Approximating Upper Degree-Constrained Partial Orientations*

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Abstract

In the Upper Degree-Constrained Partial Orientation (UDPO) problem we are given an undirected graph $G = (V, E)$, together with two degree constraint functions $d^-, d^+ : V \to \mathbb{N}$. The goal is to orient as many edges as possible, in such a way that for each vertex $v \in V$ the number of arcs entering v is at most $d^-(v)$, whereas the number of arcs leaving v is at most $d^+(v)$. This problem was introduced by Gabow [SODA'06], who proved it to be MAXSNP-hard (and thus APX-hard). In the same paper Gabow presented an LP-based iterative rounding 4/3-approximation algorithm.

As already observed by Gabow, the problem in question is a special case of the classic 3-Dimensional Matching, which in turn is a special case of the k-Set Packing problem. Back in 2006 the best known polynomial time approximation algorithm for 3-Dimensional Matching was a simple local search by Hurkens and Schrijver [SIDMA'89], the approximation ratio of which is $(3 + \varepsilon)/2$; hence the algorithm of Gabow was an improvement over the approach brought from the more general problems.

In this paper we show that the UDPO problem when cast as 3-Dimensional Matching admits a special structure, which is obliviously exploited by the known approximation algorithms for k-Set Packing. In fact, we show that already the local-search routine of Hurkens and Schrijver gives $(4 + \varepsilon)/3$-approximation when used for the instances coming from UDPO. Moreover, the recent approximation algorithm for 3-Set Packing [Cygan, FOCS'13] turns out to be a $(5 + \varepsilon)/4$-approximation for UDPO. This improves over 4/3 as the best ratio known up to date for UDPO.

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1 Introduction

During the last decades several graph orientation problems were studied (see Section 8.7 in [2] and Section 61.1 in [13]). One of the most recently introduced is the Upper Degree-Constrained Partial Orientation, abbreviated as UDPO. In the UDPO problem we are given an undirected graph $G = (V, E)$, together with two degree constraint functions $d^-, d^+ : V \to \mathbb{N}$. The goal is to orient as many edges as possible, in such a way that for each vertex $v \in V$ the number of arcs entering v is at most $d^-(v)$, whereas the number of arcs leaving v is at most $d^+(v)$. This problem was introduced by Gabow [9], motivated by a variant of the maximum bipartite matching problem arising when planning a two-day event

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with several parallel sessions and each participant willing to attend one chosen session each day, but without a particular order on the two selected sessions (for the exact definition, see [9]).

UPPER DEGREE-CONSTRAINED PARTIAL ORIENTATION (UDPO) **Input:** Undirected graph G, degree constraints $d^+, d^- : V(G) \to \mathbb{Z}_{\geq 0}$ **Find:** A subset $\overline{F} \subseteq E(G)$ which admits an orientation F satisfying $\deg^+_F(v) \leq d^+(v)$ and $\deg^-_F(v) \leq d^-(v)$ for each $v \in V(G)$. **Maximize:** $\lvert F \rvert$

Gabow proved the problem to be MAXSNP-hard (thus also APX-hard), and showed an LP-based iterative rounding 4/3-approximation algorithm. As he already observed, UDPO is a special case of the 3-DIMENSIONAL MATCHING problem, which in turn is a special case of the k-SET PACKING problem defined as follows.

k-SET PACKING **Input:** A family $\mathcal{F}$ of subsets of a finite universe U, such that $\lvert F \rvert \leq k$ for every $F \in \mathcal{F}$ **Find:** A subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ of pairwise-disjoint subsets **Maximize:** $\lvert \mathcal{F}_0 \rvert$

Note that at the time when these results were published, the best known approximation ratio for 3-SET PACKING was the simple local search of Hurkens and Schrijver [12] with approximation ratio $(3+\varepsilon)/2$. Therefore the 4/3 ratio achieved by Gabow improved upon the results obtained using the algorithm for the more general problem.

1.1 Our results

In this paper we analyze the behaviour of two known approximation algorithms for the k-SET PACKING problem as solutions for UDPO. We prove that these algorithms obliviously exploit hidden structural properties present in every 3-SET PACKING instance obtained via the natural reduction from UDPO. Consequently, when cast as algorithms for the UDPO problem, these local-search routines attain better approximation ratios than they do for the worst-case instances of the 3-SET PACKING or 3-Dimensional Matching problems.

First, we show that already the simple local-search routine of Hurkens and Schrijver [12] is a $(4+\varepsilon)/3$-approximation when used for the instances coming from UDPO. Next, we prove that the recent algorithm for 3-SET PACKING [7], again, used as a black box, turns out to be a $(5+\varepsilon)/4$-approximation for UDPO. This way we derive the best known ratio for UDPO, improving over 4/3 obtained by the algorithm of Gabow. In fact, our approximation ratio matches the 5/4 lower bound on the integrality gap of the underlying natural LP relaxation [9].

Technical contribution of our paper is based on the analysis of *simple* instances, where all the degree bounds are either zero or one, which means that each vertex can have only zero or one incoming and outgoing arcs. Interestingly, for a wide class of local-search routines, simple instances are actually no easier than arbitrary ones. The properties of these instances give rise to a 4-SET PACKING-like structure which can be used in the analysis, though it is not explicitly used by the algorithms.

1.2 Organization of the paper

In the following subsection we discuss related work on the subject. Next, in Section 2.1 we recall the reduction from UDPO to 3-SET PACKING, followed by Section 2.2 where we describe the local-search algorithms from previous work on k-SET PACKING. In Section 3 we state the main properties behind the analyses of the approximation ratios of both algorithms and slightly strengthen both these results.

In the remaining sections we provide an improved analysis of the performance of both algorithms on instances obtained via the reduction from UDPO. In Section 4 we prove that the worst-case approximation ratio is already attained by simple instances (with all degree bounds at most one). The properties of simple instances are applied in Section 5 to obtain a performance guarantee complementary to those in Section 3. Finally, in Section 6 we combine the two to derive our main results.

