be included in F in Step (2) of the algorithm. On the other hand, all edges colored in Step (4) have weight more than α, they can not be in T. Hence, only edges in S are colored in Step (4). Edges in $L \cup M \cup T$ are colored in Step (3). In Step (3) of the algorithm, we include at most one edge of F incident at any vertex in each F_i. Thus each color class obtains at most one edge incident at each vertex from F and therefore, from $L \cup M \cup T$. ◀

Using the observation and the claim, we itemize the configuration of each of the tight colors depending on the size of edges with that color. Note that in this case tight colors have weight more than $1 - \alpha \geq 1 - 1/3 = 2/3$.

1. If the tight color has a single edge f. Then we have that $w_f > 2/3$ and only possibility is $i)(L)$; Here by (L), we again denote that the bin contains only one item and that item is an item from the set L.
2. If the tight color contains exactly two edges. From Claim 16, the bin can contain at most one item from $L \cup M \cup T$. On the other hand, (S, S) or (T, S) has weight $\leq 2/3$. So the bin can contain at most one item from S and one item from $L \cup M$. Thus the possible size types of these edges are $ii)(L, S); iii)(M, S)$; As above, by (L, S) we denote that the bin contains only two items: exactly one item from set L and exactly one item from S.
3. If the tight color contains three edges. From Claim 16, the bin can contain at most one item from $L \cup M \cup T$. However if the bin contains an item from L, the sum of weights of an item from L and two items from S exceeds one. Thus the possible size types of these edges are $iv)(M, S, S); v)(S, S, S); vi)(T, S, S)$.

Now consider the following configuration LP based on the items at v: $LP_{bin}(v)$:

$$min \sum_{i=1}^{19} y_i$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq \beta m \quad (13)$$

$$y_{14} \geq x_1 \quad (14)$$

$$y_8 + y_9 + y_{10} + y_{15} \geq x_2 \quad (15)$$

$$y_3 + y_7 + y_9 + y_{11} + 2y_{12} + y_{16} \geq x_3 \quad (16)$$

$$y_2 + 2y_4 + y_6 + y_{10} + y_{11} + 2y_{13} + y_{17} \geq x_4 \quad (17)$$

$$3y_1 + 2y_2 + 2y_3 + y_4 + 2y_5 + y_6 + y_7 + y_8 + y_{18} \geq x_2 + x_3 + 2x_4 + 3x_5 + 2x_6 \quad (18)$$

$$(3y_5 + 3y_6 + 3y_7 + y_8 + y_9 + y_{10} + 3y_{11} + 3y_{12} + 3y_{13} + 3y_{14} + 4y_{15} + 6y_{16} + 6y_{17} + 6y_{18} + 9y_{19}) \geq x_6 \quad (19)$$

$$y_j, x_k \geq 0 \quad \forall j \in [19], k \in [6] \quad (20)$$

▶ **Lemma 17.** *The optimal number of unit sized bins needed to pack all edges incident at u is at least the optimum value of $LP_{bin}(v)$.*

Proof. Given a feasible packing of edges incident at v in at most m unit sized bins, we construct a feasible solution $(\bar{x}, \bar{y})$ to the linear programming relaxation whose objective is at most the number of unit sized bins needed in the packing. In the feasible solution $(\bar{x}, \bar{y})$, the variables $\bar{x}$ are constructed using the coloring given by the algorithm. The variables $\bar{y}$ are constructed using the optimal bin packing.

We first define the variables $\bar{x}$. Let $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6$ be the number of tight colors at v of type $(L), (L, S), (M, S)$, $(M, S, S), (S, S, S), (T, S, S)$, respectively. Since the coloring of the edges incident at v is one of the six types described above and there are at least βm tight colors at v, we have that $\sum_{i=1}^{6} \bar{x}_i \geq \beta m$ and thus the solution satisfies constraint (13).

To construct the solution $\bar{y}$, we will group the bins in the optimal bin packing solutions depending on the subset of items present in them into nineteen classes and the number of bins in each class will define the variables $\bar{y}$. The constraints (14)-(19) will correspond to making sure that the optimal bin packing solution has appropriate number of items of each size type.

