

Reconstruction/Non-reconstruction Thresholds for Colourings of General Galton-Watson Trees*

Charilaos Efthymiou

Georgia Institute of Technology, College of Computing
266 Ferst Drive, Atlanta, GA-30332, USA
efthymiou@gmail.com

Abstract

The broadcasting models on trees arise in many contexts such as discrete mathematics, biology, information theory, statistical physics and computer science. In this work, we consider the k -colouring model. A basic question here is whether the assignment at the root affects the distribution of the colourings at the vertices at distance h from the root. This is the so-called *reconstruction problem*. For the case where the underlying tree is d -ary it is well known that $d/\ln d$ is the *reconstruction threshold*. That is, for $k = (1 + \epsilon)d/\ln d$ we have non-reconstruction while for $k = (1 - \epsilon)d/\ln d$ we have reconstruction.

Here, we consider the largely unstudied case where the underlying tree is chosen according to a predefined distribution. In particular, we consider the well-known Galton-Watson trees. The corresponding model arises naturally in many contexts such as the theory of spin-glasses and its applications on random Constraint Satisfaction Problems (rCSP). The study on rCSP focuses on Galton-Watson trees with offspring distribution $\mathcal{B}(n, d/n)$, i.e. the binomial with parameters n and d/n , where d is fixed. Here we consider a *broader* version of the problem, as we assume *general offspring distribution* which includes $\mathcal{B}(n, d/n)$ as a special case.

Our approach relates the corresponding bounds for (non)reconstruction to certain *concentration properties* of the offspring distribution. This allows to derive reconstruction thresholds for a very wide family of offspring distributions, which includes $\mathcal{B}(n, d/n)$. A very interesting corollary is that for distributions with expected offspring d , we get reconstruction threshold $d/\ln d$ under *weaker concentration* conditions than what we have in $\mathcal{B}(n, d/n)$.

Furthermore, our reconstruction threshold for the random colorings of Galton-Watson with offspring $\mathcal{B}(n, d/n)$, implies the reconstruction threshold for the random colourings of $G(n, d/n)$.

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1 Introduction

The broadcasting models on trees and the closely related reconstruction problem are studied in statistical physics, biology, communication theory, e.g. see [8, 21, 14]. Our work is motivated from the study of *random Constraint Satisfaction Problems* (rCSP) such as random graph colouring, random k -SAT etc. This is mainly because the models on random trees capture some of the most fundamental properties of the corresponding models on random (hyper)graphs, e.g. [7, 15, 20].

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The most fundamental problem in the study of broadcasting models is to determine the reconstruction/non-reconstruction threshold. I.e. whether the configuration of the root biases the distribution of the configuration of distant vertices. The transition from non-reconstruction to reconstruction can be achieved by adjusting appropriately the parameters of the model. Typically, this transition exhibits a *threshold behaviour*. So far, the main focus of the study was to determine the precise location of this threshold for various models when the underlying graph is a fixed tree, mostly regular. In these cases, typically the reconstruction threshold is expressed in terms of the maximum degree of the underlying tree, e.g. [3, 26, 2, 4].

In a lot of applications, e.g. phylogeny reconstruction, rCSP, usually the underlying tree is random. Motivated by such problems, in this work we study the reconstruction problem for the colouring model when the underlying tree is chosen according to some predefined probability distribution. In particular, we consider *Galton-Watson* trees (GW-trees) with some *general* offspring distribution.

In our setting, the main technical challenge is to deal with the so-called “effect of high degrees”. That is, we expect to have vertices in the tree which are of degree much higher than the expected offspring. The deviation from the expected degree is so large that expressing the (non)reconstruction bounds in terms of maximum degree leads to highly suboptimal results. Similar challenges appear in related problems in random graphs $G(n, d/n)$ e.g. sampling colourings [11, 10, 13, 27].

It is a folklore conjecture that when the offspring distribution is “reasonably” concentrated about its expectation, then the reconstruction threshold can be expressed in terms of the expected offspring of the underlying tree. Somehow, the concentration makes the high degree vertices sufficiently rare, such that their effect on the phenomenon is negligible. Our aim is to make the intuitive base of this relation *rigorous* by adopting the most generic assumptions about the offspring distribution.

More specifically, our result summarizes as follows: We provide a concentration criterion for the distributions over the non-negative integers about the expectation. For a GW-tree with offspring distribution that satisfies this criterion, the transition from non-reconstruction to reconstruction exhibits a threshold behaviour at the critical point $d/\ln d$, where d is the expected offspring. Interestingly, the aforementioned concentration criterion is much weaker than the standard tail bounds we have for many natural distributions, e.g. $\mathcal{B}(n, d/n)$ with fixed d .

On the other hand, when the concentration of the offspring distribution is not sufficiently high to provide thresholds, we still get upper and lower bounds for reconstruction and non-reconstruction, respectively. These bounds are expressed in terms of the tails of the offspring distribution.

Concluding, let us remark that the reconstruction threshold we get for the random colourings of GW-tree with offspring $\mathcal{B}(n, d/n)$, allows to compute the corresponding threshold for the random colourings of $G(n, d/n)$ [7, 15, 20]. See Section 2.1 for more discussion.

2 Definitions and Results

For the sake of brevity, we define the colouring model and the reconstruction problem, first, in terms of a fixed complete Δ -ary T of height h , where $\Delta, h > 0$ are integers. Later we will extend these definitions w.r.t. GW trees.

The broadcasting models on a tree T are models where information is sent from the root over the edges to the leaves. For some finite set of spins (colours) $S = \{1, 2, \dots, k\}$, a

configuration on T is an element in $S^{V(T)}$, i.e. it is an assignment of spins to the vertices of T . The spin of the root r is chosen according to some initial distribution over S . The information propagates along the edges of the tree as follows: There is a $k \times k$ stochastic matrix M such that if the vertex v is assigned spin i , then its child u is assigned spin j with probability $M_{i,j}$. The k -colouring model we consider here corresponds to having M such that

$$M_{i,j} = \begin{cases} \frac{1}{k-1} & \text{for } i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

We let μ be the *uniform distribution* over the k -colourings of T . We also refer to μ as the Gibbs distribution. The broadcasting process gives rise to Gibbs distributions on the underlying tree T . Fixing the spin (colour assignment) at the root of T , the configuration we get after the broadcasting process has finished is distributed as in μ conditional the spin of the root.

The reconstruction problem can be cast very naturally in terms of the corresponding Gibbs distribution. More specifically, let $r(T)$ (or r_T) denote the root of the tree T . Also, let $L_h(T)$ be the set of vertices at distance h from the root $r(T)$. Finally, we let μ^i be the distribution μ conditional that the spin at r_T is $i \in S$. Reconstructibility is defined as follows:

► **Definition 1.** For any $i, j \in S$ let $\|\mu^i - \mu^j\|_{L_h}$ denote the total variation distance of the projections of μ^i and μ^j on L_h . We say that a model is *reconstructible* on a tree T if there exists $i, j \in S$ for which

$$\lim_{h \rightarrow \infty} \|\mu^i - \mu^j\|_{L_h(T)} > 0.$$

When the above limit is zero for every i, j , then we say that the model has *non-reconstruction*.

Non-reconstruction implies, also, that *typical* colourings of the vertices at level h of the tree have a vanishing effect on the distribution of the colouring of $r(T)$, as h grows.

For the colouring model on Δ -ary trees it is well-known that the reconstruction threshold is at the critical value $\Delta / \ln \Delta$, see [3, 22, 25, 26]. That is, for any given fixed $\epsilon > 0$ and sufficiently large Δ , we have non-reconstruction when $k \geq (1 + \epsilon)\Delta / \ln \Delta$ while for $k \leq (1 - \epsilon)\Delta / \ln \Delta$ we have reconstruction.

Rather than considering a fixed tree, here, we consider a Galton Watson tree (GW-trees) with some *general* offspring distribution. In particular, we let the following:

► **Definition 2.** Let ξ be a distribution over the non negative integers. We let \mathcal{T}_ξ denote a Galton-Watson tree with offspring distribution ξ . Also, given some integer $h > 0$, we let \mathcal{T}_ξ^h denote the restriction of \mathcal{T}_ξ to its first h levels¹.

