Final Coalgebras from Corecursive Algebras

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Abstract

We give a technique to construct a final coalgebra in which each element is a set of formulas of modal logic. The technique works for both the finite and the countable powerset functors. Starting with an injectively structured, corecursive algebra, we coinductively obtain a suitable subalgebra called the "co-founded part". We see – first with an example, and then in the general setting of modal logic on a dual adjunction – that modal theories form an injectively structured, corecursive algebra, so that this construction may be applied. We also obtain an initial algebra in a similar way.

We generalize the framework beyond **Set** to categories equipped with a suitable factorization system, and look at the examples of **Poset** and $\mathbf{Set}^{\mathrm{op}}$.

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1 Introduction

1.1 The Problem

Consider image-countable labelled transition systems, i.e. coalgebras for the Set functor $B : X \mapsto (\mathcal{P}_c\, X)^{\mathcal{A}}$. Here $\mathcal{A}$ is a fixed set (not necessarily countable) of labels and $\mathcal{P}_c\, X$ is the set of countable subsets of X. It is well-known [25] that, in order to distinguish all pairs of non-bisimilar states, Hennessy-Milner logic with finitary conjunction is not sufficiently expressive, and we instead require infinitary conjunction. For example, we may take all formulas

$$\phi ::= \quad \bigwedge_{i \in I} \phi_i \;\mid\; \neg\phi \;\mid\; [a]\phi$$

where the indexing sets I are countable; and write $\bigvee$ and $\langle a \rangle$ for the de Morgan duals of $\bigwedge$ and $[a]$ respectively. Alternatively, it is sufficient to take the following $\Diamond$*-layered formulas.*

$$\phi ::= \quad \langle a \rangle \, (\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg\phi_j) \tag{1}$$

For a B-coalgebra (X, ζ), the semantics of these formulas is given by

$$u \models \langle a \rangle \, (\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg\phi_j) \iff \exists x \in (\zeta(u))_a .\, (\forall i \in I . x \models \phi_i \wedge \forall j \in J .\; x \not\models \psi_j) \tag{2}$$

Following [15, 22], we obtain a final B-coalgebra in which states are sets of formulas, or, alternatively, sets of $\Diamond$-layered formulas. Specifically, if $(\!|x|\!)_{X,\zeta}$ is the set of $\Diamond$-layered

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formulas satisfied by a state x within the coalgebra (X, ζ), then the final coalgebra has carrier

$$M = \{ (\!| x |\!)_{X,\zeta} \mid (X, \zeta) \text{ a } T\text{-coalgebra, } x \in X \}$$

and its structure sends $(\!| x |\!)_{X,\zeta}$ to the image of x along the function

$$X \xrightarrow{\zeta} FX \xrightarrow{F(\!| - |\!)_{X,\zeta}} FM$$

It may, however, be argued that this construction is not quite satisfactory, because it is couched in terms of all B-coalgebras. We might as well just form the sum of all B-coalgebras[1] and then take the strongly extensional quotient, i.e. the quotient by bisimilarity. The modal logic is not playing any real role.

We therefore ask: is it possible to construct the final coalgebra purely out of the logic, without referring to other coalgebras? In particular, we shall need to characterize when a set of formulas is of the form $(\!| x |\!)_{X,\zeta}$.

In the case of coalgebras of the *finite* powerset functor – for which finite conjunctions are expressive enough to distinguish non-bisimilar states – this question was answered in [4, Theorem 5.9] following [1, 23] and [29, Theorem 7.4]. The first step is to construct a transition system, called the "canonical model of modal logic K" [7], consisting of sets of modal formulas closed under certain inference rules. Then the subsystem consisting of hereditarily image-finite elements is a final coalgebra.

It is, however, not evident whether or how this construction could be adapted to logic with infinite conjunctions. We shall not consider that question in this paper. Instead we present a different solution, which is applicable quite generally.

Our solution treats sets of modal formulas as elements not of a transition system but of an *algebra*. We then cut down that algebra by a novel "co-founded part" construction, and this gives the final coalgebra.

1.2 Structure of Paper

The paper is in three sections.

In Section 2, we introduce our main construction: the co-founded part of an algebra. We see how this construction, applied to a suitable algebra, gives a final coalgebra.

In Section 3 we generalize our work to any modal logic on a dual adjunction. We see how such a logic, if it is expressive, will always give a suitable corecursive algebra so that our final coalgebra construction can be applied.

In Section 4 we further generalize our results, from **Set** to other categories equipped with a factorization system. We look at two examples of particular interest:

- **Poset**, giving a model of similarity;
- $\mathbf{Set}^{\mathsf{op}}$, giving a new construction of initial algebras.

1.3 Notation

Let X be a set.

- We write $\mathcal{P}X$ for the poset of subsets of X, ordered by inclusion.
- We write $\mathrm{EqRel}(X)$ for the poset of equivalence relations on X, ordered by inclusion.

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[1] The sum is a proper class, but this may be avoided e.g. by including only coalgebras carried by a subset of $\mathbb{N}$.

- For $U \in \mathcal{P}X$, we write ${}^{\circ}U$ for U regarded as a set, and $i_U : {}^{\circ}U \to X$ for the inclusion.
- For $(\equiv) \in \mathrm{EqRel}(X)$ we write $X/\equiv$ for the quotient set, and $e_{\equiv} : X \to (X/\equiv)$ for $x \mapsto [x]_{\equiv}$.
- For $U \subseteq V \in \mathcal{P}X$ we write $i_{U,V} : {}^{\circ}U \to{}^{\circ} V$ for the inclusion.
- For $(\equiv) \subseteq (\equiv') \in \mathrm{EqRel}(X)$ we write $e_{\equiv,\equiv'} : (X/\equiv) \to (X/\equiv')$ for $[x]_{\equiv} \mapsto [x]_{\equiv'}$.

In diagrams, $\rightarrowtail$ indicates an injection and $\twoheadrightarrow$ a surjection.

A *partial function* from a set X to a set Y is a pair (U, f) of $U \in \mathcal{P}X$ and $f : U \to Y$. We write $(U, f) \sqsubseteq (V, g)$ is when $U \subseteq V$ and

$$\begin{tikzcd} U \arrow[r, "i_{U,V}"] \arrow[dr, "f"'] & V \arrow[d, "g"] \\ & Y \end{tikzcd}$$

. We write $X \rightharpoonup Y$ for the poset of partial functions ordered by $\sqsubseteq$.

2 Solving the Problem

This section solves the problem set out in Section 1.1. We construct an algebra of *theories*. Then we describe how every algebra has a special subalgebra called the *co-founded part*. The co-founded part of our algebra of theories provides a final coalgebra as required.

2.1 The B-Algebra of Theories

Our first step is to obtain a B-algebra from the modal logic, where B is our endofunctor $X \mapsto (\mathcal{P}_c X)^{\mathcal{A}}$.