1.3 Related work on k-set packing

Prior to the the recent improvements for the k-SET PACKING problem [7, 14], quasipolynomial-time algorithms with approximation ratios $(k+2)/3$ [10] and $(k+1+\varepsilon)/3$ [8] were obtained.

There is also a line of research on the weighted variant of k-SET PACKING, where we want to select a maximum-weight family of pairwise-disjoint sets from $\mathcal{F}$. Arkin and Hassin [1] gave a $(k-1+\varepsilon)$-approximation algorithm, later Chandra and Halldórsson [6] improved it to a $(2k+2+\varepsilon)/3$-approximation. Currently, the best-known approximation ratio is $(k+1+\varepsilon)/2$ due to Berman [3]. All the mentioned results are based on local search.

For the standard (unweighted) k-SET PACKING problem, Chan and Lau [5] also presented a strengthened LP relaxation with integrality gap $(k+1)/2$.

On the other hand, Hazan et al. [11] proved that k-SET PACKING is hard to approximate within a factor of $\mathcal{O}(k/\log k)$. Concerning small values of k, Berman and Karpinski [4] obtained a $(98/97-\varepsilon)$-hardness for 3-DIMENSIONAL MATCHING, which implies the same lower bound for 3-SET PACKING.

2 Preliminaries

Let G be an undirected (multi)graph. We sometimes treat G as a directed graph where each *edge* $e \in E(G)$ is represented by a pair of oppositely directed *arcs* in $A(G)$. For an arc $e \in A(G)$ we denote by $\overline{e}$ the corresponding edge in $E(G)$, and by e^R the reverse arc. We also define $\overline{A} = \{\overline{e} : e \in A\}$ and $A^R = \{e^R : e \in A\}$ for an arbitrary subset $A \subseteq A(G)$.

A *partial orientation* of G can be defined as a subset $F \subseteq A(G)$ such that $F^R \cap F = \emptyset$. It is called *feasible* (for degree constraints $d = (d^+, d^-)$), if $\deg_F^+(v) \leq d^+(v)$ and $\deg_F^-(v) \leq d^-(v)$ for each $v \in V(G)$, that is, if the number of arcs leaving v and the number of arcs entering v do not violate the upper bounds. Now, UDPO can be reformulated as the problem of finding a maximum feasible partial orientation F, rather than the corresponding set of undirected edges $\overline{F}$.

For an undirected (multi)graph G and a set $U \subseteq V(G)$ we also define $N_G(U)$ as the set of vertices $v \notin U$ adjacent to some $u \in U$; we also set $N_G[U] = N_G(U) \cup U$. The subgraph induced by a subset $X \subseteq V(G)$ is denoted as $G[X]$. A bipartite graph H with a fixed bipartition $V(H) = A \cup B$ is often represented as a triple $(A, B, E(H))$. The subgraph induced by $A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$ is then dented as $H[A', B']$.

2.1 Reduction to 3-set packing

The following reduction to 3-SET PACKING was introduced by Gabow [9]. Let $I = (G, d)$ be an instance of UDPO. We construct an equivalent instance of the 3-SET PACKING problem, i.e., a set family $\mathcal{F}$ over a universe U.

The universe U is a disjoint union of three sets: V^+, V^- and E. The set V^+ contains $d^+(v)$ copies v_i^+ of each $v \in V(G)$, V^- contains $d^-(v)$ copies v_i^- of each $v \in V(G)$, and E is defined as $E(G)$. The family $\mathcal{F}$ consists of sets $\{u_i^+, v_j^-, e\}$ and $\{v_j^+, u_i^-, e\}$ for each edge $e = \{u, v\}$ and all possible indices i, j.

Given a feasible partial orientation F, the degree constraints clearly let us choose for each arc $e = uv$ two copies u_i^+ and v_j^-, so that the choices are distinct across all arcs leaving u and entering v, respectively. Consequently, the sets $\{u_i^+, v_j^-, \bar{e}\}$ form a disjoint subfamily of $\mathcal{F}$. Similarly, given any disjoint set-family $\mathcal{F}_0 \subseteq \mathcal{F}$ it is easy to see that orienting $\bar{e}$ from u to v for any $\{u_i^+, v_j^-, \bar{e}\} \in \mathcal{F}_0$ gives a feasible partial orientation.

2.2 Local search for k-set packing

In this section we recall and reinterpret some of the results behind two local-search approaches to the k-SET PACKING problem: the classic one yielding a $(k+\varepsilon)/2$-approximation [12] and the recent $(k+1+\varepsilon)/3$-approximation by Cygan [7].

For an instance $(U, \mathcal{F})$ of the k-SET PACKING problem, we build an undirected conflict graph $G = G(\mathcal{F})$ with $V(G) = \mathcal{F}$ and vertices F, F' made adjacent if $F \cap F' \neq \emptyset$. Observe that solutions to this instance of k-SET PACKING correspond to independent sets in this graph. The algorithms maintain a solution $\mathcal{F}_0 \subseteq \mathcal{F}$ and try to replace it with a larger, but similar solution. They try to use a disjoint family $X \subseteq \mathcal{F} \setminus \mathcal{F}_0$ and replace $\mathcal{F}_0$ by $\mathcal{F}_0' = (\mathcal{F}_0 \setminus N_G(X)) \cup X$, where $G = G(\mathcal{F})$ is the conflict graph. Note that $N_G(X) \cap \mathcal{F}_0$ consists exactly of those members of $\mathcal{F}_0$ which cannot be present together with X in a single disjoint family. It is reasonable to preform this operation if the resulting family $\mathcal{F}_0'$ is larger than $\mathcal{F}_0$, or equivalently $|N_G(X) \cap \mathcal{F}_0| < |X|$. This leads to a notion of *improving sets*, defined for $\mathcal{F}_0 \subseteq \mathcal{F}$ as disjoint families $X \subseteq \mathcal{F} \setminus \mathcal{F}_0$ such that $|N_G(X) \cap \mathcal{F}_0| < |X|$. The classic approach to the k-SET PACKING problem is to search for improving sets of sufficiently large constant size.