For the sake of brevity any distribution ξ on the non-negative integers is represented as a stochastic vector. That is, for Z distributed as in ξ and any integer $i \geq 0$, it holds that $\Pr[Z = i] = \xi(i)$ (or ξ_i).

For the case of a random tree, e.g. Galton-Watson tree, the notion of reconstructibility, extends as follows:

¹ In other words, \mathcal{T}_ξ^h is the induced subtree of \mathcal{T}_ξ which contains all the vertices within graph distance h from the root.

► **Definition 3.** We say that a model is *reconstructible* on \mathcal{T}_ξ if there exists $i, j \in S$ for which

$$\lim_{h \rightarrow \infty} \mathbb{E} \|\mu^i - \mu^j\|_{L_h} > 0,$$

where the expectation is w.r.t. the instances of the tree. When the above limit is zero for every $i, j \in S$, then we say that the model has *non-reconstruction*.

So as to have a threshold behavior for reconstruction, it is natural to require a certain kind of parametrization for the offspring distribution ξ . This parametrization allows to adjust the expectation from low to high. In this work we assume that we are dealing with such distribution.

► **Definition 4.** Consider \mathcal{T}_ξ for some offspring distribution ξ with expected offspring d_ξ . For the k -colouring model on \mathcal{T}_ξ we have a *reconstruction threshold* θ for some function $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, if the following holds: For any $\alpha > 0$ and $d_\xi > d_\xi(\alpha)$, we have non-reconstruction when $k \geq (1 + \alpha)\theta(d_\xi)$, while we have reconstruction when $k \leq (1 - \alpha)\theta(d_\xi)$.

One of the main results of this work is to show that we have a threshold behaviour for the reconstruction/non-reconstruction transition for the k -colourings of \mathcal{T}_ξ when ξ is *well concentrated*. The notion of well concentration is defined as follows:

► **Definition 5.** A distribution ξ over the positive integers with expectation d_ξ is defined to be “well concentrated” if the following is true: There is an absolute constant $c > 0$ such that for any fixed $\gamma > 0$, sufficiently large d_ξ and any $x \geq (1 + \gamma)d_\xi$ it holds that

$$\sum_{j \geq x} \xi_j \leq x^{-c} \quad \text{and} \quad \sum_{j \leq (1-\gamma)d_\xi} \xi_j \leq (d_\xi)^{-c}. \quad (1)$$

The quantity c is independent of the distribution ξ . We do not compute the exact value of c but it is implicit from our analysis.

The following theorem is one of the main results in our work.

► **Theorem 6.** *Let ξ be a well concentrated distribution over the non-negative integers. Then, the colouring model on \mathcal{T}_ξ , with expected offspring d_ξ , has reconstruction threshold $d_\xi / \ln d_\xi$.*

The above theorem follows as a corollary of a more general and more technical result, Theorem 10. This theorem is more general as it covers non-threshold cases, too.

It is not hard to show that $\mathcal{B}(n, d/n)$ is well concentrated. This follows trivially by just using standard Chernoff bounds (e.g. [24]). Then, Theorem 6 implies the following corollary.

► **Corollary 7.** *Consider \mathcal{T}_ξ where ξ is the distribution $\mathcal{B}(n, d/n)$, where d is fixed. Then, the colouring model on \mathcal{T}_ξ , has reconstruction threshold $d / \ln d$.*

As a matter of fact, it is elementary to verify that $\mathcal{B}(n, d/n)$ is, by no means, the less well concentrated offspring distribution we can have. That is, a distribution with less heavy tails than $\mathcal{B}(n, d/n)$ can be well concentrated.

2.1 From Galton-Watson trees to Random Graphs

The non-reconstruction phenomenon in rCSP seems to be central in the algorithmic problems. In particular, it has been related to the *efficiency* of local algorithms which search for satisfying solutions. That is, when we have non-reconstruction, usually there is an efficient (simple) local algorithm which finds satisfying assignments efficiently e.g. [6, 16]. On the other hand, in the reconstruction regime there is no efficient algorithm which finds solutions.

For this reason, the transition from non-reconstruction to reconstruction on rCSPs has been attributed the name “algorithmic barrier”², see [1].

The ingenious, however, mathematically non-rigorous *Cavity Method*, introduced by physicists [18, 17], makes very impressive predictions about the fundamental properties of rCSP. One of the most interesting parts of these predictions involves the Gibbs distribution and its spatial mixing properties like reconstructibility. The Cavity Method predicts that the spatial mixing properties of the Gibbs distribution over the colouring of $G(n, d/n)$ can be studied by means of the Gibbs distribution of the k -colourings over a Galton-Watson tree with offspring distribution $\mathcal{B}(n, d/n)$, where d is fixed independent of n . That is, choose some vertex v in $G(n, d/n)$ and some fixed radius neighborhood around v . The projection of Gibbs distribution on this neighborhood is, somehow, “similar” to the corresponding Gibbs distribution over the Galton-Watson tree. The above line of arguments, led to conjecture that the colouring model on a random graph $G(n, d/n)$ has the same reconstruction threshold as that of the GW tree with offspring $\mathcal{B}(n, d/n)$.

All the above considerations from the Cavity Method have been studied on a rigorous basis in [7, 15, 20]. We have a quite accurate picture of the relation between the local projection of Gibbs distribution on $G(n, d/n)$ and the Gibbs distribution on the $\mathcal{B}(n, d/n)$ Galton-Watson tree. In particular, we have mathematically rigorous arguments which imply that indeed the reconstruction thresholds for $G(n, d/n)$ and GW-tree coincide as far as the colouring model is concerned³. That is, Corollary 7 implies that, indeed, the reconstruction threshold for the colouring model on $G(n, d/n)$ is $d/\ln d$.

3 High Level Description

In this section, we give a high level overview of how do we derive upper and lower bounds for reconstruction and non-reconstruction, respectively. Consider an instance of \mathcal{T}_ξ^h for some distribution ξ over the non-negative integers and some integer $h > 0$.

► **Remark.** For a set of vertices Λ in the tree, we use the term *random colouring of Λ* to indicate the following way of colouring Λ : Take a random colouring of the tree and keep only the colouring of the vertices in Λ . Also, when we refer to a *typical colouring of vertex set Λ* , we imply that this colouring is typical w.r.t. the aforementioned distribution.

Depending on the tails of ξ we choose appropriate quantities Δ_+ and Δ_- such that $\Delta_- \leq d_\xi \leq \Delta_+$. Given these two quantities we show that we have non-reconstruction for $k \geq (1 + \alpha)\Delta_+/\ln \Delta_+$ and we have reconstruction for $k \leq (1 - \alpha)\Delta_-/\ln \Delta_-$, for the colouring model on \mathcal{T}_ξ^h , where $\alpha > 0$ is fixed. We show (non)reconstruction by arguing about the *structure of \mathcal{T}_ξ^h* .

3.1 Non Reconstruction

First, we focus on non-reconstruction. Given Δ_+ , we define a set of structural specifications such that if \mathcal{T}_ξ^h satisfies them, then we have non-reconstruction for $k \geq (1 + \alpha)\Delta_+/\ln \Delta_+$. We should consider Δ_+ to be a parameter for the specifications.

In particular, given Δ_+ , we introduce the notion of *mixing vertex*. Roughly speaking, a vertex $v \in \mathcal{T}_\xi^h$ is mixing if the following is true: A typical k -colouring of the vertices at level h

² We should mention that this observation is empirical as there is no corresponding (rigorous) computational hardness result.

³ For more details on the convergence between the distribution on the GW-tree and $G(n, d/n)$, see [7].

(e.g. above remark) does not bias the colouring of v by too much when $k \geq (1 + \alpha)\Delta_+ / \ln \Delta_+$. A vertex is biased if it is forced to choose from a relatively small set of colours. Perhaps a simple example of a vertex u *not* being mixing is when the subtree rooted at u has minimum degree much larger than Δ_+ .

Whether some vertex in \mathcal{T}_ξ^h is mixing or not depends on the subtree that hangs below it. An inductive definition of a mixing vertex, roughly, is as follows: A non leaf vertex v is mixing if the number of its children is at most Δ_+ while no more than $o(\Delta_+)$ of its children are non-mixing vertices. We consider the leaves of the tree to be mixing vertices, by default.