Say that a *theory* is any set of $\Diamond$-layered formulas; this is a crude notion of theory, with no requirement of deductive closure. Let Form be the set of all theories. Our B-algebra is (Form, α) where $\alpha : B\,\mathsf{Form} \to \mathsf{Form}$ can be thought of as describing how the theory of a state x can be obtained from the theories of its successors. Explicitly, α sends $\mathcal{M} \in B\,\mathsf{Form}$ to the set of formulas $\langle a \rangle\ (\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg\psi_j)$ for which there exists $M \in \mathcal{M}_a$ such that $\forall i \in I.\ \phi_i \in M$ and $\forall j \in J.\ \psi_j \notin M$.

This B-algebra has two key properties. Firstly it is *corecursive*, which we explain in the next section. Secondly it is *injectively structured* i.e. α is an injection; we defer the proof of this until Section 3.4.

2.2 Corecursive Algebras

We reprise here the basic concepts of recursive coalgebras and corecursive algebras.

Let B be an endofunctor on a category $\mathcal{C}$. We write $\mathrm{Alg}(B)$ and $\mathrm{Coalg}(B)$ for the categories of B-algebras and B-coalgebras respectively. The evident bijection between isomorphically structured B-algebras and isomorphically structured B-coalgebras will be written $(-)^{-1}$, in either direction.

As explained in [33], a common patten for recursively defining a function $f : X \to Y$ is to first parse $x \in X$ into constituent parts, then apply f to each part, then combine the results. This motivated the following definition.

▶ **Definition 1.** [10, 11, 14, 32] A *B-coalgebra-to-algebra map* from a B-coalgebra (X, ζ) to a B-algebra (Y, θ) is a morphism $f : X \to Y$ satisfying

$$\begin{tikzcd}
BX \arrow[r, "Bf"] & BY \arrow[d, "\theta"] \\
X \arrow[u, "\zeta"] \arrow[r, "f"'] & Y
\end{tikzcd}$$

Equivalently, it is a fixpoint of the endofunction

$$\begin{tikzcd}
\mathcal{C}(X,Y) \arrow[r, "B_{X,Y}"] & \mathcal{C}(BX,BY) \arrow[r, "\mathcal{C}(\zeta,\theta)"] & \mathcal{C}(X,Y)
\end{tikzcd}$$

Such a map may be composed with a B-algebra map $(X', \zeta') \to (X, \zeta)$ or a B-coalgebra map $(Y, \theta) \to (Y', \theta')$ in the evident way.

► **Definition 2.**

1. A B-coalgebra is *recursive* when there is a unique map from it to each B-algebra.
2. Dually, a B-algebra is *corecursive* when there is a unique map from each B-coalgebra to it.

► **Proposition 3.**

1. $(-)^{-1}$ *gives a bijection between initial* B*-algebras and isomorphically structured recursive coalgebras.*
2. *Dually,* $(-)^{-1}$ *gives a bijection between final* B*-coalgebras and isomorphically structured corecursive algebras.*

Proof. By Lambek's lemma. ◄

Recursive coalgebras are an easily grasped concept, thanks to Taylor's characterization of recursive coalgebras as *well-founded* coalgebras in the case where $\mathcal{C} = \mathbf{Set}$ and B preserves inverse images [33, 32].

Corecursive algebras (other than free ones [2]) appear not to have such a simple characterization [11]. Still, it is evident that our B-algebra of theories in Section 2.1 is corecursive. The unique map from a B-coalgebra (X, ζ) to our algebra is $(\!|-|\!)_{X,\zeta}$.

2.3 The Co-founded Part of an Algebra

Certain elements of a B-algebra are said to be *co-founded*. This is a coinductively defined predicate. To get some intuition, consider first the case where B is presented by operations. For an element of a B-algebra to be co-founded, it must be of the form $c(y_i \mid i \in I)$ where each y_i is co-founded.

Now for the general case. Let B be an endofunctor on **Set**, and (Y, θ) a B-algebra.

► **Definition 4.** We define an endofunction p on $\mathcal{P}Y$ as follows. For $U \in \mathcal{P}Y$ we define $p(U) \subseteq Y$ to be the range of the composite

$$\begin{tikzcd}
B^{\circ}U \arrow[r, "Bi_U"] & BY \arrow[r, "\theta"] & Y
\end{tikzcd}$$

This gives a square

$$\begin{tikzcd}
B^{\circ}U \arrow[r, "Bi_U"] \arrow[d, two heads, "r_U"'] & BY \arrow[d, "\theta"] \\
{}^{\circ}p(U) \arrow[r, tail, "i_{p(U)}"'] & Y
\end{tikzcd}$$

We next see that p is monotone and r is natural.

▶ **Proposition 5.** *If $U \subseteq V \in \mathcal{P}Y$, then $p(U) \subseteq p(V)$ and*

$$\begin{tikzcd} B^{\circ}U \arrow[r, "Bi_{U,V}"] \arrow[d, two heads, "r_U"'] & B^{\circ}V \arrow[d, two heads, "r_V"] \\ {}^{\circ}p(U) \arrow[r, tail, "i_{p(U),p(V)}"'] & {}^{\circ}p(V) \end{tikzcd}$$

writing $i_{U,V}$ for the inclusion of U in V.

Proof. The diagram

$$\begin{tikzcd} {}^{\circ}U \arrow[rr, tail, "i_{U,V}"] \arrow[dr, tail, "i_U"'] & & {}^{\circ}V \arrow[dl, tail, "i_V"] \\ & Y & \end{tikzcd}$$

commutes, so

$$\begin{tikzcd} B^{\circ}U \arrow[rr, "Bi_{U,V}"] \arrow[d, two heads, "r_U"'] \arrow[dr, "Bi_U"] & & B^{\circ}V \arrow[d, two heads, "r_V"] \arrow[dl, "Bi_V"'] \\ {}^{\circ}p(U) \arrow[dr, tail, "i_{p(U)}"'] & BY \arrow[d, "\theta"] & {}^{\circ}p(V) \arrow[dl, tail, "i_{p(V)}"] \\ & Y & \end{tikzcd}$$

commutes.

Diagonal fill-in gives

$$\begin{tikzcd} B^{\circ}U \arrow[rr, "Bi_{U,V}"] \arrow[d, two heads, "r_U"'] & & B^{\circ}V \arrow[d, two heads, "r_V"] \\ {}^{\circ}p(U) \arrow[rr, dotted, no head, "n"'] \arrow[dr, tail, "i_{p(U)}"'] & & {}^{\circ}p(V) \arrow[dl, tail, "i_{p(V)}"] \\ & Y & \end{tikzcd}$$

So $p(U) \subseteq p(V)$ and $n = i_{p(U),p(V)}$. ◀

▶ **Definition 6.**

1. A *subalgebra* of (Y, θ) is $U \in \mathcal{P}Y$ for which there exists a (necessarily unique) function

$$\begin{tikzcd} B^{\circ}U \arrow[r, "Bi_U"] \arrow[d, dotted] & BY \arrow[d, "\theta"] \\ {}^{\circ}U \arrow[r, "i_U"'] & Y \end{tikzcd}$$

. Equivalently, it is a prefixpoint of p.