To define the 19 different classes of bin types in the optimal solution, we need to further classify items according to size. Let $L_1, L_2 \subseteq L$ be the set of large edges that appear in the configurations of the type (L) and (L, S), respectively, in the tight colors.

As for any item $l_1 \in L_1$, $w_{l_1} + \alpha > 1$. We get,

$$w_{l_1} > 1 - \alpha \geq 1 - 1/3 = 2/3. \tag{21}$$

Let M_1, M_2 be the set of medium edges that appear in the tight colors of type (M, S) and (M, S, S), respectively.

We now have the following claim where we characterize the possible bin configurations. We show that each bin contains items which correspond to one of 19 possible configurations or their subsets.

▶ **Claim 18.** *Consider any feasible bin packing of edges incident at v restricted to edges in $L \cup M \cup S \cup T$. Then each bin must contain items which correspond to a subset of one of the following 19 configurations.*

$C_1 : (S, S, S)$	$C_2 : (M_2, S, S)$	$C_3 : (M_1, S, S)$
$C_4 : (M_2, M_2, S)$	$C_5 : (S, S, T, T, T)$	$C_6 : (M_2, S, T, T, T)$
$C_7 : (M_1, S, T, T, T)$	$C_8 : (L_2, S, T)$	$C_9 : (L_2, M_1, T)$
$C_{10} : (L_2, M_2, T)$	$C_{11} : (M_1, M_2, T, T, T)$	$C_{12} : (M_1, M_1, T, T, T)$
$C_{13} : (M_2, M_2, T, T, T)$	$C_{14} : (L_1, T, T, T)$	$C_{15} : (L_2, T, T, T, T)$
$C_{16} : (M_1, T, T, T, T, T, T)$	$C_{17} : (M_2, T, T, T, T, T, T)$	$C_{18} : (S, T, T, T, T, T, T)$
$C_{19} : (T, T, T, T, T, T, T, T, T)$		

Proof. Observe that since item in L have weight more than $\frac{1}{2}$, items in M have weight more than $\frac{1}{3}$, items in S have weight more than $\frac{1}{4}$ and items in T have weight more than $\frac{1}{10}$, there can be at most one item from L, two items items from $L \cup M$, at most three items in total from $L \cup M \cup S$ and at most nine items from $L \cup M \cup S \cup T$ in any feasible packing.

1. Bins with three items from $D := L \cup M \cup S$. As the sum of weights of three elements from D is more than $3\alpha > 0.9$, elements from T can not appear in these bins as $3\alpha + w_f > 1$ for any $f \in T$. Moreover, configurations which contain at least one item of L cannot have three items from D without the weight exceeding one. Thus, the packing can contain only items from M and S. When the bin contains only S items, it corresponds to configuration C_1. Packings which contain one item from $M_1 \cup M_2$ and two items from S are exactly the configurations C_2, C_3.

Now let us consider the case when we have two items from $M_1 \cup M_2$. First observe that for each $h \in M_1$, there exists a $s \in S$ such that (h, s) are the only edges colored with a tight color. Thus we have that $w_h + w_s > 1 - \alpha$. But then for any other $h' \in M_1 \cup M_2$ and $s' \in S$, we have that

$$w_h + w_{h'} + w_{s'} \geq w_h + w_s + \alpha > 1 \tag{22}$$

where the inequality follows since $w_{h'} \geq w_s$ and $w_{s'} \geq \alpha$. This implies that configurations of type (M_1, M_1, S) or (M_1, M_2, S) are not feasible. Hence, the only possible remaining configuration is C_4.