Furthermore, our specifications require that the mixing vertices are *sufficiently many* and *well spread* over the tree. To be more specific, we want the following: Every path from the root of \mathcal{T}_ξ^h to a vertex at level h contains a sufficiently large number of vertices which are mixing. Additionally, we would like that the number of vertices at level h should not deviate significantly from their expectation.

Then, we argue that non-reconstruction holds for the colouring model on any, *arbitrary*, instance of \mathcal{T}_ξ^h which satisfies the aforementioned specifications when $k \geq (1 + \alpha)\Delta_+ / \ln \Delta_+$. The choice of $\Delta_+ \geq d_\xi$ is the smallest possible that guarantees that \mathcal{T}_ξ^h satisfies the structural specifications with probability that tends to 1 as $h \rightarrow \infty$.

For showing non-reconstruction, given a fixed tree of the desired structure, we use an idea introduced in [4]. The authors there show non-reconstruction by upper bounding appropriately the second moment of a quantity called “magnetization of the root”. This approach has turned out to be quite popular for showing non-reconstruction bounds for various models on fixed trees e.g. [3, 26, 2, 4]. Additionally to [4], our approach builds on the very elegant combinatorial formalization from [3], which uses the notion of *unbiasing boundary* to deal with the magnetization of the root.

The approach in [3], for Δ -ary trees, shows non-reconstruction by arguing that the typical colourings of the vertices at level h do not bias the colouring of the vertices in the largest part of the underlying (regular) tree. The additional challenge here is that the trees we consider are highly non-regular. So as to get an effect similar to that of an unbiasing boundary from the colorings at level h , we need to argue about the subtree structure of each vertex in the tree. At this point we use the specification requirement. In other words, the setting we develop here with the mixing vertices somehow allows, to a certain extent, to apply the idea of unbiasing boundaries to control the magnetization of the root of the non-regular trees we deal with.

3.2 Reconstruction

As opposed to non-reconstruction, the reconstruction bound is well known in the special case where the offspring distribution is $\mathcal{B}(n, d/n)$, e.g. [19, 25]. Our approach deviates from both [19, 25] in that it applies to GW-trees with a general offspring distributions, while it focuses on the structural properties of the underlying tree, i.e. as we do for the non-reconstruction case.

We are based on the following observation. Consider some fixed tree T of height h and some positive integer k . Take a random k -colouring of the vertices at level h of that tree. Consider, now, the probability that this colouring “freezes” the colouring of the root of T . The assignment at the root gets frozen when the colouring of the vertices at level h specifies *uniquely* the colouring at the root. A sufficient condition for reconstruction is that the probability that the colouring of the root gets frozen is bounded away from zero for any $h > 0$. The reconstruction bound for a Δ -ary tree follows exactly from this argument. That is, for $k \leq (1 - \alpha)\Delta / \ln \Delta$, a random colouring of $L_h(T)$ freezes the colour assignment of the root with probability bounded away from zero for any $h > 0$, see [25, 22].

The above argument extends naturally to the case of a non-regular tree T' of height h . More specifically, if T' has a h -level, Δ -ary subtree, rooted at $r(T')$, then the colouring model on T' has reconstruction for $k \leq (1 - \alpha)\Delta / \ln \Delta$.

As far as the reconstruction for \mathcal{T}_ξ is regarded, we work as follows: We consider some parameter Δ_- which depends on the offspring distribution ξ . We show that \mathcal{T}_ξ^h has a Δ_- -ary subtree with h levels rooted at $r(\mathcal{T}_\xi^h)$, with probability bounded away from zero for any $h > 0$. The considerations in the previous paragraphs and Definition 3, imply that the colouring model in \mathcal{T}_ξ has reconstruction for $k \leq (1 - \alpha)\Delta_- / \ln \Delta_-$. Our choice of Δ_- is the largest possible that guarantees exactly the subtree specification for \mathcal{T}_ξ^h .

4 Upper and Lower Bounds

We start our analysis by focusing on the upper and the lower bounds for reconstruction and non-reconstruction, respectively. Consider \mathcal{T}_ξ^h and the k -colouring model on this tree. We define appropriate quantities Δ_- and Δ_+ which depend on the statistics of the offspring distribution ξ . As far as Δ_+ is concerned, we have the following:

► **Definition 8.** Consider a distribution ξ over the non negative integers with expectation d_ξ . Given some fixed $\delta \in (0, 1/10)$, we let $\Delta_+ = \Delta_+(\delta) \geq d_\xi$ be the minimum integer such that the following holds: There are $q \in [0, 3/4)$ and $\beta \geq 4$, independent of d_ξ , such that

$$q \geq \sum_{i > \Delta_+} \xi_i + \Pr [\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}] \quad (2)$$

and

$$\sum_{t > \Delta_+} t \cdot \xi_t \leq \exp(-2\beta \ln d_\xi), \quad \Pr [\mathcal{B}(\Delta_+, q) > (\Delta_+)^{\delta}] \leq \exp(-2\beta \ln d_\xi). \quad (3)$$

We discuss how do we choose δ in the range $(0, 1/10)$ a bit later. Given ξ and δ we choose the minimum Δ_+ that satisfy the above requirements. Then we use Δ_+ as a parameter to specify a set of structural specifications for trees (roughly described in Section 3). For any instance of \mathcal{T}_ξ which satisfies these specification we have non-reconstruction for any $k \geq (1 + \alpha)\Delta_+ / \ln \Delta_+$, where α is fixed.

► **Remark.** It turns out that there is a relation between the quantities α and δ . This means that given α we should choose appropriately δ , or the other way around.

The notion of mixing vertex is related to Definition 8 in the following way: A vertex v in \mathcal{T}_ξ^h is mixing if the number of its children is at most Δ_+ , while at most $(\Delta_+)^{\delta}$ of them are non-mixing.

To get further intuition, perhaps, it is useful to consider the condition in (2) and its implication in terms of the mixing vertices. Since we need the tree to have sufficiently many and well-spread mixing vertices, it is natural to require that the probability of a vertex in \mathcal{T}_ξ^h to be mixing is sufficiently large regardless of its level in the tree. We satisfy exactly this requirement from (2).

Let q be an upper bound for the probability of each child of some $v \in \mathcal{T}_\xi^h$ to be non-mixing⁴. It is straightforward to show that the r.h.s. of (2) is an upper bound for v to be non-mixing. Moreover, if (2) holds, then clearly q is an upper bound for v to be non-mixing,

⁴ The probability of a vertex being non-mixing depends only on the subtree rooted at this vertex.

too. That is, if some vertex at some level l of the tree is non-mixing with probability at most q , then (2) guarantees that for any vertex at level $l - 1$ the probability of it being non-mixing has the same upper bound q . This implies that regardless of its level at the tree, each vertex v is mixing with probability at least $1 - q$. For further details see in Section 8.

As far as Δ_- is concerned, we have the following.

► **Definition 9.** Let ξ be a distribution over the non negative integers. Given some $\delta \in (0, 1/10)$, we let $\Delta_- = \Delta_-(\delta) \leq d_\xi$ be the maximum integer such that the following holds: There is $g \in [0, 3/4)$ such that

$$g \geq \sum_{i < \Delta_-} \xi_i + \sum_{i \geq \Delta_-} \xi_i \cdot \Pr[\mathcal{B}(i, 1 - g) < (\Delta_-) - (\Delta_-)^\delta]. \tag{4}$$

The arguments for reconstruction are based on showing that with probability bounded away from zero for any h , the following holds for \mathcal{T}_ξ^h : The root of \mathcal{T}_ξ^h has a subtree of height h such that each non leaf vertex has at least $\Delta_- - (\Delta_-)^\delta$ many children. In the full version of this extended abstract, in [12], we show that the condition in (4) guarantees that \mathcal{T}_ξ^h has exactly this property.

The following theorem is the main technical result of this work. The trees considered in Theorem 10 do not necessarily have well concentrated offspring distribution ξ .