2. The least prefixpoint μp is called the *least subalgebra*.
3. The greatest postfixpoint μp is called the *co-founded part* of (Y, θ).

To summarize, we have B-algebra morphisms:

$$\begin{tikzcd} B^{\circ}\mu p \arrow[r, "Bi_{\mu p,\nu p}"] \arrow[d, two heads, "r_{\mu p}"'] & B^{\circ}\nu p \arrow[r, "Bi_{\nu p}"] \arrow[d, two heads, "r_{\nu p}"'] & BY \arrow[dd, "\theta"] \\ {}^{\circ}p(\mu p) \arrow[r, tail, "i_{p(\mu p),p(\nu p)}"'] \arrow[d, equal] & {}^{\circ}p(\nu p) \arrow[d, equal] & \\ {}^{\circ}\mu p \arrow[r, tail, "i_{\mu p,\nu p}"'] & {}^{\circ}\nu p \arrow[r, tail, "i_{\nu p}"'] & Y \end{tikzcd}$$

Clearly the least subalgebra and co-founded parts of (Y, θ) are both surjectively structured B-algebras. (More generally, a surjectively structured subalgebra is precisely a fixpoint of p.)

We next see that any map to (Y, θ) from either a surjectively structured algebra or a coalgebra has range contained in the co-founded part.

▶ **Lemma 7.**

1. *Any B-algebra homomorphism $f : (X, \phi) \to (Y, \theta)$ with ϕ surjective, factorizes uniquely as* $(X, \phi) \xrightarrow{g} ({}^\circ\nu p, r_{\nu p}) \overset{i_{\nu p}}{\rightarrowtail} (Y, \theta)$
2. *Any B-coalgebra-to-algebra-map $f : (X, \zeta) \to (Y, \theta)$ factorizes uniquely as*

$$(X, \zeta) \xrightarrow{g} ({}^\circ\nu p, r_{\nu p}) \overset{i_{\nu p}}{\rightarrowtail} (Y, \theta)$$

Proof. We encompass both cases by supposing a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "\zeta"] \arrow[dr, two heads, "\phi"'] & BX \arrow[r, "Bf"] & BY \arrow[d, "\theta"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Writing U for the range of f gives

$$\begin{tikzcd}
Z \arrow[r, "\zeta"] \arrow[d, two heads, "\phi"'] & BX \arrow[r, "Be"] \arrow[dr, "Bf"'] & B^\circ U \arrow[r, two heads, "r_U"] \arrow[d, "Bi_{U,Y}"] & {}^\circ p(U) \arrow[dd, tail, "i_{p(U)}"] \\
X \arrow[drrr, "f"] \arrow[d, two heads, "e"'] & & BY \arrow[dr, "\theta"] & \\
{}^\circ U \arrow[rrr, tail, "i_U"'] & & & Y
\end{tikzcd}$$

Diagonal fill-in then gives

$$\begin{tikzcd}
Z \arrow[r, "\zeta"] \arrow[d, two heads, "\phi"'] & BX \arrow[r, "Be"] & B^\circ U \arrow[r, two heads, "r_U"] & {}^\circ p(U) \arrow[dd, tail, "i_{p(U)}"] \\
X \arrow[d, two heads, "e"'] & & & \\
{}^\circ U \arrow[rrr, tail, "i_U"'] \arrow[uurrr, dotted] & & & Y
\end{tikzcd}$$

so U is a postfixpoint of p, so $U \subseteq \nu p$. There is a morphism

$$\begin{tikzcd}
& {}^\circ\nu p \arrow[dr, "i_{\nu p}"] & \\
X \arrow[ur, dotted, "g"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

viz. the composite $X \overset{e}{\twoheadrightarrow} {}^\circ U \overset{i_{U,\nu p}}{\rightarrowtail} \nu p$ because

$$\begin{tikzcd}
& {}^\circ\nu p \arrow[dr, bend left, "i_{\nu p}"] & \\
X \arrow[ur, "g"] \arrow[r, two heads, "e"] \arrow[rr, bend right, "f"'] & {}^\circ U \arrow[u, tail, "i_{U,\nu p}"'] \arrow[r, tail, "i_U"] & Y
\end{tikzcd}$$

Since $i_{\nu p}$ is monic, g is unique and

$$\begin{tikzcd}
Z \arrow[r, "\zeta"] \arrow[dr, "\phi"'] & BX \arrow[r, "Bg"] & B^\circ\nu p \arrow[d, "r_{\nu p}"] \\
& X \arrow[r, "g"'] & {}^\circ\nu p
\end{tikzcd}$$

commutes.

◀

▶ **Corollary 8.**

1. *The co-founded part of* (Y, θ) *is its coreflection into the full subcategory of* $\mathrm{Alg}(B)$ *on surjectively structured algebras.*
2. *If* (Y, θ) *is corecursive then so is its co-founded part.*

Proof. Each part follows from the corresponding part of Lemma 7. ◀

2.4 Injectively Structured Algebras

Let B be an endofunctor on **Set** preserving injections.

▶ **Lemma 9.** *Let* (Y, θ) *be an injectively structured* B*-algebra. For any* $U \in \mathcal{P}Y$*, the map* $r_U : B^\circ U \rightarrow^\circ p(U)$ *is an isomorphism.*

Proof. Def. 4 is factorizing an injection.. ◀

▶ **Theorem 10.** *The (co-founded part)*$^{-1}$ *of an injectively structured, corecursive* B*-algebra is a final* B*-coalgebra.*

Proof. The co-founded part is a corecursive B-algebra by Corollary 8(2) and isomorphically structured by Lemma 9. So we apply Proposition 3(2). ◀

To obtain an initial algebra, we may apply an old result [34, Theorem II.4]

▶ **Theorem 11.** *The least subalgebra of an injectively structured* B*-algebra is an initial* B*-algebra.*

Proof. Consider the endofunction q on $Y \rightharpoonup Z$ that sends a partial function (U, f) to the partial function $({}^\circ p(U),\ {}^\circ p(U) \xrightarrow{r_U^{-1}} B^\circ U \xrightarrow{Bf} BZ \xrightarrow{\phi} Z\ )$.