▶ **Fact 1** (Weak rule). *There exists an algorithm that, given a k-SET-PACKING instance $\mathcal{F}$ and a disjoint family $\mathcal{F}_0 \subseteq \mathcal{F}$, in $|\mathcal{F}|^{\mathcal{O}(r)}$ time determines whether there exists an improving set $X \subseteq \mathcal{F} \setminus \mathcal{F}_0$ of size at most r and if so, finds such an improving set.*

The novel idea of [7] was to consider larger improving sets satisfying structural properties, which allow for efficient detection of these sets. This is achieved using a structural parameter of a graph called *pathwidth*. In this paper we only use some results of [7] as a black-box, so we do not need to recall the relatively complex definition of pathwidth. Pathwidth of an undirected graph G, denoted as $\mathrm{pw}(G)$, does not exceed the number of vertices in any connected component of G. Pathwidth of an improving set X is defined as $\mathrm{pw}(G[X \cup \mathcal{F}_0])$ where $G = G(\mathcal{F})$ is the conflict graph. The following theorem uses techniques of fixed-parameter tractability to find improving sets of logarithmic size and constant pathwidth.

▶ **Theorem 2** (Strong rule: [7], Theorem 3.6). *There is an algorithm that, given a k-SET-PACKING instance $\mathcal{F}$ and a disjoint family $\mathcal{F}_0 \subseteq \mathcal{F}$, in $|\mathcal{F}|^{\mathcal{O}(C \cdot k)}$ time determines whether there exists an improving set $X \subseteq \mathcal{F} \setminus \mathcal{F}_0$ of size at most $C \log |\mathcal{F}|$ and pathwidth at most C, and if so, finds such an improving set.*

Note that $\mathrm{pw}(G[X \cup \mathcal{F}_0]) \leq |X \cup (\mathcal{F}_0 \cap N(X))| < 2|X|$ for any improving set. Thus, the strong rule is able to find all improving sets discovered by the weak rule if only we set $C \geq 2r$. Moreover, let us note that both rules are *monotone* in a certain sense.

▶ **Observation 3.** *If no improving set can be found using Theorem 2 for $\mathcal{F}_0 \subseteq \mathcal{F}$, then one still cannot find an improving set if the instance $\mathcal{F}$ is restricted to any $\mathcal{F}'$ such that $\mathcal{F}_0 \subseteq \mathcal{F}' \subseteq \mathcal{F}$. The weak rule of Fact 1 enjoys the same property.*

We say that a partial orientation F is a *local optimum* if the underlying family $\mathcal{F}_0$ cannot be improved using the reduction rule in question. For the weak rule of Fact 1 (with fixed r) we call such orientations *weak local optima* and for the strong rule of Theorem 2 (with fixed C) — *strong local optima*.

The remaining part of this paper is devoted to the analysis how large these local optima can be compared to the global optimum. More precisely, we show that for every $\varepsilon > 0$ there is an appropriate choice of parameters such that $|F| \geq (\frac{3}{4} - \varepsilon)|OPT|$ for any weak local optimum F and global optimum OPT, while for any strong local optimum this can be improved to $|F| \geq (\frac{4}{5} - \varepsilon)|OPT|$. The parameters $r = r_\varepsilon$ and $C = C_\varepsilon$ do not depend on the instance, so the running time of the implementations of both local-search rules is polynomial.

3 Tools from k-set packing

In this section we recall and reinterpret several pieces of the analyses of the local-search algorithms for k-SET PACKING; see [12, 7]. We focus on the subgraph of the conflict graph $G(\mathcal{F})$ induced by two solutions: a local and a global optimum. Sets belonging to both families can be ignored, which leads to a bipartite graph with degrees bounded by k. The following results are stated in the language of abstract bipartite graphs so that we can later use some of them in a slightly different context.

▶ **Definition 4.** Let $H = (A, B, E(H))$ be a bipartite graph. A set $X \subseteq B$ is called *improving*, if $|N_H(X)| < |X|$.

A slightly simpler version of the following lemma is a part of the analysis of the classic $(k+\varepsilon)/2$-approximation local search, which goes back to Hurkens and Schrijver [12]. Here, we observe that the worst-case $(k+\varepsilon)/2$ ratio can be attained only if (almost) all vertices in A are of degree k. If a constant fractions of vertices does not satisfy this property, our variant lets us derive a better bound.

▶ **Lemma 5.** *Fix a positive integer $k \geq 3$. For every $\varepsilon > 0$ there exists a constant c_ε satisfying the following property. Let $H = (A, B, E(H))$ be a bipartite graph with degrees not exceeding k. If there is no improving set $X \subseteq B$ with $|X| \leq c_\varepsilon$, then*

$$|B| \leq \tfrac{k-1+\varepsilon}{2}|A| + \tfrac{1}{2}|\{a \in A : \deg_H(a) = k\}|.$$

The proof below is based on the proof of Lemma 3.11 in [7], where the following auxiliary lemma is (implicitly) proved. For completeness, we provide its proof in the Appendix.

▶ **Lemma 6** ([7]). *Fix a positive integer $k \geq 3$ and a real number $\varepsilon > 0$. Let $H = (A, B, E(H))$ be a bipartite graph with degrees not exceeding k. If there is no improving set $X \subseteq B$ with $|X| \leq 2(k+1)^{\varepsilon^{-1}}$, then there exists an induced subgraph $H' = H[A', B']$ such that:*

(a) $|A \setminus A'| = |B \setminus B'|$,
(b) *there are no vertices $b \in B'$ with* $\deg_{H'}(b) = 0$,
(c) *there are at most $\varepsilon|A|$ vertices $b \in B'$ with* $\deg_{H'}(b) = 1$.