2. Bins with two items from D. Here we consider maximal configurations which are not subsets of configurations which contain three items from D. When the configuration contains two items $s_1, s_2 \in S$, we have that $w_{s_1} + w_{s_2} > 0.6$ and thus the only maximal configuration is C_5. Configurations C_6, C_7 cover the case when the configuration contains one item from S and one item from M. Let $l \in L_1$. Since l appears alone in a tight color, we have that $w_l + \alpha > 1$. Since every item in $M \cup S$ has weight at least α, there is no valid configuration with two items from D such that one of them is in L_1. If the configuration contains $l_2 \in L_2$ and $g \in M_1 \cup M_2 \cup S$, it can at most contain one element $t \in T$ as $w_{l_2} + w_g + w_t > 0.9$. Thus configurations C_8 cover the case when there is one item from S and one item from L. Now we are left with cases when there are no S items in the bin. If there is one L item and one M item, C_9, C_{10} cover such possibilities. Similarly if the configuration contains two items from M, it can contain at most 3 elements from T. Configurations C_{11}, C_{12}, C_{13} cover all such the possibilities.
3. Bins with one item from D. Here we consider configurations which are not subsets of configurations which contain at least two items from D. Note that as for any item $l_1 \in L_1$, from inequality (21), $w_{l_1} > 2/3$. Thus (L_1, T, T, T) is the maximal configuration containing one L_1 item. C_{14} is the corresponding configuration. The other four possible configurations are $C_{15}, C_{16}, C_{17}, C_{18}$ where the bins contain one item from L_2, M_1, M_2, S respectively. In these case the number of T items are upper bounded by $4, 6, 6, 6$ respectively from the lower bound of size of items in the corresponding classes in D.
4. Bins with no item from D. Only possible maximal configuration is C_{19}. ◀

We map each configuration in the optimal bin packing solution to one of types C_i where the configuration is either C_i or its subset. Let $\bar{y}_i$ denote the number of bins mapped to type C_i for each $1 \leq i \leq 19$. We now count the number of items of each type to show feasibility of the constraints of the linear program.

Constraint (14). Items of type L_1 equal $\bar{x}_1$ and can only be contained in configuration C_{14}. Thus we have $\bar{y}_{14} \geq \bar{x}_1$.

Constraint (15). Similarly, items of type L_2 equal $\bar{x}_2$. They are contained in configurations C_8, C_9, C_{10} and C_{15}. Thus we have $\bar{y}_8 + \bar{y}_9 + \bar{y}_{10} + \bar{y}_{15} \geq \bar{x}_2$.

Constraint (16). Items of type M_1 equal $\bar{x}_3$ and are contained once in configurations $C_3, C_7, C_9, C_{11}, C_{16}$ and twice in configuration C_{12}. Thus we have $\bar{y}_3 + \bar{y}_7 + \bar{y}_9 + \bar{y}_{11} + 2\bar{y}_{12} + \bar{y}_{16} \geq \bar{x}_3$.

Constraint (17). Items of type M_2 equal $\bar{x}_4$ and are contained once in configurations $C_2, C_6, C_{10}, C_{11}, C_{17}$ and twice in configurations C_4, C_{13}. Thus we have $\bar{y}_2 + 2\bar{y}_4 + \bar{y}_6 + \bar{y}_{10} + \bar{y}_{11} + 2\bar{y}_{13} + \bar{y}_{17} \geq \bar{x}_4$ satisfying constraint (17).

Constraint (18). Items of type S equal $\bar{x}_2 + \bar{x}_3 + 2\bar{x}_4 + 3\bar{x}_5 + 2\bar{x}_6$ and occur once in configurations $C_4, C_6, C_7, C_8, C_{18}$, twice in configurations C_2, C_3, C_5 and thrice in C_1. Thus, we have $3\bar{y}_1 + 2\bar{y}_2 + 2\bar{y}_3 + \bar{y}_4 + 2\bar{y}_5 + \bar{y}_6 + \bar{y}_7 + \bar{y}_8 + \bar{y}_{18} \geq \bar{x}_2 + \bar{x}_3 + 2\bar{x}_4 + 3\bar{x}_5 + 2\bar{x}_6$.

Constraint (19). Items of type T equal $\bar{x}_6$ and occur once in configurations C_8, C_9, C_{10}, thrice in configurations $C_5, C_6, C_7, C_{11}, C_{12}, C_{13}, C_{14}$, four times in configuration C_{15}, six times in configurations C_{16}, C_{17}, C_{18} and nine times in configurations C_{19}. Thus, we have $(3\bar{y}_5+3\bar{y}_6+3\bar{y}_7+\bar{y}_8+\bar{y}_9+\bar{y}_{10}+3\bar{y}_{11}+3\bar{y}_{12}+3\bar{y}_{13}+3\bar{y}_{14}+4\bar{y}_{15}+6\bar{y}_{16}+6\bar{y}_{17}+6\bar{y}_{18}+9\bar{y}_{19}) \geq \bar{x}_6$.