To show q monotone, if $(U, f) \sqsubseteq (V, g)$ i.e.

$$\begin{tikzcd}
& {^\circ U} \arrow[dl, hook', "i_U"'] \arrow[dr, "f"] \arrow[dd, "i_{U,V}"] & \\
Y & & Z \\
& {^\circ V} \arrow[ul, hook, "i_V"] \arrow[ur, "g"'] &
\end{tikzcd}$$

then Proposition 5 gives

$$\begin{tikzcd}
& {^\circ p(U)} \arrow[dl, hook', "i_{p(U)}"'] \arrow[r, "r_U^{-1}"'] \arrow[dd, "i_{p(U),p(V)}"] \arrow[rrr, bend left, "q(U,f)"] & B^\circ U \arrow[dr, "Bf"] \arrow[dd, "Bi_{U,V}"] & & \\
Y & & & BZ \arrow[r, "\phi"] & Z \\
& {^\circ p(V)} \arrow[ul, hook, "i_{p(V)}"] \arrow[r, "r_V^{-1}"] \arrow[rrru, bend right, "q(V,g)"'] & B^\circ V \arrow[ur, "Bg"'] & &
\end{tikzcd}$$

Now we have

$$\begin{tikzcd}
(Y \rightharpoonup Z) \arrow[r, "q"] \arrow[d, "\mathrm{dom}"'] & (Y \rightharpoonup Z) \arrow[d, "\mathrm{dom}"] \\
\mathcal{P}Y \arrow[r, "p"'] & \mathcal{P}Y
\end{tikzcd}$$

Since $Y \rightharpoonup Z$ has and dom preserves suprema of ordinal chains, we obtain $\mathrm{dom}(\mu q) = \mu p$. Therefore μq is the unique fixpoint of q whose domain is μp, i.e. the unique B-algebra homomorphism from $(\mu p, r_{\mu p})$ to (Z, ϕ). ◀

Returning to our example: we began in Section 2.1 with a corecursive B-algebra of theories that is injectively structured (though we have still to prove that). By Theorem 10, its (co-founded part)$^{-1}$ is a final coalgebra; and by Theorem 11, its least subalgebra is an initial algebra. Both are constructed purely from the logic, as stipulated in Section 1.1.

3 Final Coalgebras From Modal Logic on a Dual Adjunction

In the previous section, we saw an example where modal formulas give rise to an injectively structured corecursive algebra, as required for Theorem 10. We shall now see how this arises in the general setting of modal logic over a dual adjunction [9, 12, 19, 20, 27]. We begin with an explanation of this formulation of modal logic, based on [20].

3.1 Dual Adjunctions

An adjunction $F \dashv G \ : \ \mathcal{D} \to \mathcal{C}$ may be described by an isomorphism

$$\mathcal{C}(X, G\Phi) \cong \mathcal{D}(FX, \Phi) \qquad \text{natural in } X \in \mathcal{C}^{\mathrm{op}}, \Phi \in \mathcal{D}.$$

This gives a functor $\mathcal{O} \ : \ \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ (also known as a “bimodule” or “profunctor”), sending (X, Φ) to either $\mathcal{C}(X, G\Phi)$ or $\mathcal{D}(FX, \Phi)$; it does not matter which, since they are isomorphic. This suggests an alternative (equivalent) definition of adjunction: as a functor $\mathcal{O}$ together with two isomorphisms

$$\mathcal{C}(X, U\Phi) \cong \mathcal{O}(X, \Phi) \cong \mathcal{D}(FX, \Phi) \qquad \text{natural in } X \in \mathcal{C}^{\mathrm{op}}, \Phi \in \mathcal{D}.$$

For example, we can describe the adjunction $\mathcal{P} \dashv \mathcal{P} \ : \ \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ by the isomorphisms

$$\mathbf{Set}(X, \mathcal{P}\Phi) \cong \mathbf{Rel}(X, \Phi) \cong \mathbf{Set}(\Phi, \mathcal{P}X) \qquad \text{natural in } X \in \mathbf{Set}^{\mathrm{op}}, \Phi \in \mathbf{Set}^{\mathrm{op}}.$$

where $\mathbf{Rel}(X, \Phi)$ is the set of relations between X and Φ. In particular, if X is the set of states of a transition system (X, ζ) and Φ is the set of formulas of our logic, the satisfaction relation $\models$ is an element of $\mathbf{Rel}(X, \Phi)$. It corresponds to a map $X \to \mathcal{P}\Phi$ viz. $(\!|-|\!)_{X,\zeta}$ and also to a map $\Phi \to \mathcal{P}X$ sending each formula to its satisfying states. This example is a dual adjunction between $\mathbf{Set}$ and itself; more generally we want a dual adjunction between a category $\mathcal{C}$, whose objects we think of as sets of states, and a category $\mathcal{D}$, whose objects we think of as sets of formulas. We summarize as follows.

▶ **Definition 12.** A *dual adjunction* for a category $\mathcal{C}$ consists of
- a category $\mathcal{D}$
- a functor $\mathcal{O} \ : \ \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$
- functors $\mathcal{O}_* \ : \ \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $\mathcal{O}^* \ : \ \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, and isomorphisms

$$\mathcal{C}(X, \mathcal{O}^*\Phi) \ \cong \ \mathcal{O}(X, \Phi) \ \cong \ \mathcal{D}(\Phi, \mathcal{O}_*X) \qquad \text{natural in } X \in \mathcal{C}^{\mathrm{op}}, \Phi \in \mathcal{D}^{\mathrm{op}}.$$

natural in $X \in \mathcal{C}^{\mathrm{op}}$ and $\Phi \in \mathcal{D}^{\mathrm{op}}$.

3.2 Modal Logic on a Dual Adjunction

Recall that, for a set X of states, BX is the set of *single-step behaviours* ending in a state in X. In our example $BX = \mathcal{P}_c(\mathcal{A} \times X)$.

As explained in [20], there are two essential ingredients required to build a modal logic.

Firstly the syntax. For a set Φ of atoms, let $L\Phi$ be the set of *single-layer formulas* with atoms drawn from Φ. In our example, following (1), $L\Phi$ is the set of formulas

$$\langle a \rangle \; (\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg \psi_j) \qquad\qquad (\phi_i, \psi_j \in \Phi)$$

More succinctly $L\Phi = \mathcal{A} \times \mathcal{P}_c\Phi \times \mathcal{P}_c\Phi$. General formulas form an initial L-algebra.

Secondly the semantics. Given a relation $\models$ between X and Φ saying which states satisfy which atoms, let $\rho_{X,\Phi}(\models)$ be the induced relation between single-step behaviours (BX) and single-layer formulas ($L\Phi$). In our example, following (2), we have for $s \in BX$

$$s \; (\rho_{X,\Phi}(\models)) \; \langle a \rangle \; (\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg \psi_j) \iff \exists x \in s_a. \; (\forall i \in I. x \models \phi_i \wedge \forall j \in J. \; x \not\models \psi_j)$$

The general situation is summarized as follows.