► **Theorem 10.** *Let some fixed $\alpha > 0$. Consider an instance of \mathcal{T}_ξ^h such that the expected offspring d_ξ is sufficiently large. Set $\delta = \min\{\alpha/2, 1/10\}$, i.e. the variable that specifies both Δ_+ and Δ_- .*

For μ , the Gibbs distribution over the k -colourings of \mathcal{T}_ξ^h , the following is true:

non-reconstruction: *For $k = (1 + \alpha)\Delta_+ / \ln \Delta_+$ and any $i, j \in [k]$ it holds that*

$$\mathbb{E} \|\mu^i - \mu^j\|_{L_h} \leq 8k^2(2\Delta_+)^{-0.45\delta h}.$$

reconstruction: *For $k = (1 - \alpha)\Delta_- / \ln \Delta_-$ there are $i, j \in [k]$ such that*

$$\mathbb{E} \|\mu^i - \mu^j\|_{L_h} \geq \frac{1}{4} \left(1 - \frac{2}{\log k}\right).$$

Both of the expectations above are taken w.r.t. the tree instances.

The whole proof of Theorem 10 appears in the full version of this work in [12]. In this extended abstract we provide a sketch for the proof of the most interesting part of the theorem, the non-reconstruction part. See in Section 5.

Given Theorem 10, it is elementary to show that Theorem 6 holds. I.e. given that the offspring distribution is well concentrated (Definition 5), we to show that Δ_- and Δ_+ are sufficiently close to each other. The derivations are simple they can be found in the full version of this paper in [12].

Notation. For any tree T we let $r(T)$ or r_T denote its root. Let $L_h(T)$ denote the set of vertices at graph distance h from $r(T)$. For every vertex $v \in T$, we define \tilde{T}_v the subtree of T as follows: Delete the edge between v and its parent in T . Then \tilde{T}_v is the connected component that contains v . We use the convention that $r(\tilde{T}_v) = v$.

We use capital letters of the Latin alphabet to indicate random variables which are colourings of the tree T , e.g. X, Y , etc. We use small letters of the greek alphabet to indicate fixed colourings, e.g. σ, τ , etc. We use the notation σ_Λ or $X(\Lambda)$ to indicate that the vertices in Λ have a colour assignment specified by the colourings σ, X , respectively.

Given a tree T , we let μ denote the Gibbs distribution for its k -colourings. Usually we consider μ under certain boundary conditions. That is, given some $\Lambda \subset T$, and σ , a k -colouring of T , we consider the Gibbs distribution conditional that the vertices in Λ have

fixed colouring σ_Λ . In this case we denote the corresponding Gibbs distribution as μ^{σ_Λ} . For $\Xi \subseteq T$ we let μ_Ξ denote the *marginal* of the Gibbs distribution for the vertices in Ξ . We denote marginals over the vertex set Ξ of a Gibbs distribution with boundary σ_Λ in the natural way, i.e. $\mu_\Xi^{\sigma_\Lambda}$.

5 Proof of Theorem 10 – Non Reconstruction

First, consider a fixed tree T of height h and we let $L = L_h(T)$. From [23] we have that

$$\|\mu^i - \mu\|_{r_T} \leq k \cdot \sum_{\sigma(L) \in [k]^L} \mu_L(\sigma_L) \cdot \|\mu^{\sigma(L)} - \mu\|_{r_T}. \tag{5}$$

Furthermore, from the definition of the total variation distance we have that

$$\begin{aligned} \sum_{\sigma(L) \in [k]^L} \mu_L(\sigma_L) \cdot \|\mu^{\sigma(L)} - \mu\|_{r_T} &= \frac{1}{2} \sum_{\sigma(L) \in [k]^L} \mu_L(\sigma_L) \cdot \sum_{c \in [k]} \left| \mu_{r_T}^{\sigma(L)}(c) - 1/k \right| \\ &= \frac{1}{2} \sum_{c \in [k]} \sum_{\sigma(L) \in [k]^L} \mu_L(\sigma_L) \cdot \left| \mu_{r_T}^{\sigma(L)}(c) - 1/k \right|. \end{aligned} \tag{6}$$

The quantity $\left| \mu_{r_T}^{\sigma(L)}(c) - 1/k \right|$, is usually called *magnetization of the root $r(T)$* , e.g. see [5]. The inner sum is the average magnetization at the root, w.r.t. boundary conditions at the set L . We bound this average magnetization by using the following standard result.

► **Proposition 11.** *Consider a fixed tree T of height h and some integer $k > 0$. For every $c \in [k]$ the following is true: Let X be a random k -colouring of T conditional that $X(r_T) = c$. It holds that*

$$\sum_{\sigma(L) \in [k]^L} \mu_L(\sigma(L)) \cdot \left| \mu_{r_T}^{\sigma(L)}(c) - 1/k \right| \leq \sqrt{\frac{1}{k} \cdot \left\| \mu^{X_L}(\cdot) - \mu^{Z^q_L}(\cdot) \right\|_{\{r_T\}}}, \tag{7}$$

where Z^q is random colouring of T conditional that $Z^q(r_T) = q$, where q maximizes the r.h.s. of (7).

The proof of Proposition 11, which is very similar to the proof of Lemma 1 in [4], appears also in the full version of this work in [12].

The quantity on the r.h.s. of (7) is a deterministic one, i.e. it depends only the tree T , c and k . We let

$$\mathbb{G}_{c,k}(T) = \left\| \mu^{X_L}(\cdot) - \mu^{Z^q_L}(\cdot) \right\|_{\{r_T\}}.$$

Consider \mathcal{T}_ξ^h as in the statement of Theorem 10. The quantity $\mathbb{G}_{c,k}(\mathcal{T}_\xi^h)$ is a random variable. In the light of (6), (5) and Proposition 11, it suffices to show that $\mathbb{E} \left[\mathbb{G}_{c,k}(\mathcal{T}_\xi^h) \right]$ tends to zero with h sufficiently fast, for any $c \in [k]$.

► **Definition 12 (Mixing Root).** Let Δ_+ and δ be as in the statement of Theorem 10. For a tree T of height h , its root is called mixing if the following holds: When $h = 0$, then $r(T)$ is mixing, by default. When $h > 0$, $r(T)$ is mixing if and only if $\text{deg}(r_T) \leq \Delta_+$ and there are at most $(\Delta_+)^{\delta}$ many vertices v children of $r(T)$ such that \tilde{T}_v does not have a mixing root.

► **Definition 13.** Given $\zeta \in [0, 1]$ and some integer $t > 0$, we let $\mathcal{A}_{t,\zeta}$ denote the set of trees T of height at most t such that the following holds: Every path \mathcal{P} of length t from $r(T)$ to $L_t(T)$ contains at least $(1 - \zeta)t$ vertices v such that \tilde{T}_v has a mixing root.

Before presenting our next result, we need to do the following remark. In Definition 8, given ξ and δ , among others the following inequality should hold for Δ_+ ,

$$\sum_{t \geq \Delta_+} t \cdot \xi_t < \exp(-2\beta \ln d_\xi),$$

where $\beta \geq 4$. Given Δ_+ and ξ the exact value of the parameter β is already specified. That is, when we define Δ_+ and ξ , the value of β is implicit.

► **Proposition 14.** *Assume that the distribution ξ, δ, Δ_+ are as defined in the statement of Theorem 10. Let $\mathcal{C} = \beta \ln d_\xi$. Also, let $\zeta \in (0, 1)$ and $\theta = \theta(\zeta) > 1$ be such that $(1 - \zeta)\theta < 1$ and $\beta(1 - \theta) < -1$. Then, for every $h \geq 1$ it holds that*

$$\Pr[\mathcal{T}_\xi^h \in \mathcal{A}_{h,\zeta}] \geq 1 - \exp[-(1 - \theta(1 - \zeta))\mathcal{C} \cdot h].$$

The proof of Proposition 14 appears in Section 8.

► **Theorem 15.** *Let ξ, δ, Δ_+ and α be as in the statement of Theorem 10. Also, let $\zeta \in (0, 1)$ and let the integer $h \geq 1$. For $k = (1 + \alpha)\Delta_+ / \ln \Delta_+$, it holds that*

$$\mathbb{E}[\mathbb{G}(\mathcal{T}_\xi^h) | \mathcal{T}_\xi^h \in \mathcal{A}_{h,\zeta}] \leq \frac{4(2\Delta_+)^{-0.9(3/4-\zeta)\delta h}}{\Pr[\mathcal{T}_\xi^h \in \mathcal{A}_{h,\zeta}]}.$$

The proof of Theorem 15 appears in Section 6.