This implies that $(\bar{x}, \bar{y})$ is a feasible solution to $LP_{bin}(v)$ and its objective equals the number of bins needed to pack the edges incident at v in unit sized bins. Thus we have the lemma. ◀

We now show a contradiction by showing the optimal value of the $LP_{bin}(v)$ is more than m.

▶ **Lemma 19.** *The optimal solution to the $LP_{bin}(v)$ is strictly more than m.*

Proof. We prove this by considering the dual linear program of the $LP_{bin}(v)$. Since every feasible solution to the dual LP gives a lower bound on the objective of the primal $LP_{bin}(v)$, it is enough to exhibit a feasible dual solution of objective strictly more than m to prove the lemma. Now the dual of the $LP_{bin}(v)$ is given below:

$$\begin{array}{ll}
max \quad \beta m \cdot v_1 & \\
\text{Subject to:} & \\
v_1 - v_2 \leq 0, & v_1 - v_3 - v_6 \leq 0, \\
v_1 - v_4 - v_6 \leq 0, & v_1 - v_5 - 2v_6 \leq 0, \\
v_1 - 3v_6 \leq 0, & v_1 - 2v_6 - v_7 \leq 0, \\
3v_6 \leq 1, & v_5 + 2v_6 \leq 1, \\
v_4 + 2v_6 \leq 1, & 2v_5 + v_6 \leq 1, \\
2v_6 + 3v_7 \leq 1, & v_5 + v_6 + 3v_7 \leq 1, \\
v_4 + v_6 + 3v_7 \leq 1, & v_3 + v_6 + v_7 \leq 1, \\
v_3 + v_4 + v_7 \leq 1, & v_3 + v_5 + v_7 \leq 1, \\
v_4 + v_5 + 3v_7 \leq 1, & 2v_4 + 3v_7 \leq 1, \\
2v_5 + 3v_7 \leq 1, & v_2 + 3v_7 \leq 1, \\
v_3 + 4v_7 \leq 1, & v_4 + 6v_7 \leq 1, \\
v_5 + 6v_7 \leq 1, & v_6 + 6v_7 \leq 1, \\
9v_7 \leq 1, & v_i \geq 0 \quad \forall i \in [7].
\end{array}$$

A feasible dual solution is: $v_1 = \frac{9}{13}, v_2 = \frac{9}{13}, v_3 = \frac{7}{13}, v_4 = \frac{5}{13}, v_5 = \frac{1}{13}, v_6 = \frac{4}{13}, v_7 = \frac{1}{13}$. Thus dual optima is at least $\beta m \cdot \frac{9}{13} > m$. Thus, we need more than m bins to pack all items incident at v, a contradiction. ◀

This completes the proof of Lemma 14. ◀

Therefore, the proof of Theorem 2 is complete. ◀

If we assume that all edges have weight more than 1/4, then similar analysis will attain $2.2m$ colors are sufficient. For the proof, we refer the readers to [13].

▶ **Theorem 20.** *If all edges have weight more than $1/4$, then there is a polynomial time algorithm for the* WEIGHTED BIPARTITE EDGE COLORING *problem which returns a proper weighted coloring using at most $\lceil 2.2m \rceil$ colors where m is denotes the maximum over all the vertices of the number of unit-sized bins needed to pack the weights of incident edges, i.e., $\mu_{(\frac{1}{4},1]}(m, r) \leq \lceil 2.2m \rceil$.*

3 Conclusion

Considering the case $1/4 \geq \alpha > 1/5$ separately, might improve the bound by more case analysis. However we can attain at most $35m/16 \approx 2.19m$ by that. Finding a better approximation algorithm (independent of m) or inapproximability, and extending our techniques to general graphs will be interesting.

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