▶ **Definition 13.** For an endofunctor B on $\mathcal{C}$, a *modal logic on a dual adjunction*, or just *modal logic*, consists of
- a dual adjunction $(\mathcal{D}, \mathcal{O})$ for $\mathcal{C}$
- an endofunctor L on $\mathcal{D}$, the *syntax functor*
- a map $\rho_{X,\Phi} \; : \; \mathcal{O}(X, \Phi) \to \mathcal{O}(BX, L\Phi)$ natural in $X \in \mathcal{C}^{\mathsf{op}}, \Phi \in \mathcal{D}^{\mathsf{op}}$, called the *semantics*.

We have expressed the semantics ρ in terms of $\mathcal{O}$, but could alternatively express it in terms of $\mathcal{O}_*$ or $\mathcal{O}^*$.

▶ **Proposition 14.** *Let $(\mathcal{D}, \mathcal{O}, L, \rho)$ be a modal logic for an endofunctor B on $\mathcal{C}$.*

1. *There is a unique natural transformation*

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{op}} \arrow[r, "\mathcal{O}_*"] \arrow[d, "B"'] \arrow[dr, phantom, "\Downarrow \rho_*"] & \mathcal{D} \arrow[d, "L"] \\
\mathcal{C}^{\mathsf{op}} \arrow[r, "\mathcal{O}_*"'] & \mathcal{D}
\end{tikzcd}$$

from which $\rho_{X,\Phi}$ may be recovered via

$$\begin{tikzcd}
\mathcal{O}(X, \Phi) \arrow[rr, "\rho_{X,\Phi}"] \arrow[d, dash, "\cong"'] & & \mathcal{O}(BX, L\Phi) \arrow[d, dash, "\cong"] \\
\mathcal{D}(\Phi, \mathcal{O}_*X) \arrow[r, "L_{\Phi,\mathcal{O}_*X}"'] & \mathcal{D}(L\Phi, L\mathcal{O}_*X) \arrow[r, "\mathcal{D}(L\Phi, \rho_*^X)"'] & \mathcal{D}(L\Phi, \mathcal{O}_*BX)
\end{tikzcd}$$

2. *There is a unique natural transformation*

$$\begin{tikzcd}
\mathcal{D}^{\mathsf{op}} \arrow[r, "\mathcal{O}^*"] \arrow[d, "L"'] \arrow[dr, phantom, "\Downarrow \rho^*"] & \mathcal{C} \arrow[d, "B"] \\
\mathcal{D}^{\mathsf{op}} \arrow[r, "\mathcal{O}^*"'] & \mathcal{C}
\end{tikzcd}$$

from which $\rho_{X,\Phi}$ may be recovered via

$$\begin{tikzcd}
\mathcal{O}(X, \Phi) \arrow[rr, "\rho_{X,\Phi}"] \arrow[d, dash, "\cong"'] & & \mathcal{O}(BX, L\Phi) \arrow[d, dash, "\cong"] \\
\mathcal{C}(X, \mathcal{O}^*\Phi) \arrow[r, "B_{X,\mathcal{O}^*\Phi}"'] & \mathcal{C}(BX, B\mathcal{O}^*\Phi) \arrow[r, "\mathcal{C}(BX, \rho^*_\Phi)"'] & \mathcal{C}(BX, \mathcal{O}^*L\Phi)
\end{tikzcd}$$

Before proving this, let us explain these maps in logical terms.

- ρ_*^X associates, to each single-layer formula ϕ built from properties on X, the property on single-step behaviours ending in X that ϕ describes. In our example logic, it sends $\langle a\rangle\ (\bigwedge_{i\in I}\phi_i \wedge \bigwedge_{j\in J}\neg\psi_j)$, where ϕ_i and ψ_j are subsets of X, to the set of $s \in BX$ such that $\exists x \in s_a.\ (\forall i \in I.\ x \in \phi_i \wedge \forall j \in J.\ x \notin \psi_j)$
- ρ_Φ^* associates, to each single-step behaviour ending in theories on Φ, the theory consisting of single-layer formulas constructed from Φ satisfied by that behaviour. In our example, it sends $s \in B\mathcal{P}\Phi$ to the set of formulas $\langle a\rangle\ (\bigwedge_{i\in I}\phi_i \wedge \bigwedge_{j\in J}\neg\psi_j)$, with $\phi_i, \psi_j \in \Phi$, such that $\exists M \in s_a.\ (\forall i \in I.\ \phi_i \in M \wedge \forall j \in J.\ \psi_j \notin M)$

Proof. (of Proposition 14) For part (1), we use calculus of ends.

$$\begin{aligned}
& \text{Maps}\quad \mathcal{O}(X,\Phi) \to \mathcal{O}(BX, L\Phi)\quad \text{natural in } X,\ \Phi\\
\cong\ & \text{Maps}\quad \mathcal{D}(\Phi,\mathcal{O}_*X) \to \mathcal{D}(L\Phi,\mathcal{O}_*BX)\quad \text{natural in } X,\ \Phi\\
\cong\ & \int_{X,\Phi} \mathbf{Set}(\mathcal{D}(\Phi,\mathcal{O}_*X), \mathcal{D}(L\Phi,\mathcal{O}_*BX))\\
\cong\ & \int_X \mathcal{D}(L\mathcal{O}_*X, \mathcal{O}_*BX)\qquad \text{(by the Yoneda Lemma)}\\
\cong\ & \text{Maps}\quad L\mathcal{O}_*X \to \mathcal{O}_*BX\quad \text{natural in } X
\end{aligned}$$

Tracing through the bijection backwards gives the constructions described. Part (2) is proved similarly. ◀

Explicitly, ρ_*^X is the image of $\mathsf{id}_{\mathcal{O}_*X}$ in the composite

$$\mathcal{D}(\mathcal{O}_*X,\mathcal{O}_*X)\ \cong\ \mathcal{O}(X,\mathcal{O}_*X)\ \xrightarrow{\rho_{X,\mathcal{O}_*X}}\ \mathcal{O}(BX, L\mathcal{O}_*X)\ \cong\ \mathcal{D}(L\mathcal{O}_*X,\mathcal{O}_*BX)$$

and ρ_Φ^* is the image of $\mathsf{id}_{\mathcal{O}^*\Phi}$ in the composite

$$\mathcal{C}(\mathcal{O}^*\Phi,\mathcal{O}^*\Phi)\ \cong\ \mathcal{O}(\mathcal{O}^*\Phi,\Phi)\ \xrightarrow{\rho_{\mathcal{O}^*\Phi,\Phi}}\ \mathcal{O}(B\mathcal{O}^*\Phi, L\Phi)\ \cong\ \mathcal{C}(B\mathcal{O}^*\Phi,\mathcal{O}^*L\Phi)$$

3.3 Relating States to Modal Formulas

In this section, let $(\mathcal{D},\mathcal{O},L,\rho)$ be a modal logic for an endofunctor B on $\mathcal{C}$.