Set $\zeta = 1/4$, and $\theta = 1.3$, applying Proposition 14 we get that

$$\Pr[\mathcal{T}_\xi^h \notin \mathcal{A}_{h,\zeta}] \leq d_\xi^{-0.1h}. \tag{8}$$

For the same values of ζ, θ as above, (8) with Theorem 15 gives that

$$\mathbb{E}[\mathbb{G}(\mathcal{T}_\xi^h) | \mathcal{T}_\xi^h \in \mathcal{A}_{h,\zeta}] \leq 8(2\Delta_+)^{-0.45\delta h}. \tag{9}$$

Since we always have $0 \leq \mathbb{G}(T) \leq 1$, for ζ and θ as above, we get that

$$\mathbb{E}[\mathbb{G}(\mathcal{T}_\xi^h)] \leq \mathbb{E}[\mathbb{G}(\mathcal{T}_\xi^h) | \mathcal{T}_\xi^h \in \mathcal{A}_{h,1/4}] + \Pr[\mathcal{T}_\xi^h \notin \mathcal{A}_{h,1/4}] \leq 16(2\Delta_+)^{-0.45\delta h},$$

where the last inequality follows from (8) and (9). The theorem follows.

6 Proof of Theorem 15

Consider first the quantity $\mathbb{G}_{c,k}(T)$, for some fixed tree T . Then, it holds that

$$\mathbb{G}_{c,k}(T) = \left\| \mu^{X_L}(\cdot) - \mu^{Z_L^q}(\cdot) \right\|_{r_T}. \tag{10}$$

An important remark from Proposition 11 is that it allows to use any kind of correlation between the X, Z^q . For this reason we assume that (X, Z^q) is distributed as in $\nu_{c,q}^T$. We are going to specify this distribution later. First we get the following result.

► **Proposition 16.** *Let ξ, δ, Δ_+ and α be as in the statement of Theorem 15. Also let $0 \leq \gamma \leq \delta$. Then for $k = (1 + \alpha)\Delta_+ / \ln \Delta_+$, it holds that*

$$\begin{aligned} \mathbb{E}[\mathbb{G}_{c,k}(\mathcal{T}_\xi^h) | \mathcal{T}_\xi^h \in \mathcal{A}_{h,\zeta}] &\leq \frac{1}{\Pr[\mathcal{T}_\xi^h \in \mathcal{A}_{h,\zeta}]} \left(2 \exp\left(-\frac{1}{8}(\Delta_+)^{\frac{h/4-1}{2}\delta + \frac{7}{8}\frac{\alpha}{1+\alpha}}\right) \cdot \mathbb{E}[|L_h(\mathcal{T}_\xi^h)|] \right. \\ &\quad \left. + 2(2(\Delta_+)^{-\gamma})^{(3/4-\zeta)h} \cdot \mathbb{E}[H(X_L, Z_L^q)] \right). \end{aligned} \tag{11}$$

For the above proposition we remark the following: On the r.h.s. of (11) the rightmost expectation term is w.r.t. both the joint distribution of X, Z^q and the distribution over the tree \mathcal{T}_ξ^h . The rest expectations are w.r.t. the distributions over trees only, i.e. \mathcal{T}_ξ^h . The proof of Proposition 16 appears in Section 7.

For showing the theorem we bound appropriately the two expectations on the r.h.s. of (11). It is elementary that

$$\mathbb{E} [|L_h(\mathcal{T}_\xi^h)|] = (d_\xi)^h. \tag{12}$$

For bounding $\mathbb{E} [H(X_L, Z_L^q)]$ we need to specify a coupling between the random variables X and Z^q which minimizes their expected Hamming distance. Observe that the expected hamming distance is both w.r.t. the coupling and the randomness of the trees.

The coupling of X and Z^q we use, can be defined inductively as follows: We colour the vertices from the root down to the leaves. For a vertex v whose father w is such that $X(w) = Z^q(w)$ we couple $X(v)$ and $Z^q(v)$ identically, i.e. $X(v) = Z^q(v)$. On the other hand, when $X(w) \neq Z^q(w)$ we set $X(v) = Z^q(v)$ unless $X(v) = Z^q(w)$, then we set $Z^q(v) = X(v)$.

Let w be a vertex in the tree and let u be a child of w . Then, for the coupling above, it holds that

$$\Pr [X(u) \neq Z^q(u) | X(w) \neq Z^q(w)] = k^{-1}.$$

In \mathcal{T}_ξ^h , the expected number of children per (non-leaf) vertex is d_ξ . Then, it is elementary to show that for a disagreeing vertex, the expected number of disagreeing children is $d_\xi/k \leq \frac{\ln \Delta_+}{1+\alpha}$, since $\Delta_+ > d_\xi$. Furthermore, it holds that

$$\mathbb{E} [H(X_L, Y_L)] \leq \left(\frac{\ln \Delta_+}{1+\alpha} \right)^h. \tag{13}$$

Observe that the above expectation is w.r.t. *both* tree instances and the joint distribution of the two random colourings.

The theorem follows by combining (13), (12) and Proposition 16.

7 Proof of Proposition 16

The previous setting allows to use ideas based on the notion of biasing-unbiasing boundary (introduced in [3]) to prove Proposition 16. To be more precise, the definition of biasing non-biasing boundaries we use here is slightly different than that [3], but the approach is similar.

► **Definition 17 (Non-Biasing Boundary).** For $\alpha, \gamma, \delta, \Delta_+$ as in the statement of Proposition 16, we let $k = (1 + \alpha)\Delta_+ / \ln \Delta_+$, and let some integer $t \geq 1$. Consider a tree H of height t such that $r(H)$ is mixing. For σ , a k -colouring of H , we say that σ_L does not bias the root if the following holds:

- if $t = 1$, then $\sigma(L_t(G))$ uses all but at least $(\Delta_+)^{\gamma}$ many colours.
- if $t > 1$, then the following holds: We let v_1, \dots, v_s be the children of the root of H , where $s \leq \Delta_+$. Also, let $\mathbb{S} \subseteq \{ \tilde{H}_{v_1}, \tilde{H}_{v_2}, \dots, \tilde{H}_{v_s} \}$ contain only the subtrees whose roots are mixing. Then, there are at most Δ_+^{δ} many subtrees $\tilde{H}_{v_i} \in \mathbb{S}$ such that $\sigma(L_{t-1}(\tilde{H}_{v_i}))$ biases the root $r(\tilde{H}_{v_i})$.

Also, we let $\mathcal{U}(T)$ denote the set of all boundary conditions on L which are not biasing.

Note the notion of non-biasing boundary condition makes sense only for trees with mixing root.

► **Lemma 18.** *Let γ, α, Δ_+ be as in the statement of Proposition 16. Let $k = (1 + \alpha) \frac{\Delta_+}{\ln \Delta_+}$, also let some integer $t \geq 1$. Consider a fixed tree T of height t and let $L = L_t(T)$. For σ , a k -colouring of T , such that σ_L is biasing for the root of T the following is true: There is at least one $c \in [k]$ such that for X , a random k -colouring of T , it holds that*

$$\Pr[X_{r(T)} = c | X_L = \sigma_L] \geq (\Delta_+)^{-\gamma}.$$

For a proof of Lemma 18 see in the full version of this work in [12].

► **Definition 19.** Let $\alpha, \gamma, \delta, \Delta_+, h$ be as in the statement of Proposition 16. Consider a tree T of height h and let $L = L_h(T)$. For every vertex $w \in L$ we define the set of boundaries $\mathcal{U}_w \subseteq [k]^L$ as follows: Let \mathcal{P} denote the path that connects r_T and w and we let

$$\mathcal{M} = \{v \in \mathcal{P} : \text{dist}(r_T, v) \leq (3/4)h, \tilde{T}_v \text{ has mixing root}\}.$$

Then \mathcal{U}_w contains the boundary conditions on L which do not bias the root of any of the subtrees \tilde{T}_v where $v \in \mathcal{M}$.