Proof of Lemma 5. Lemma 6 applied to the graph H gives its subgraph $H' = H[A', B']$. For every integer d define $B'_d = \{b \in B' : \deg_{H'}(b) = d\}$ and $B'_{d+} = \{b \in B' : \deg_{H'}(b) \geq d\}$. Let us count edges of H': clearly, $|E(H')| \leq (k-1)|A'| + |\{a \in A' : \deg_{H'}(a) = k\}|$ since the degrees do not exceed k. On the other hand, $|E(H')| \geq |B'_1| + 2|B'_{2+}|$, and consequently $|B'_1| + 2|B'_{2+}| \leq (k-1)|A'| + |\{a \in A' : \deg_{H'}(a) = k\}|$. Combining this inequality with the properties of H' following from Lemma 6, we get

$$\begin{aligned}
&2|B| = 2|B \setminus B'| + 2|B'| = 2|A \setminus A'| + 2|B'_1| + 2|B'_{2+}| \leq \\
&2|A \setminus A'| + |B'_1| + (k-1)|A'| + |\{a \in A' : \deg_{H'}(a) = k\}| \leq \\
&\qquad\qquad\qquad\qquad (k-1+\varepsilon)|A| + |\{a \in A : \deg_H(a) = k\}|,
\end{aligned}$$

that is, $|B| \leq \frac{k-1+\varepsilon}{2}|A| + \frac{1}{2}|\{a \in A : \deg_H(a) = k\}|$. ◀

The following lemma is a slightly stronger variant of Lemma 3.11 in [7]. Again we provide a better bound whenever a constant fraction of vertices in A does not have full degree.

▶ **Lemma 7.** *Fix a positive integer $k \geq 3$. For every $\varepsilon > 0$ there exists a constant C_ε satisfying the following property. Let $H = (A, B, E(H))$ be a bipartite graph with degrees not exceeding k. If there is no improving set $X \subseteq B$ such that $|X| \leq C_\varepsilon \log |V(H)|$ and $\mathrm{pw}(H[N_G[X]]) \leq C_\varepsilon$, then*

$$|B| \leq (\tfrac{k}{3} + \varepsilon)|A| + \tfrac{1}{3}|\{a \in A : \deg_H(a) \geq k\}|.$$

Proof. The proof is similar to that of Lemma 5. Instead of Lemma 6, we use the following auxiliary result proved in [7]. The construction of the subgraph H' is the same as in the proof of Lemma 6 (see Appendix). However, to derive point (4), formulated as Claim 3.12 in [7], the wider range of possibilities for X is exploited.

▶ **Claim 8** ([7]). *For every $\varepsilon > 0$ the constant C_ε can be chosen so that there exists an induced subgraph $H' = H[A', B']$ such that:*
(a) $|A \setminus A'| = |B \setminus B'|$,
(b) *there are no vertices $b \in B'$ with $\deg_{H'}(b) = 0$,*
(c) *there are at most $\varepsilon|A|$ vertices $b \in B'$ with $\deg_{H'}(b) = 1$,*
(d) *there are at most $(1+\varepsilon)|A'|$ vertices $b \in B'$ with $\deg_{H'}(b) = 2$.*

Again, for every integer d define $B'_d = \{b \in B' : \deg_{H'}(b) = d\}$ and $B'_{d+} = \{b \in B' : \deg_{H'}(b) \geq d\}$. As before, we count edges $E(H')$. We have $|E(H')| \geq |B'_1| + 2|B'_2| + 3|B'_{3+}|$ and $|E(H')| \leq (k-1)|A'| + |\{a \in A' : \deg_{H'}(a) = k\}|$. Summing up, we obtain

$$\begin{aligned}
&3|B| = 3|B \setminus B'| + 3|B'_1| + 3|B'_2| + 3|B'_{3+}| = 3|A \setminus A'| + 2|B'_1| + |B'_2| + |E(H')| \leq \\
&3|A \setminus A'| + 2\varepsilon|A| + (1+\varepsilon)|A'| + (k-1)|A'| + |\{a \in A' : \deg_{H'}(a) = k\}| \leq \\
&3|A \setminus A'| + k|A'| + 3\varepsilon|A| + |\{a \in A' : \deg_H(a) = k\}| \leq \\
&\qquad\qquad\qquad\qquad (k+3\varepsilon)|A| + |\{a \in A : \deg_H(a) = k\}|,
\end{aligned}$$

that is, $|B| \leq (\frac{k}{3} + \varepsilon)|A| + \frac{1}{3}|\{a \in A : \deg_H(a) = k\}|$. ◀

4 Reduction to simple instances

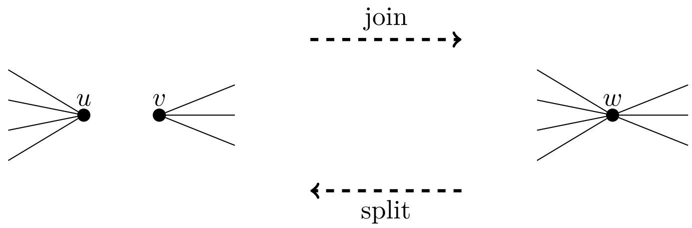
An instance $I = (G, d)$ of UDPO is called *simple* if $d^+(v), d^-(v) \in \{0, 1\}$ for every $v \in V(G)$ and *proper* if $\deg_G(v) \geq \max(d^+(v), d^-(v)) > 0$ for every $v \in V$. Clearly, any instance can be easily reduced to an equivalent proper instance by decreasing the degree constraints. In this section we show that it suffices to analyze the local-search algorithms for simple instances. More precisely, we prove that the worst-case ratio between the sizes of a local and a global optima is attained already for simple instances. Although this is stated below as an existential result, our reduction is constructive and it could be efficiently implemented.

▶ **Theorem 9.** *Fix a monotone local-search rule for* 3-SET PACKING. *Suppose that there exists an instance I of* UDPO *with a locally-optimum partial orientation F such that $|F| = \alpha|OPT_I|$. Then there exists a simple instance I' of* UDPO *with a locally-optimum partial orientation F' satisfying $|F'| = \alpha|OPT_{I'}|$.*

Let $I = (G, d)$ be an arbitrary instance. For a pair of distinct non-adjacent vertices $u, v \in V(G)$ we define the operation of *joining* u and v as follows: u and v are *identified* in G into a single vertex w and their degree constraints for w are obtain by summing the respective constraints for u and v.