We want to relate B-coalgebras (transition systems) to the initial L-algebra (the set of formulas).

▶ **Definition 15.**

1. An $(\mathcal{O}|\rho)$-*connection* between a B-coalgebra (X,ζ) and an L-coalgebra (Φ,γ) is a fixpoint of the endofunction $\mathcal{O}(X,\Phi) \xrightarrow{\rho_{X,\Phi}} \mathcal{O}(BX, L\Phi) \xrightarrow{\mathcal{O}(\zeta,\gamma)} \mathcal{O}(X,\Phi)$.
2. Let $(\mathcal{O}|\rho)_*\ :\ \mathrm{Coalg}(B) \to \mathrm{Alg}(L)$ be the functor sending (X,ζ) to
$$(\mathcal{O}_*X,\ \ L\mathcal{O}_*X \xrightarrow{\rho_*^X} \mathcal{O}_*BX \xrightarrow{\mathcal{O}_*\zeta} \mathcal{O}_*X\ \ )$$
3. Let $(\mathcal{O}|\rho)^*\ :\ \mathrm{Coalg}(L) \to \mathrm{Alg}(B)$ be the functor sending (Φ,γ) to
$$(\mathcal{O}^*\Phi,\ \ B\mathcal{O}^*\Phi \xrightarrow{\rho_\Phi^*} \mathcal{O}^*L\Phi \xrightarrow{\mathcal{O}^*\gamma} \mathcal{O}^*\Phi\ \ )$$

▶ **Proposition 16.** *Let (X,ζ) be a B-coalgebra and (Φ,γ) an L-coalgebra. For $f \in \mathcal{O}(X,\Phi)$ corresponding to $f_*\ :\ \Phi \to \mathcal{O}_*X$ and $f^*\ :\ X \to \mathcal{O}^*\Phi$, the following are equivalent.*

- *f is an $(\mathcal{O}|\rho)$-connection between (X,ζ) and (Φ,γ).*
- *f_* is an L-coalgebra-to-algebra map $(\Phi,\gamma) \to (\mathcal{O}|\rho)_*(X,\zeta)$.*
- *f^* is a B-coalgebra-to-algebra map $(X,\zeta) \to (\mathcal{O}|\rho)^*(\Phi,\gamma)$.*

Proof. The following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}(X,\mathcal{O}^*\Phi) \arrow[r, "B_{X,\mathcal{O}^*\Phi}"] \arrow[d, dash, "\cong"'] & \mathcal{C}(BX,B\mathcal{O}^*\Phi) \arrow[r, "\mathcal{C}(BX,\rho^*_\Phi)"'] \arrow[rr, bend left, "\mathcal{C}(\zeta,(\rho^*_\Phi;\mathcal{O}^*\gamma))"] & \mathcal{C}(BX,\mathcal{O}^*L\Phi) \arrow[r, "\mathcal{C}(\zeta,\mathcal{O}^*\gamma)"'] \arrow[d, dash, "\cong"'] & \mathcal{C}(X,\mathcal{O}^*\Phi) \arrow[d, dash, "\cong"] \\
\mathcal{O}(X,\Phi) \arrow[rr, "\rho_{X,\Phi}"] \arrow[d, dash, "\cong"'] & & \mathcal{O}(BX,L\Phi) \arrow[r, "\mathcal{O}(\zeta,\gamma)"] \arrow[d, dash, "\cong"'] & \mathcal{O}(X,\Phi) \arrow[d, dash, "\cong"] \\
\mathcal{D}(\Phi,\mathcal{O}_*X) \arrow[r, "L_{\Phi,\mathcal{O}_*X}"'] & \mathcal{D}(L\Phi,L\mathcal{O}_*X) \arrow[r, "\mathcal{D}(L\Phi,\rho^X_*)"] \arrow[rr, bend right, "\mathcal{D}(\gamma,(\rho^X_*;\mathcal{O}_*\zeta))"'] & \mathcal{D}(L\Phi,\mathcal{O}_*BX) \arrow[r, "\mathcal{D}(\gamma,\mathcal{O}_*\zeta)"] & \mathcal{D}(\Phi,\mathcal{O}_*X)
\end{tikzcd}$$

An $(\mathcal{O}|\rho)$-connection between (X,ζ) and (Φ,γ) is a fixpoint of the central line.

An L-coalgebra-to-algebra map $(\Phi,\gamma) \to (\mathcal{O}|\rho)_*(X,\zeta)$ is a fixpoint of the bottom line.

A B-coalgebra-to-algebra map $(X,\zeta) \to (\mathcal{O}|\rho)^*(\Phi,\gamma)$ is a fixpoint of the top line. ◀

▶ **Corollary 17.**
1. *The functor $(\mathcal{O}|\rho)_*$ sends recursive B-coalgebras to corecursive L-algebras.*
2. *The functor $(\mathcal{O}|\rho)^*$ sends recursive L-coalgebras to corecursive B-algebras.*

Now suppose we have an initial L-algebra μL, and regard this as the set of all L-formulas. Let (X,ζ) be a B-coalgebra.
- The unique $(\mathcal{O}|\rho)$-connection between (X,ζ) and $(\mu L)^{-1}$ is regarded as the satisfaction relation $\models$ between states and formulas.
- The unique L-coalgebra-to-algebra map $(\mu L)^{-1} \to (\mathcal{O}|\rho)_*(X,\zeta)$ can be described more simply as the unique L-algebra homomorphism $\mu L \to (\mathcal{O}|\rho)_*(X,\zeta)$. We regard this as the function sending each L-formula to the set of states that satisfy it.
- The unique B-coalgebra-to-algebra map $(X,\zeta) \to (\mathcal{O}|\rho)^*((\mu L)^{-1})$ is regarded as the function $(\!|-|\!)_{X,\zeta}$ sending each state to the set of formulas it satisfies.

We have now seen that $(\mathcal{O}|\rho)^*((\mu L)^{-1})$ is a corecursive B-algebra. The other requirement of Theorem. 10, the injective structure, will be addressed in the next section.

▶ Remark. Proposition 16 and Corollary 17, as well as corresponding results for primitive recursion and corecursion, have recently appeared (for covariant adjunctions) as part of a general account of recursion schemes [17, Theorems 3.4 and 5.6].

3.4 Expressive Modal Logics

The key notion of [20] is the following abstract definition of an expressive modal logic.

▶ **Definition 18.** A modal logic $(\mathcal{D},\mathcal{O},L,\rho)$ for an injection-preserving endofunctor B on **Set** is said to be *expressive* when ρ^*_Φ is injective for every $\Phi \in \mathcal{D}$.

For such a logic, we can state our main theorem.