► **Proposition 20.** *Let $\alpha, \gamma, \delta, \Delta_+, h, \zeta$ be as in the statement of Proposition 16. Let some fixed tree $T \in \mathcal{A}_{h, \zeta}$ and let $L = L_h(T)$. Consider σ, τ to be two k -colourings of T such that $H(\sigma_L, \tau_L) = 1$. Furthermore, assume that $\sigma(w) \neq \tau(w)$ for some $w \in L$, while both $\sigma_L, \tau_L \in \mathcal{U}_w$. Then it holds that*

$$\|\mu^{\sigma_L} - \mu^{\tau_L}\|_{r(T)} \leq \Delta_{\zeta, h}^* = (2\Delta_+^{-\gamma})^{(3/4-\zeta)h}.$$

Proof. For showing the proposition we use disagreement percolation coupling construction. This approach is somehow standard and it has been used in different contexts, e.g. [9, 11]. For the full proof of the propositions see in the full version of this work in [12]. ◀

► **Proposition 21.** *Let $\alpha, \gamma, \delta, \Delta_+, h, \zeta$ be as in the statement of Proposition 16. Consider a fixed tree $T \in \mathcal{A}_{h, \zeta}$. Let X be a random k -colouring of T . For $k = (1 + \alpha)\Delta_+ / \ln \Delta_+$ and any $w \in L_h(T)$ it holds that*

$$\Pr[X_L \notin \mathcal{U}_w] \leq 2 \exp\left(-\frac{1}{8}(\Delta_+)^{\frac{h/4-1}{2}\delta + \frac{7}{8}\frac{\alpha}{1+\alpha}}\right).$$

For the proof of Proposition 21 see in the full version of this work in [12].

Proof of Proposition 16. First, consider some fixed tree $T \in \mathcal{A}_{h, \zeta}$ and we let $L = L_h(T)$. Usually we fix a colouring of L and we call it (the colouring) boundary condition. We also use the term “free” boundary to indicate the absence of any boundary condition on L or some of its vertices.

Consider two colourings of the leaves $\sigma(L)$ and $\tau(L)$. We let m be the Hamming distance between $\sigma(L)$ and $\tau(L)$, i.e. $m = H(\sigma_L, \tau_L)$. Let v_1, \dots, v_m be the vertices in L for which σ_L and τ_L disagree. Consider the sequence of boundary conditions $Z_0, \dots, Z_{2m} \in [k]^L$ such that $\sigma_L = Z_1, \tau_L = Z_{2m}$ while the rest of the members are as follows: For $i \leq m$, we get Z_i from Z_{i-1} by substituting the assignment of v_i from $\sigma(v_i)$ to “free”. Also, for $i \geq m$ we get Z_{i+1} from Z_i by substituting $Z(v_{i-m})$ from “free” to $\tau(v_{i-m})$. It is direct that $H(Z_i, Z_{i+1}) = 1$.

From triangle inequality, it holds that

$$\|\mu^{\sigma_L} - \mu^{\tau_L}\|_{r(T)} \leq \sum_{i=0}^{2m-1} \|\mu^{Z_i} - \mu^{Z_{i+1}}\|_{r(T)}. \tag{14}$$

Also, it is not hard to see that for every $w \in L$ the following is true: if $\sigma_L \in \mathcal{U}_w$, then $Z_i \in \mathcal{U}_w$ for every $i = 1, \dots, m$. Similarly, if $\tau_L \in \mathcal{U}_w$, then $Z_i \in \mathcal{U}_w$ for every $i = m, \dots, 2m$.

Let the event $\mathbb{U}_{v_i}^{\sigma, \tau} = \{\sigma_L \notin \mathcal{U}_{v_i} \cup \tau_L \notin \mathcal{U}_{v_i}\}$. Then it holds that

$$\|\mu^{Z_i} - \mu^{Z_{i+1}}\|_{r(T)} \leq \mathbb{I}_{\{\mathbb{U}_{v_i}\}} + (1 - \mathbb{I}_{\{\mathbb{U}_{v_i}\}}) \Delta_{\zeta, h}^*, \tag{15}$$

where $\Delta_{\zeta, h}^*$ is defined in the statement of Proposition 20. In words, the above inequality states the following: if at least one of the σ_L, τ_L are not in \mathcal{U}_{v_i} , then the l.h.s. of (15) is at most 1. On the other hand, if both $\sigma_L, \tau_L \in \mathcal{U}_{v_i}$ then the total variation distance on the l.h.s. can be upper bounded by using Proposition 20.

Plugging (15) into (14) we have that

$$\|\mu^{\sigma_L} - \mu^{\tau_L}\|_{r(T)} \leq 2 \cdot \sum_{v \in L_h(T)} \mathbb{I}_{\{\sigma_v \neq \tau_v\}} \cdot [\mathbb{I}_{\{\mathbb{U}_v\}} + (1 - \mathbb{I}_{\{\mathbb{U}_v\}}) \cdot \Delta_{\zeta, h}^*]. \tag{16}$$

Now, we consider the quantity $\mathbb{G}_{c, k}(T)$, i.e. $\mathbb{G}_{c, k}(T) = \|\mu^{X_L} - \mu^{Z_L^q}\|_{r(T)}$. For bounding $\mathbb{G}_{c, k}(T)$ we are going to use (16). That is

$$\begin{aligned} \mathbb{G}_{c, k}(T) &= \|\mu^{X_L} - \mu^{Z_L^q}\|_{r(T)} \leq \sum_{\sigma_L, \tau_L \in [k]^L} \Pr[X_L = \sigma_L, Z_L^q = \tau_L] \cdot \|\mu^{\sigma_L} - \mu^{\tau_L}\|_{r(T)} \\ &\leq 2 \cdot \sum_{\sigma_L, \tau_L \in [k]^L} \Pr[X_L = \sigma_L, Z_L^q = \tau_L] \\ &\quad \cdot \sum_{v \in L_h(T)} \mathbb{I}_{\{\sigma_v \neq \tau_v\}} \cdot (\mathbb{I}_{\{\mathbb{U}_v^{\sigma, \tau}\}} + (1 - \mathbb{I}_{\{\mathbb{U}_v^{\sigma, \tau}\}}) \Delta_{\zeta, h}^*) \quad [\text{from (16)}] \\ &\leq 2 \cdot \sum_{v \in L_h(T)} (\Pr[X(v) \neq Z^q(v), \mathbb{U}_v^{X_L, Z_L^q}] + \Pr[X(v) \neq Z^q(v)] \cdot \Delta_{\zeta, h}^*) \\ &\leq 2 \cdot \sum_{v \in L_h(T)} \Pr[\mathbb{U}_v^{X_L, Z_L^q}] + 2 \cdot \sum_{v \in L_h(T)} \Pr[X(v) \neq Z^q(v)] \cdot \Delta_{\zeta, h}^*. \end{aligned}$$

Due to symmetry it holds that $\Pr[X(L) \notin \mathcal{U}_v] = \Pr[Z^q(L) \notin \mathcal{U}_v]$. Using this observation and a union bound, the above inequality implies that

$$\begin{aligned} \mathbb{G}_{c, k}(T) &\leq 4 \sum_{v \in L} \Pr[X(L) \notin \mathcal{U}_v] + \Delta_{\zeta, h}^* \sum_{v \in L} \Pr[X(v) \neq Z^q(v)] \\ &\leq 2 \exp\left(-\frac{1}{8}(\Delta_+)^{\frac{h/4-1}{2}\delta + \frac{7}{8}\frac{\alpha}{1+\alpha}}\right) \cdot |L_h(T)| + 2\Delta_{\zeta, h}^* \cdot \mathbb{E}_{\nu_{c, q}}[H(X_L, Z_L^q)], \end{aligned}$$

where in the last inequality we used Proposition 21 to bound $\Pr[X(L) \notin \mathcal{U}_v]$. $\mathbb{E}_{\nu_{c, q}}[H(X(L), Z^q(L))]$ is the expected Hamming distance between X_L and Z_L^q and depends only on the joint distribution of X, Z^q , which is denoted as $\nu_{c, q}$.