Let us analyze how joining can be interpreted in terms of 3-SET PACKING instances obtained through the reduction of Section 2.1. Let $\mathcal{F}$ and $\mathcal{F}'$ families of 3-sets produced from instances I and I', respectively before and after joining. The universes $U = V^+ \cup V^- \cup E$ and $U' = V'^+ \cup V'^- \cup E'$ of both families can regarded as equal. This is because identifying non-adjacent vertices preserves the edge-set of the graph and because $d^+(u)$ copies of u and $d^+(v)$ copies of v in V^+ can be identified with $d^+(w) = d^+(u) + d^+(v)$ copies of w in V'^+ (similarly for V^- and V'^-). In this setting all 3-sets in $\mathcal{F}$ also belong to $\mathcal{F}'$ (though $\mathcal{F}'$ might be a strict superset of $\mathcal{F}$). Consequently, if a partial orientation is feasible in I, it is also feasible in the resulting instance I', but the converse does not necessarily hold.

If I' is obtained from I by joining u and v into w, we say that I can be obtained from I' by *splitting* w. Splitting is said to *preserve* a partial orientation A, if A is feasible in I' and remains feasible in I.



▶ **Lemma 10.** *Let $I = (G, d)$ be a proper instance with two feasible partial orientations A, B. If $\max(d^+(v), d^-(v)) \geq 2$ for some $v \in V(G)$, then one can split v so that both A and B are preserved and the resulting instance I' is proper.*

Proof. First, let us introduce an auxiliary vertex v' connected to v by $d^+(v) + d^-(v)$ parallel edges. We extend d to v' setting the constraints in v' large enough to accommodate all edges incident to v'. Note that this operation does not alter the original edges of G and the constraints at their endpoints. Hence, it has no effect on feasibility of A or B, in particular on whether one can split v.

Now, let us modify A to obtain A' by orienting $d^+(v) - \deg^+_A(v)$ edges from v to v' and $d^-(v) - \deg^-_A(v)$ edges from v' to v. Note that A' is feasible in the extended graph and the degree constraints for v are tight. Analogously, we extend B to B'. A larger partial orientation may only be harder to preserve, so it suffices to prove that one can split v preserving A' and B'. Equivalently, the construction in this paragraph lets us assume that $\deg^+_A(v) = \deg^+_B(v) = d^+(v)$ and $\deg^-_A(v) = \deg^-_B(v) = d^-(v)$.

Both for A and B we classify edges of G incident to v into three types: oriented towards v $(-)$, oriented towards the other endpoint $(+)$ and not included in the orientation (0). In total, we get a partition of the set $\delta(v)$, consisting of edges incident to v, into nine sets E_{ab}

with $a, b \in \{+, -, 0\}$; here a corresponds to the orientation in A and b to the orientation in B.

In some situations, one can clearly take a few edges incident to v, and split v into two vertices, one *new* vertex $\tilde{v}$ incident to the selected edges, and the other, still denoted as v, incident to the remaining edges. We refer to this operation as *splitting out* some edges. Note that in order to preserve both A and B, we need to split out edges so that for $\tilde{v}$ the number incoming edges is the same in both orientations, similarly for the outgoing arcs. We shall make sure that this number is always 0 or 1, i.e., $(d^+(\tilde{v}), d^-(\tilde{v})) \in \{(0,1),(1,0),(1,1)\}$. The constraints at v are decreased accordingly.

1. If $E_{++} \neq \emptyset$, one can split out a single edge $e \in E_{++}$ setting constraints $(1, 0)$; symmetrically if $E_{--} \neq \emptyset$ one sets $(0, 1)$.
2. If $E_{+-} \neq \emptyset$ and $E_{-+} \neq \emptyset$, one can split out two edges – one of each type, setting constraints $(1, 1)$.
3. If $E_{0+} \neq \emptyset$ and $E_{+0} \neq \emptyset$, one can split out two edges – one of each type, setting constraints $(1, 0)$; symmetrically if $E_{0-} \neq \emptyset$ and $E_{-0} \neq \emptyset$ one sets $(0, 1)$.
4. If $E_{+-} \neq \emptyset$, $E_{0+} \neq \emptyset$, and $E_{-0} \neq \emptyset$, one can split out three edges – one of each type, setting constraints $(1, 1)$; symmetrically, if $E_{-+} \neq \emptyset$, $E_{+0} \neq \emptyset$, and $E_{0-} \neq \emptyset$, one also sets $(1, 1)$.

We shall prove that one of these rules is always applicable. Note that the resulting instance is guaranteed to be proper as we have $\max(d^+(v), d^-(v)) \geq 2$, so it is impossible to leave v with both constraints equal to 0, which is forbidden in proper instances.

We proceed by contradiction, showing that if no rule is applicable, then $d^+(v) = d^-(v) = 0$, which is impossible because I is proper. Let $n_{ab} = |E_{ab}|$. Recall that we have made an assumption that $\deg^+_A(v) = \deg^+_B(v) = d^+(v)$ and $\deg^-_A(v) = \deg^-_B(v) = d^-(v)$, which implies the following equalities:

$$n_{0+} + n_{++} + n_{-+} = d^+(v) = n_{+0} + n_{++} + n_{+-},$$
$$n_{0-} + n_{+-} + n_{--} = d^-(v) = n_{-0} + n_{-+} + n_{--}.$$

If $n_{++} > 0$ or $n_{--} > 0$ we could apply rule 1. Therefore

$$n_{0+} + n_{-+} = d^+(v) = n_{+0} + n_{+-},$$
$$n_{0-} + n_{+-} = d^-(v) = n_{-0} + n_{-+}.$$