▶ **Theorem 19.** *Let $(\mathcal{D},\mathcal{O},L,\rho)$ be an expressive modal logic for an injection-preserving endofunctor B on* **Set** *. Let μL be an initial algebra for L.*
1. *The (co-founded part)$^{-1}$ of $(\mathcal{O}|\rho)^*((\mu L)^{-1})$ is a final B-coalgebra.*
2. *The least subalgebra of $(\mathcal{O}|\rho)^*((\mu L)^{-1})$ is an initial B-algebra.*

Proof. Prop. 2(1) tells us that $(\mu L)^{-1}$ is an isomorphically structured recursive coalgebra. So $(\mathcal{O}|\rho)^*((\mu L)^{-1})$ is a corecursive B-algebra by Corollary 17(2), and injectively structured by the definition of $(\mathcal{O}|\rho)^*$. So part (1) follows from Theorem 10, and part (2) from Theorem 11. ◀

It remains to establish that our example of a modal logic from described in Sect. 3.2 is expressive.

Proof. (essentially the same as [20, Section 6.1])

Let $s, t \in B\mathcal{O}^*\Phi$ with $\rho^*_\Phi s = \rho^*_\Phi t$. For $a \in \mathcal{A}$, we want to show $s_a = t_a$.

Let $M \in s_a$. Define the sets $I = \{N \in t_a \mid M \not\subseteq N\}$ and $J = \{N \in t_a \mid N \not\subseteq M\}$, which are countable since t_a is. For $N \in I$ choose $\phi_N \in M \setminus N$, and for $N \in J$ choose $\psi_N \in N \setminus M$. The formula $\langle a \rangle \left(\bigwedge_{N \in I} \phi_N \wedge \bigwedge_{N \in J} \neg\psi_N \right)$ is in $\rho^*_\Phi s = \rho^*_\Phi t$, so there is $P \in t_a$ such that

1. for all $N \in I$, $\phi_N \in P$ (implying $P \neq N$);
2. for all $N \in J$, $\psi_N \notin P$ (implying $P \neq N$).

(1) gives $P \notin I$, so $M \subseteq P$. (2) gives $P \notin J$, so $P \subseteq M$. Thus $P = M$, giving $M \in t_a$.

Likewise $M \in t_a$ implies $M \in s_a$. ◀

4 Beyond Set

In this section we generalize our results to categories other than **Set**. We give our general results in Section 4.1, and examine the special cases of **Poset** in Section 4.2 and $\mathbf{Set}^{\mathrm{op}}$ in Section 4.3.

4.1 General Results

We work with a category $\mathcal{C}$ equipped with an *orthogonal factorization system* $(\mathcal{E}, \mathcal{M})$. This consists of two lluf subcategories $\mathcal{E}$ and $\mathcal{M}$ of $\mathcal{C}$, containing all isomorphisms, with every $\mathcal{C}$-morphism $X \xrightarrow{f} Y$ having an essentially unique factorization into an $\mathcal{E}$-morphism $X \overset{e}{\twoheadrightarrow} U$ and a $\mathcal{M}$-morphism $U \overset{m}{\rightarrowtail} Y$ See e.g. [3] for an account of these systems. Here are some examples:

- on **Set**, let $\mathcal{E}$ consist of surjections, and $\mathcal{M}$ of injections;
- on **Poset**, let $\mathcal{E}$ consist of surjective maps, and $\mathcal{M}$ of order-reflecting (hence injective) maps;
- on $\mathbf{Set}^{\mathrm{op}}$, let $\mathcal{E}$ consists of injections, and $\mathcal{M}$ of surjections.

We further require that all $\mathcal{M}$-morphisms are monic, and the $\mathcal{M}$-subobjects of any object form a small complete lattice. (A stronger assumption, which apparently includes all examples of interest, is that $\mathcal{C}$ is equipped with a well-powered sink factorization system. See [3] for an account of source and sink factorization.)

Let B be an $\mathcal{M}$-preserving endofunctor on $\mathcal{C}$. We adapt our results as follows; the proofs are essentially unchanged.

▶ **Theorem 20** (generalizing Theorem 10)**.** *Let (Y, θ) be an $\mathcal{M}$-structured, corecursive B-algebra. Then its (co-founded part)$^{-1}$ is a final B-coalgebra.*

▶ **Theorem 21** ([34, Theorem II.4], generalizing Theorem 11)**.** *Suppose that $\mathcal{M}$ has, and the inclusion $\mathcal{M} \subseteq \mathcal{C}$ preserves, colimits of ordinal chains. Then the least subalgebra of an injectively structured B-algebra is an initial B-algebra.*

Note the extra condition imposed here, needed to ensure the poset $Y \rightharpoonup Z$ has suprema of ordinal chains. The condition is true for **Poset**, but false for $\mathbf{Set}^{\mathrm{op}}$, where $\mathcal{M}$ lacks an initial object because it is the opposite of the category of surjections.

▶ **Theorem 22** (generalizing Theorem 19). *Let $(\mathcal{D}, \mathcal{O}, L, \rho)$ be a modal logic for B that is* $\mathcal{M}$-expressive, *i.e.* $\rho^*_\Phi \in \mathcal{M}$ *for every $\Phi \in \mathcal{D}$ [19]. Let μL be an initial algebra for L.*

1. *The (co-founded part)$^{-1}$ of $(\mathcal{O}|\rho)^*((\mu L)^{-1})$ is a final B-coalgebra.*
2. *Suppose that $\mathcal{M}$ has, and the inclusion $\mathcal{M} \subseteq \mathcal{C}$ preserves, colimits of ordinal chains. Then the least subalgebra of $(\mathcal{O}|\rho)^*((\mu L)^{-1})$ is an initial B-algebra.*

All the proofs of the above theorems are essentially the same as the ones we gave for **Set**.

We now look at two examples of this more general theory.

4.2 Poset Example

Notation. For a poset X we write

- $\mathsf{Up}\,X$ for the set of upsets
- $\mathsf{Down}\,X$ for the set of downsets
- $\mathsf{Up}_c\,X$ for the set of countably generated upsets
- $\mathsf{Down}_c\,X$ for the set of countably generated downsets.

all ordered by inclusion.

It was shown in [24] following [16, 18, 36] that the collection of states of image-countable transition systems can be characterized modulo *similarity* as a final coalgebra for the endofunctor B on **Poset** sending X to $(\mathsf{Down}_c\,X)^{\mathcal{A}}$. Similarity is also characterized by modal formulas of the form

$$\phi ::= \quad \langle a \rangle \bigwedge_{i \in I} \phi_i$$

Coalgebraic accounts of logic for similarity have been given in [5, 13, 35].

Once again we ask how to construct a final coalgebra directly from the logic. We answer this with the following modal logic $(\mathcal{D}, \mathcal{O}, L, \rho)$ for B.