The proposition follows by averaging over \mathcal{T}_ξ^h , conditional that we have a tree in $\mathcal{A}_{h, \zeta}$. That is

$$\begin{aligned} \mathbb{E}[\mathbb{G}_{c, k}(\mathcal{T}_\xi^h) \mid \mathcal{T}_\xi^h \in \mathcal{A}_{h, \zeta}] &\leq \frac{1}{\Pr[\mathcal{T}_\xi^h \in \mathcal{A}_{h, \zeta}]} \left(2 \exp\left(-\frac{1}{8}(\Delta_+)^{\frac{h/4-1}{2}\delta + \frac{7}{8}\frac{\alpha}{1+\alpha}}\right) \cdot \mathbb{E}[|L_h(\mathcal{T}_\xi^h)|] \right. \\ &\quad \left. + 2(2\Delta_+^{-\gamma})^{(3/4-\zeta)h} \cdot \mathbb{E}[H(X_L, Z_L^q)] \right). \end{aligned}$$

The rightmost expectation term is w.r.t. both $\nu_{c,q}$ and the distribution of random trees \mathcal{T}_ξ^h . In the above derivations we used the following, easy to derive, inequality

$$\mathbb{E} [f(\mathcal{T}_\xi^h) | \mathcal{T}_\xi^h \in \mathcal{A}_{h,\zeta}] \leq \mathbb{E} [f(\mathcal{T}_\xi^h)] / \Pr [\mathcal{T}_\xi^h \in \mathcal{A}_{h,\zeta}],$$

where f is any non-negative functions on the support of the distribution \mathcal{T}_ξ^h . The proposition follows. \blacktriangleleft

8 Proof of Proposition 14

For $i = (1 - \zeta)h$ we let $Q_{h,i} = \Pr [\mathcal{T}_\xi^h \notin \mathcal{A}_{h,\zeta}]$. Also, we let $Q_{h,i}^t = \Pr [\mathcal{T}_\xi^h \notin \mathcal{A}_{h,\zeta} | \deg(r(\mathcal{T}_\xi^h)) = t]$ Using a simple union bound we get the following: For $t \leq (\Delta_+)^{\delta}$ it holds that

$$Q_{h,i}^t \leq t \cdot Q_{h-1,i-1}. \tag{17}$$

Intuitively, the above is implied by the following: If $\deg(r(\mathcal{T}_\xi^h)) \leq (\Delta_+)^{\delta}$, then, regardless of its children, the root $r(\mathcal{T}_\xi^h)$ is mixing. Conditional that $\deg(r(\mathcal{T}_\xi^h)) \leq (\Delta_+)^{\delta}$ holds, so as to have $\mathcal{T}_\xi^h \notin \mathcal{A}_{h,\zeta}$, there should be a vertex v , child of $r(\mathcal{T}_\xi^h)$ such that the following is true: The subtree \tilde{T}_v has a path from its root to its vertices of at level $h - 1$ which contain less than $i - 1$ mixing vertices.

Using similar arguments, for $(\Delta_+)^{\delta} \leq t \leq \Delta_+$, we get the following lemma, whose proof appear in Section 8.1.

► **Lemma 22.** For $(\Delta_+)^{\delta} < t \leq \Delta_+$, it holds that

$$Q_{h,i}^t \leq 2t (Q_{h-1,i-1} + Q_{h-1,i} \cdot \Pr [\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}]).$$

Finally, using a simple union bound we get that for $t > \Delta_+$ it holds that

$$Q_{h,i}^t \leq t \cdot Q_{h-1,i}. \tag{18}$$

The above follows by a line of arguments similar to those we used for (17) and by noting that if $\deg(r(\mathcal{T}_\xi^h)) \geq \Delta_+$, then the root of \mathcal{T}_ξ^h is non-mixing.

We are bounding $Q_{h,i}$ by using (17), (18) and Lemma 22. We have that

$$\begin{aligned} Q_{h,i} &= \sum_{t=0}^n Q_{h,i}^t \xi_t \\ &= Q_{h-1,i-1} \cdot \sum_{t=0}^{(\Delta_+)^{\delta}} t \cdot \xi_t + 2Q_{h-1,i-1} \cdot \sum_{t=(\Delta_+)^{\delta}+1}^{\Delta_+} t \cdot \xi_t + \\ &\quad + 2Q_{h-1,i} \cdot \Pr [\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}] \cdot \sum_{t=(\Delta_+)^{\delta}+1}^{\Delta_+} t \cdot \xi_t + Q_{h-1,i} \cdot \sum_{t \geq (\Delta_+)+1} t \cdot \xi_t \\ &\leq 2Q_{h-1,i-1} \sum_{t=0}^{\Delta_+} t \cdot \xi_t + Q_{h-1,i} \left(2 \Pr [\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}] \sum_{t=(\Delta_+)^{\delta}}^{\Delta_+} t \cdot \xi_t \right. \\ &\quad \left. + \sum_{t \geq (\Delta_+)+1} t \cdot \xi_t \right) \\ &\leq 2d_\xi \cdot Q_{h-1,i-1} + Q_{h-1,i} \left(2d_\xi \cdot \Pr [\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}] + \sum_{t \geq (\Delta_+)+1} t \cdot \xi_t \right). \tag{19} \end{aligned}$$

The following lemma uses (19) to derive an upper bound on $Q_{h,i}$.

► **Lemma 23.** *Let h, β, \mathcal{C} be as in the statement of Proposition 14. Also, let $\lambda \in (0, 1)$ and $\theta' > 1$ be a fixed numbers such that $\beta(1 - \theta') < -1$ and $\lambda\theta' < 1$. Then for $i = \lambda h$ and $Q_{h,i}$ that satisfies the inequality in (19), it holds that*

$$Q_{h,i} \leq \exp[-(1 - \lambda\theta') \cdot \mathcal{C} \cdot h]. \quad (20)$$

The proof of Lemma 23 appears in Section 8.2

The proposition follows by using the above lemma and setting $\lambda = (1 - \zeta)$ and $\theta' = \theta$, where ζ and θ are defined in the statement of Proposition 14.

8.1 Proof of Lemma 22

Let q_{h-1} be the probability for each child of $r(\mathcal{T}_\xi^h)$ to be non-mixing. Conditional that $r(\mathcal{T}_\xi^h)$ has degree t , the number of non-mixing children of $r(\mathcal{T}_\xi^h)$ is binomially distributed with parameters, t, q_{h-1} , i.e. $\mathcal{B}(t, q_{h-1})$. Letting $Q_{h,i}^M = \Pr[\mathcal{T}_\xi^h \notin \mathcal{A}_{h,\zeta} \mid r(\mathcal{T}_\xi^h) \text{ is mixing}]$ and $Q_{h,i}^N = \Pr[\mathcal{T}_\xi^h \notin \mathcal{A}_{h,\zeta} \mid r(\mathcal{T}_\xi^h) \text{ is not mixing}]$, it holds that

$$\begin{aligned} Q_{h,i}^t &\leq \sum_{j=0}^{(\Delta_+)^{\delta}} \binom{t}{j} q_{h-1}^j (1 - q_{h-1})^{t-j} [(t-j)Q_{h-1,i-1}^M + jQ_{h-1,i-1}^N] + \\ &\quad + \sum_{j=(\Delta_+)^{\delta}+1}^t \binom{t}{j} q_{h-1}^j (1 - q_{h-1})^{t-j} [(t-j)Q_{h-1,i}^M + jQ_{h-1,i}^N]. \end{aligned}$$

Using the standard equality that $(t-j)\binom{t}{j} = t\binom{t-1}{j}$, we get that

$$\begin{aligned} Q_{h,i}^t &\leq t(1 - q_{h-1})Q_{h-1,i-1}^M \sum_{j=0}^{(\Delta_+)^{\delta}} \binom{t-1}{j} q_{h-1}^j (1 - q_{h-1})^{t-1-j} \\ &\quad + tq_{h-1}Q_{h-1,i-1}^N \sum_{j=1}^{(\Delta_+)^{\delta}} \binom{t-1}{j-1} q_{h-1}^{j-1} (1 - q_{h-1})^{t-j} \\ &\quad + t(1 - q_{h-1})Q_{h-1,i}^M \sum_{j=(\Delta_+)^{\delta}+1}^{t-1} \binom{t-1}{j} q_{h-1}^j (1 - q_{h-1})^{t-1-j} \\ &\quad + tq_{h-1}Q_{h-1,i}^N \sum_{j=(\Delta_+)^{\delta}+1}^t \binom{t-1}{j-1} q_{h-1}^{j-1} (1 - q_{h-1})^{t-j}. \end{aligned}$$