If $n_{+-} > 0$ and $n_{-+} > 0$ we could apply rule 2; without loss of generality we assume $n_{+-} = 0$ and thus

$$n_{0+} + n_{-+} = d^+(v) = n_{+0},$$
$$n_{0-} = d^-(v) = n_{-0} + n_{-+}.$$

Consequently, we have $n_{+0} \geq n_{0+}$ and $n_{0-} \geq n_{-0}$. Therefore, if $n_{0+} > 0$ or $n_{-0} > 0$, we could apply rule 3, which means that both these values are equal to 0 and

$$n_{0-} = n_{+0} = n_{-+} = d^+(v) = d^-(v).$$

However, if the common value of these variables was not equal to 0, we could apply rule 4. This way we get the announced contradiction. ◀

▶ **Corollary 11.** *If I is a proper instance with feasible partial orientations A and B, then with a finite sequence of vertex splitting preserving both A and B, one can obtain a simple proper instance I'.*

Proof. It suffices to exhaustively apply Lemma 10. Observe that this process must terminate, as vertex splitting increases the number of vertices and changes neither $D^+ = \sum_{v\in V(G)} d^+(v)$ nor $D^- = \sum_{v\in V(G)} d^-(v)$, while $|V(G)| \le D^+ + D^-$ for every proper instance. ◀

For a proof of Theorem 9, it suffices to apply Corollary 11 for $A = F$ and $B = OPT_I$. Vertex splitting may only reduce the family of feasible partial orientations, so OPT_I is still a global optimum. Also, this operation preserves F as a local optimum with respect any fixed monotone local-search rule. This follows from the fact that vertex splitting can be seen as removing sets in the underlying instance of 3-SET PACKING (without changing the size of the universe), and monotonicity means that removing sets from the universe does not make finding an improving set easier.

Therefore, Corollary 11 gives a simple instance I' for which F and OPT_I are still a local and a global optimum, respectively.

5 Another conflict graph for simple instances

We start the analysis of the local-search algorithms with a different construction of a conflict graph for a pair of feasible partial orientations A and B in a simple instance I of UDPO. The construction exploits the properties of simple instances and does not naturally generalize to arbitrary ones.

Let us consider an undirected graph $G' = (V(G), \overline{A} \cap \overline{B})$ and let $\mathcal{C}$ be the family of connected components of G'. For a connected component $C \in \mathcal{C}$ we define $\delta_A[C]$ as the set of arcs $e \in A$ incident to *exactly* one vertex of C; analogously we define $\delta_B[C]$.

▶ **Fact 12.** *For every component $C \in \mathcal{C}$ we have $|\delta_A[C]| \le 2$ and $|\delta_B[C]| \le 2$.*

Proof. As I is a simple instance, all the vertices in G' are of degree at most two, which means that C is a path or a cycle. Consequently, in either case, again by the assumption that I is simple, we have $|\delta_A[C]| \le 2$ and $|\delta_B[C]| \le 2$, because, both in A and in B, at most $2|C|$ arc endpoints can be incident to C and at least $2(|C| - 1)$ of these are endpoints of arcs induced by C. ◀

Let $A' = \{e \in A : \bar{e} \in \overline{A} \setminus \overline{B}\}$ and $B' = \{e \in B : \bar{e} \in \overline{B} \setminus \overline{A}\}$. We construct a bipartite graph $H = (A', B', E_H)$ so that $a \in A'$ is adjacent to $b \in B'$ whenever there is a component $C \in \mathcal{C}$ such that simultaneously $a \in \delta_A[C]$ and $b \in \delta_B[C]$. Since every arc in A' or B' is incident to exactly two components, Fact 12 lets us easily bound the degrees in H.

▶ **Corollary 13.** *The degree of every vertex in H is at most 4.*

The following lemma lets us interpret H as a conflict graph between A and B.

▶ **Lemma 14.** *For any $X \subseteq B'$ the following three-step procedure modifies A into another feasible partial orientation A_X:*
(a) *remove all arcs in $N_H(X)$,*
(b) *add all arcs in X,*
(c) *reverse all arcs in components C such that $X \cap \delta_B[C] \ne \emptyset$.*

Proof. We shall prove that that resulting orientation A_X satisfies the degree constraints for every vertex $v \in V(G)$. Let C be the component of v in G' (possibly $C = \{v\}$).

If $X \cap \delta_B[C] \ne \emptyset$, we shall prove that arcs incident to v in A_X form a subset of arcs incident to v in B. Indeed, by construction of H, all arcs $e \in A'$ incident to C were removed in step (1). Moreover, all arcs induced by C were reoriented in step (3) (from the orientation consistent with A to the orientation consistent with B). We might have added some arcs $e \in X$ incident to v in step (2) but these arcs are also present in B. Similarly, if $X \cap \delta_B[C] = \emptyset$, we shall prove that arcs incident to v in A_X form a subset of arcs incident to v in A. Indeed, no arcs incident to C could have been added in step (1) and the arcs induced by C were not reoriented in step (3). The only possible changes were removals in step (1).

In both cases we have shown that the arcs incident to v in A_X form a subset of arcs incident to v in another feasible partial orientation. Hence, A_X is feasible at any vertex v. ◀

Unfortunately, improvements through Lemma 14 in general might yield reorientation of many edges, which is unfeasible for our local-search rules. Thus, we slightly restrict the graph H to make sure that small improving sets in H yield small improving sets in the underlying 3-SET PACKING instance.