- $\mathcal{D} = \mathcal{C} = \mathbf{Poset}$.
- $\mathcal{O}(X, \Phi) = \mathsf{Up}\,(X \times \Phi)$, because if $x \models \phi$ and $x \lesssim y$ and $\phi \Rightarrow \psi$ then $y \models \psi$.
- $\mathcal{O}_*$ and $\mathcal{O}^*$ are Up, with evident natural isomorphisms $\mathbf{Poset}(X, \mathsf{Up}\,\Phi) \cong \mathsf{Up}\,(X \times \Phi) \cong \mathbf{Poset}(\Phi, \mathsf{Up}\,X)$.
- L maps Φ to the set of formulas $\langle a \rangle \bigwedge_{i \in I} \phi_i$ modulo the following preorder: we have $\langle a \rangle \bigwedge_{i \in I} \phi_i \leqslant \langle b \rangle \bigwedge_{j \in J} \psi_j$ when $a = b$ and for all $j \in J$ there is $i \in I$ with $\phi_i \Rightarrow \psi_j$. More briefly L maps Φ to the poset $\mathcal{A} \times \mathsf{Up}_c \Phi$.
- $\rho_{X,\Phi}(\models)$ is the relation from BX to $L\Phi$ that relates s to $\langle a \rangle \bigwedge_{i \in I} \phi_i$ when $\exists x \in s_a.\ \forall i \in I.x \models \phi_i$.

We deduce the form of ρ_* and ρ^*.

- The function ρ^X_* maps $\langle a \rangle \bigwedge_{i \in I} \phi_i$, where ϕ_i and ψ_j are upsets of X, to the upset of $s \in BX$ such that $\exists x \in s_a.\ \forall i \in I.\ x \in \phi_i$
- The function ρ^*_Φ maps $s \in B\,\mathsf{Up}\,\Phi$ to the upset of formulas $\langle a \rangle \bigwedge_{i \in I} \phi_i$, with $\phi_i, \psi_j \in \Phi$, such that $\exists M \in s_a.\ \forall i \in I.\ \phi_i \in M$

To apply Theorem 22, we show that ρ^*_Φ is order-injective.

Proof. Suppose $s, t \in B\,\mathsf{Up}\,\Phi$ and $\rho^*_\Phi s \subseteq \rho^*_\Phi t$. For $a \in \mathcal{A}$, we want to show $s_a \subseteq t_a$.

Let $M \in s_a$. It is a downset in $\mathsf{Up}\,\Phi$ generated by $\{\phi_i \mid i \in I\}$, where I is countable. The formula $\langle a \rangle \bigwedge_{i \in I} \phi_i$ is in $\rho^*_\Phi(s)$, and hence is in $\rho^*_\Phi(t)$, so there is $N \in t_a$ such that $\forall i \in I.\, \phi_i \in N$. Hence $M \subseteq N$. Since t_a is a downset in $\mathsf{Up}\,\Phi$, we have $M \in t_a$. ◀

We therefore obtain both a final coalgebra and an initial algebra from Theorem 22.

4.3 The Dual Construction

We briefly consider the dual of Theorem 10, i.e. the case of Theorem 20 where $\mathcal{C} = \mathbf{Set}^{\mathsf{op}}$. Here the complete lattice $\mathcal{P}Y$ is replaced by the complete lattice $\mathrm{EqRel}(Y)$ of equivalence relations on Y.

Let B be an endofunctor on **Set**. We now need B to preserve surjections, not injections, but that is automatic since surjections are split epis. An injectively structured coalgebra is sometimes called an *extensional* coalgebra, after ZF set theory's Axiom of Extensionality.

Given a B-coalgebra (Y, ζ), and $(\equiv) \in \mathrm{EqRel}(Y)$, we define $p(\equiv) \in \mathrm{EqRel}(Y)$ to be the kernel of the composite

$$\begin{tikzcd} Y \arrow[r, "\zeta"] & BY \arrow[r, two heads, "Be_{\equiv}"] & B(Y/\equiv) \end{tikzcd}$$

where $e_{\equiv} : Y \to (Y/\equiv)$ sends $x \mapsto [x]_{\equiv}$. This gives a square

$$\begin{tikzcd} Y \arrow[r, two heads, "e_{p(\equiv)}"] \arrow[d, "\zeta"'] & Y/p(\equiv) \arrow[d, tail, "r_{\equiv}"] \\ BY \arrow[r, two heads, "Be_{\equiv}"'] & B(Y/\equiv) \end{tikzcd}$$

Then p is a monotone endofunction on $\mathrm{EqRel}(Y)$. Its least prefixpoint μp is called *extensional equivalence*, and the B-coalgebra $(Y/\mu p, r_{\mu p})$ is called the *extensional quotient* of (X, ζ). This is dual to the co-founded part construction. Therefore, dually to Corollary 8(1), the extensional quotient is a reflection of (Y, ζ) into the full subcategory of $\mathrm{Coalg}(B)$ on extensional coalgebras.

The dual of Theorem 10 is as follows.

▶ **Theorem 23.** *Let (X, ζ) be a surjectively structured, recursive B-coalgebra. Then its (extensional quotient)$^{-1}$ is an initial B-algebra.*

We illustrate this with the endofunctor $B : X \mapsto \mathcal{P}_c X$. Let X be the set of well-founded terms built from an ω-ary operation c and a constant d. Let ζ be the function

$$\begin{array}{rcl} c(t_i \mid i \in \mathbb{N}) & \mapsto & \{t_i \mid i \in \mathbb{N}\} \\ d & \mapsto & \{\,\} \end{array}$$

The $\mathcal{P}_c$-coalgebra (X, ζ) is surjectively structured, and it is recursive because it is well-founded [33]. Therefore, by Theorem 23 its (extensional quotient)$^{-1}$ is an initial $\mathcal{P}_c$-algebra.

5 Conclusions and Further Work

We now have a general machinery for building final coalgebras from modal formulas. Many interesting questions remain.

Having considered several least prefixpoints and greatest postfixpoints, we may ask how long it takes to reach these fixpoints.

- If the functor B preserves arbitrary intersections of subsets, then p will preserve nonempty intersections of subsets. Therefore νp will be reached at ω, cf. [37].
- If the functor B preserves κ-filtered colimits then p will do so too. Therefore μp will be reached at κ.

Our example functor $X \mapsto (\mathcal{P}_c X)^{\mathcal{A}}$ preserves intersections and ω_1-filtered colimits, so νp is reached at ω and μp at ω_1, at the latest.

But this leaves the question of functors on **Set** that do not preserve intersection, cf. [37], and also the examples in Section 4. We leave these for future work.

Another task remaining is to consider canonical models for infinitary modal logics, and the relationship with logical completeness results [26, 28, 30, 31].

Finally, there are intriguing connections to explore with the use of algebras, coalgebras and duality in [6, 8, 21], and with the recent general account of recursion schemes in [17].

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