It is not hard to see that for any h, i it holds that $q_h Q_{h,i}^N \leq Q_{h,i}$ and $(1 - q_h)Q_{h,i}^M \leq Q_{h,i}$. Using these two inequalities we get that

$$\begin{aligned} Q_{h,i}^t &\leq tQ_{h-1,i-1} (\Pr[\mathcal{B}(t-1, q_{h-1}) \leq (\Delta_+)^{\delta}] + \Pr[\mathcal{B}(t-1, q_{h-1}) \leq (\Delta_+)^{\delta} - 1]) \\ &\quad + tQ_{h-1,i} (\Pr[\mathcal{B}(t-1, q_{h-1}) \geq (\Delta_+)^{\delta} + 1] + \Pr[\mathcal{B}(t-1, q_{h-1}) \geq (\Delta_+)^{\delta}]) \\ &\leq 2tQ_{h-1,i-1} + 2tQ_{h-1,i} \Pr[\mathcal{B}(t-1, q_{h-1}) \geq (\Delta_+)^{\delta}]. \quad (21) \end{aligned}$$

Note that that $\Pr[\mathcal{B}(t-1, q_{h-1}) \geq (\Delta_+)^{\delta}]$ is increasing with t . That is, for $t \leq \Delta_+$ it holds that

$$\Pr[\mathcal{B}(t-1, q_{h-1}) \geq (\Delta_+)^{\delta}] \leq \Pr[\mathcal{B}(\Delta_+, q_{h-1}) \geq (\Delta_+)^{\delta}]. \quad (22)$$

At this point we observe that the quantity q , defined in Definition 8, is an upper bound for q_h , for every h . This follows by an inductive argument, i.e. induction on h the number of levels of \mathcal{T}_ξ^h .

Clearly, for $h = 0$, the assertion is true. The tree with zero levels consists of only one vertex, which is a leaf. By default the leaves are mixing vertices, i.e. the probability of a leaf to be non-mixing is zero. Since $q \in [0, 3/4)$, q is an upper bound for the vertex to be non-mixing.

Given $h > 0$, assume that the assertion is true for $\mathcal{T}_\xi^{h'}$, for any $h' \leq h$. We are going to show that this is true for \mathcal{T}_ξ^h . Let \mathbf{N} be the number of non-mixing children of the root of \mathcal{T}_ξ^h . It holds that

$$\Pr[r(\mathcal{T}_\xi^h) \text{ is non-mixing}] \leq \Pr[\deg(r(\mathcal{T}_\xi^h)) > \Delta_+] + \Pr[\mathbf{N} > (\Delta_+)^{\delta} | \deg(r(\mathcal{T}_\xi^h)) \leq \Delta_+].$$

Given that $\deg(r(\mathcal{T}_\xi^h)) = D$, for some integer $D \geq 0$, \mathbf{N} is a binomial variable with parameters D, q_{h-1} . Due to our induction hypothesis it holds that $q_{h-1} < q$. Since we have conditioned that $D < \Delta_+$, it is clear that \mathbf{N} is dominated by a binomial variable with parameters Δ_+, q , that is

$$\begin{aligned} \Pr[r(\mathcal{T}_\xi^h) \text{ is non-mixing}] &\leq \Pr[\deg(r(\mathcal{T}_\xi^h)) > \Delta_+] + \Pr[\mathcal{B}(\Delta_+, q) > (\Delta_+)^{\delta}] \\ &\leq \sum_{i \geq \Delta_+} \xi_i + \Pr[\mathcal{B}(\Delta_+, q) > (\Delta_+)^{\delta}] \leq q, \end{aligned}$$

where the last inequality follows from the definition of q , i.e. in Definition 8. The above inequality with (22) imply that

$$\Pr[\mathcal{B}(\Delta_+, q_{h-1}) \geq (\Delta_+)^{\delta}] \leq \Pr[\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}],$$

as $\mathcal{B}(\Delta_+, q_{h-1})$ is stochastically dominated by $\mathcal{B}(\Delta_+, q)$, since, $q_{h-1} \leq q$, for any h .

The lemma follows by plugging the above inequality into (21).

8.2 Proof of Lemma 23

We are going to use induction to prove the lemma. First we are going to show that if (20) is true for some $h > 1$ then it is also true for $h + 1$. Let $\lambda = \frac{i}{h}$, $\lambda^- = \frac{i-1}{h-1}$ and $\lambda^+ = \frac{i}{h-1}$. We rewrite (19) in terms of λ , λ^+ and λ^- as follows:

$$\begin{aligned} &Q_{\{h, \lambda h\}} \\ &\leq 2d \cdot Q_{\{h-1, \lambda^-(h-1)\}} + Q_{\{h-1, \lambda^+(h-1)\}} \left(2d \Pr[\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}] + \sum_{t \geq (\Delta_+)+1} t \cdot \xi_t \right). \end{aligned} \tag{23}$$

Using the induction hypothesis and noting that $\lambda^- = \lambda - \frac{1-\lambda}{h-1}$ we have that

$$\begin{aligned} Q_{\{h-1, \lambda^-(h-1)\}} &\leq \exp[-(1 - \theta \lambda^-)(h-1)\mathcal{C}] \\ &\leq \exp\left[-\left(1 - \theta' \left(\lambda - \frac{1-\lambda}{h-1}\right)\right)(h-1)\mathcal{C}\right] \\ &\leq \exp[-(1 - \theta' \lambda)(h-1)\mathcal{C}] \cdot \exp[-\theta'(1-\lambda)\mathcal{C}] \\ &\leq \exp[-(1 - \theta' \lambda)h\mathcal{C}] \cdot \exp[(1 - \theta')\mathcal{C}]. \end{aligned}$$

As far as $Q_{\{h-1,i\}}$ is regarded, we use the fact that $\lambda^+ = \lambda + \frac{\lambda}{h-1}$ and we get that

$$\begin{aligned} Q_{\{h-1,\lambda^+ \cdot (h-1)\}} &\leq \exp[-(1-\theta'\lambda^+)(h-1)\mathcal{C}] \\ &\leq \exp\left[-\left(1-\theta'\lambda - \frac{\theta'\lambda}{h-1}\right)(h-1)\mathcal{C}\right] \\ &\leq \exp[-(1-\theta'\lambda)(h-1)\mathcal{C}] \cdot \exp[\theta'\lambda\mathcal{C}] \\ &\leq \exp[-(1-\theta'\lambda)h\mathcal{C}] \exp[\mathcal{C}]. \end{aligned} \tag{24}$$

Substituting the bounds for $Q_{\{h-1,i-1\}}, Q_{\{h-1,i\}}$ above into (23) we get that

$$\begin{aligned} &Q_{\{h,\lambda h\}} \\ &\leq \exp[-(1-\theta'\lambda)h\mathcal{C}] \\ &\quad \times \left(2d \cdot \exp[(1-\theta')\mathcal{C}] + \exp(\mathcal{C}) \left(2d \Pr[\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}] + \sum_{t \geq (\Delta_+)+1} t \cdot \xi_t \right) \right). \end{aligned}$$

From to our assumption that $\beta(1-\theta') < -1$ it is direct that

$$2d \cdot \exp[(1-\theta')\mathcal{C}] = 2d^{1+\beta(1-\theta')} \leq 1/5.$$

Also due to our assumptions about Δ_+, δ we get that

$$\exp(\mathcal{C}) \left(2d \Pr[\mathcal{B}(\Delta_+, q) \geq (\Delta_+)^{\delta}] + \sum_{t \geq \Delta_++1} t \cdot \xi_t \right) \leq \frac{2}{5}.$$

Using the two bounds above (??) writes as follows:

$$Q_{\{h,\lambda h\}} \leq \exp[-(1-\theta' \cdot \lambda)h\mathcal{C}].$$

It remains to show the base of the induction, i.e the case $h = 1$. Since the leaves of the trees are, by default, mixing, for any fixed $\lambda \in (0, 1)$ and $h = 1$ it holds that

$$Q_{\{h,\lambda \cdot h\}} \leq \Pr[\deg(r(T)) \geq \Delta_+] = \sum_{t \geq \Delta_+} \xi_t \leq \exp[-2\mathcal{C}] \leq \exp[-(1-\theta' \cdot \lambda)\mathcal{C}],$$

as $\lambda, \theta > 0$ while $\lambda \cdot \theta' < 1$. The lemma follows.

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