▶ **Lemma 15.** *Let I be a simple instance of* UDPO *and let A, B be a pair of feasible partial orientations. For any $\varepsilon > 0$ there exists a bipartite graph $H_\varepsilon = (A', B_\varepsilon, E_{H_\varepsilon})$ such that:*
(a) *$B_\varepsilon \subseteq B'$ and $|B_\varepsilon| \geq |B'| - \varepsilon|A|$,*
(b) *degrees in H_ε do not exceed 4,*
(c) *for any $X \subseteq B_\varepsilon$ there is a feasible partial orientation A_X with $|A_X| = |A| + |X| - |N_{H_\varepsilon}(X)|$ and $|A_X \setminus A| \leq (1 + 4\varepsilon^{-1})|X|$.*

Proof. Let $\mathcal{C}_\varepsilon \subseteq \mathcal{C}$ consist of components inducing at least $2\varepsilon^{-1}$ edges in G'. Note that the total size of components $C \in \mathcal{C}$ is $|\overline{A} \cap \overline{B}| \leq |A|$, so $|\mathcal{C}_\varepsilon| \leq \frac{\varepsilon}{2}|A|$. We set B_ε as the set of arcs $e \in B'$ which are not incident to any component $C \in \mathcal{C}_\varepsilon$. Fact 12 implies $|B' \setminus B_\varepsilon| \leq 2|\mathcal{C}_\varepsilon| \leq \varepsilon|A|$ as claimed in (1). We take H_ε as the induced subgraph $H[A, B_\varepsilon]$, so (2) immediately follows from Corollary 13.

Note that $N_{H_\varepsilon}(X) = N_H(X)$ for any $X \subseteq B_\varepsilon$. Thus, we can apply Lemma 14 to obtain the orientation A_X of the desired size. For every arc $b \in X$ step (3) yields reorientation of arcs induced by at most two components $C \in \mathcal{C} \setminus \mathcal{C}_\varepsilon$. These components consist of up to $2\varepsilon^{-1}$ edges each, so in total we reorient no more than $4\varepsilon^{-1}|X|$ arcs. Together with X itself, this gives at most $(1 + 4\varepsilon^{-1})|X|$ arcs in $A_X \setminus A$. ◀

Next, we analyze this conflict graph using tools originally developed for the classic local-search $(2 + \varepsilon)$-approximation of 4-SET PACKING.

▶ **Lemma 16.** *For any $\delta > 0$ there exists a constant r_δ such that for any simple instance I of* UDPO *the following condition holds. Let F be a weak local optimum (with $r = r_\delta$) and let OPT be an optimum partial orientation. Then $|\overline{OPT} \setminus \overline{F}| \leq 2|\overline{F} \setminus \overline{OPT}| + \delta|F|$.*

Proof. We proceed with a proof by contradiction for r_δ to be specified later. We apply Lemma 15 to $A = F$, $B = OPT$ and $\varepsilon = \frac{1}{2}\delta$ to obtain a bipartite graph H_ε. Note that $|B_\varepsilon| \geq |\overline{OPT} \setminus \overline{F}| - \varepsilon|F| \geq 2|\overline{F} \setminus \overline{OPT}| + \varepsilon|F| \geq (2 + \varepsilon)|\overline{F} \setminus \overline{OPT}| = (2 + \varepsilon)|A'|$.

We plug H_ε to Lemma 5 for $k = 4$ to conclude that there is an improving set $X \subseteq B_\varepsilon$ of size at most c_ε. Lemma 15(3) implies that there exists a feasible orientation F_X such that $|F_X| > |F|$ and $|F_X \setminus F| \leq (1 + 4\varepsilon^{-1})c_\varepsilon$. Thus, setting $r_\delta = (1 + 8\delta^{-1})c_{\delta/2}$ we can make sure that the weak rule of Fact 1 is able to perform the underlying improvement. This contradicts the assumption that F is a weak local optimum. ◀

6 Analysis

Finally, we combine Lemma 16 with generic properties of 3-SET PACKING local optima.

▶ **Theorem 17.** *For every $\varepsilon > 0$ there exists a constant r_ε such that for any instance of* UDPO *and any feasible partial orientation F which is a weak local optimum (with $r = r_\varepsilon$), we have $|OPT| \leq (\frac{4}{3} + \varepsilon)|F|$, where OPT is a maximum feasible partial orientation.*

Proof. By Theorem 9, it suffices to prove the claim for simple instances only. Let $C = OPT \cap F$. Note that $F \setminus C$ and $OPT \setminus C$ induce a bipartite subgraph $H = (F \setminus C, OPT \setminus C, E(H))$ of the conflict graph in the underlying instance of 3-SET PACKING.

Suppose $\deg_H(e) = 3$ for some arc $e \in F$. Note that $e = uv$ is represented by a 3-set $\{u_i^+, v_j^-, \bar{e}\}$ for some indices $i \leq d^+(u)$ and $j \leq d^-(v)$. The three neighbours of e in H are represented by disjoint 3-sets intersecting $\{u_i^+, v_j^-, \bar{e}\}$. One of them must contain $\bar{e}$ and since $e \notin OPT$, we conclude that $e^R \in OPT$. Consequently, $|\{e \in F : \deg_H(e) = 3\}| \leq |\overline{OPT} \cap \overline{F}|$.

We set r_ε at least as large as in Lemma 16 and as c_ε in Lemma 5 for $k = 3$. The latter result yields

$$|OPT| = |C| + |OPT \setminus C| \leq |C| + (1+\varepsilon)|F \setminus C| + \tfrac{1}{2}|\{e \in F \setminus C : \deg_H(e) = 3\}| \leq (1+\varepsilon)|F| + \tfrac{1}{2}|\overline{OPT} \cap \overline{F}|.$$

If $|\overline{OPT} \cap \overline{F}| \leq \frac{2}{3}|F|$, this already concludes the proof. Otherwise $|\overline{F} \setminus \overline{OPT}| \leq \frac{1}{3}|F|$ and we apply Lemma 16 to get $|\overline{OPT} \setminus \overline{F}| \leq 2|\overline{F} \setminus \overline{OPT}| + \varepsilon|F|$, and consequently obtain

$$|OPT| = |\overline{OPT} \setminus \overline{F}| + |\overline{OPT} \cap \overline{F}| \leq 2|\overline{F} \setminus \overline{OPT}| + |\overline{OPT} \cap \overline{F}| + \varepsilon|F| = |\overline{F} \setminus \overline{OPT}| + (1+\varepsilon)|F| \leq (\tfrac{4}{3} + \varepsilon)|F|,$$

which concludes the proof. ◀

▶ **Theorem 18.** *For every $\varepsilon > 0$ there exists a constant C_ε such that for any instance of* UDPO *and any feasible partial orientation F which is a strong local optimum (with $C = C_\varepsilon$), we have $|OPT| \leq (\frac{5}{4} + \varepsilon)|F|$, where OPT is a maximum feasible partial orientation.*

Proof. We apply the same argument except that we use Lemma 7 instead of Lemma 5. We take C_ε at least as large as in Lemma 7 for $k = 3$ and so that $C_\varepsilon \geq 2r_\varepsilon$ from Lemma 16. Then

$$|OPT| \leq |C| + (1+\varepsilon)|F \setminus C| + \tfrac{1}{3}|\{e \in F \setminus C : \deg_H(e) = 3\}| \leq (1+\varepsilon)|F| + \tfrac{1}{3}|\overline{OPT} \cap \overline{F}|.$$

If $|\overline{OPT} \cap \overline{F}| \leq \frac{3}{4}|F|$, this already concludes the proof. Otherwise $|\overline{F} \setminus \overline{OPT}| \leq \frac{1}{4}|F|$ and we apply Lemma 16 to obtain $|OPT| \leq |\overline{F} \setminus \overline{OPT}| + (1+\varepsilon)|F| \leq (\frac{5}{4} + \varepsilon)|F|$. ◀

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A Proof of Lemma 6

▶ **Lemma 6.** *Fix a positive integer $k \geq 3$ and a real number $\varepsilon > 0$. Let $H = (A, B, E(H))$ be a bipartite graph with degrees not exceeding k. If there is no improving set $X \subseteq B$ with $|X| \leq 2(k+1)^{\varepsilon^{-1}}$, then there exists an induced subgraph $H' = H[A', B']$ such that:*

(a) $|A \setminus A'| = |B \setminus B'|$,

(b) *there are no vertices $b \in B'$ with $\deg_{H'}(b) = 0$,*

(c) *there are at most $\varepsilon|A|$ vertices $b \in B'$ with $\deg_{H'}(b) = 1$.*

Proof. We inductively construct a sequence of induced subgraphs $(H_i)_{i=0}^{\ell}$ with $H_0 = H$ and $H_\ell = H'$ such that each $H_i = H[A_i, B_i]$ satisfies the following properties:

1. $|A \setminus A_i| = |B \setminus B_i| \geq \varepsilon i|A|$.
2. in H_i there is no subset $X \subseteq B_i$ such that $|X| \leq 2(k+1)^{\varepsilon^{-1}-i}$ and $|N_{H_i}(X)| < |X|$,

Note that $H_0 = H$ trivially satisfies both these properties. For the inductive step, consider the graph H_i. Let us classify vertices of B_i based on their degree in H_i: we define B_i^d as the set of vertices of degree d, and B_i^{d+} as the set of vertices of degree at least d. Note that the property 1. implies $i \leq \frac{1}{\varepsilon}$, and thus $2(k+1)^{\frac{1}{\varepsilon}-i} \geq 2$. Consequently, by property 2., $B_i^0 = \emptyset$ and the vertices of B_i^1 have distinct neighbours (otherwise we would have an improving set of size one or two, respectively).

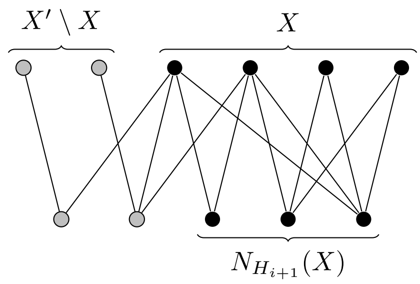

Figure 1 Lifting an improving set X in H_{i+1} to an improving set X' in H_i. Gray vertices belong to H_i but not to H_{i+1}.

We consider two cases, depending on whether $|B_i^1| \leq \varepsilon|A|$. If this inequality is satisfied, we shall prove that we can terminate at $i = \ell$ and return $H' = H_i$. The inequality directly corresponds to (3) in the statement of the lemma. Moreover, $B_i^0 = \emptyset$ is equivalent to (2) and the property 1. gives (1).

Otherwise, if $|B_i^1| > \varepsilon|A|$, we perform a further step of the construction. We build H_{i+1} setting $B_{i+1} = B_i^{2+}$ and $A_{i+1} = A_i \setminus N_{H_i}[B_i^1]$. As we have noted, vertices in B_i^1 do not share neighbours, so $|A_i \setminus A_{i+1}| = |B_i^1| = |B_i \setminus B_{i+1}|$, and consequently $|B \setminus B_{i+1}| = |A \setminus A_{i+1}|$. Also, we clearly have $|B \setminus B_{i+1}| \geq \varepsilon i|A| + |B_i^1| \geq \varepsilon(i+1)|A|$.

Hence, it suffices to show that H_{i+1} satisfies property 2. Take $X \subseteq B_{i+1}$ such that $|N_{H_{i+1}}(X)| < |X|$. We construct $X' \subseteq B_i$ with $|N_{H_i}(X')| < |X'|$ such that $|X'| \leq (k+1)|X|$. Clearly, if X then contradicts 2. for H_{i+1}, so does X' for H_i. Recall that $H_i[B_i \setminus B_{i+1}, A_i \setminus A_{i+1}]$ is a perfect matching. We denote the unique neighbour of a vertex v in this graph by $m(v)$. We simply define $X' = X \cup \{m(a) : a \in (A_i \setminus A_{i+1}) \cap N_{H_i}(X)\}$ (see also Figure 1). Then $N_{H_i}(X') = N_{H_i}(X) = N_{H_{i+1}}(X) \cup \{m(b) : b \in X' \setminus X\}$. Consequently, $|N_{H_i}(X')| = |N_{H_{i+1}}(X)| + |X' \setminus X| < |X| + |X' \setminus X| = |X'|$. Moreover, by the degree restriction in H, we have $|N_{H_i}(X)| \leq k|X|$, and thus $|X'| \leq |X| + |N_{H_i}(X)| \leq (k+1)|X|$, as claimed.

Finally, note that the property 1. implies $i \leq \varepsilon^{-1}$ so the construction terminates. ◀