Presenting a Category Modulo a Rewriting System*

Florence Clerc[1] and Samuel Mimram[2]

1 **CEA, LIST, Software Reliability Laboratory, PC 174, 91191 Gif-sur-Yvette, France**
LIX, École Polytechnique, France
PPS, Université Paris Diderot, France
florence.clerc@cea.fr
2 **LIX, École Polytechnique, France**
samuel.mimram@lix.polytechnique.fr

Abstract

Presentations of categories are a well-known algebraic tool to provide descriptions of categories by the means of generators, for objects and morphisms, and relations on morphisms. We generalize here this notion, in order to consider situations where the objects are considered modulo an equivalence relation (in the spirit of rewriting modulo), which is described by equational generators. When those form a convergent (abstract) rewriting system on objects, there are three very natural constructions that can be used to define the category which is described by the presentation: one is based on restricting to objects which are normal forms, one consists in turning equational generators into identities (i.e. considering a quotient category), and one consists in formally adding inverses to equational generators (i.e. localizing the category). We show that, under suitable coherence conditions on the presentation, the three constructions coincide, thus generalizing celebrated results on presentations of groups. We illustrate our techniques on a non-trivial example, and hint at a generalization for 2-categories.

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1 Introduction

Motivated by generalizing rewriting techniques to the setting of higher-dimensional categories, we introduce here a notion of presentation of a category modulo a rewriting system, in order to be able to present a category as generated by objects and morphisms, quotiented by relations on both morphisms *and objects*. This work can somehow be seen as an extension of traditional techniques of rewriting modulo an equational theory [1], in the case where the equational theory can itself be oriented as a convergent rewriting system, called an equational rewriting system. We provide here coherence conditions on both the original rewriting system and the equational one, so that expected properties hold: for instance, it should be "the same" to rewrite terms modulo the equational theory than to rewrite terms in normal form wrt the equational rewriting system. In this introduction, we expose our motivations, which come from higher-dimensional rewriting theory, however very little knowledge about this setting will be required in the remaining of the article.

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A *string rewriting system* P consists in an alphabet P_1 and a set $P_2 \subseteq P_1^* \times P_1^*$ of rules. Such a system induces a monoid $\|P\| = P_1^*/\overset{*}{\Leftrightarrow}$ obtained by quotienting the free monoid P_1^* on P_1 by the smallest congruence $\overset{*}{\Leftrightarrow}$ containing the rules in P_2; when the rewriting system is convergent, i.e. both confluent and terminating, normal forms provide canonical representatives of equivalence classes. Given a monoid M, we say that P is a *presentation* of M when M is isomorphic to $\|P\|$: in this case, the elements of P_1 can be seen as *generators* for M, and the elements of P_2 as a complete set of *relations* for M. For instance, the additive monoid $\mathbb{N} \times \mathbb{N}$ admits the presentation P with $P_1 = \{a, b\}$ and $P_2 = \{ba \Rightarrow ab\}$: namely, the string rewriting system is convergent, and its normal forms are words of the form $a^m b^n$, with $(m, n) \in \mathbb{N} \times \mathbb{N}$, from which it is easy to build the required isomorphism.

The notion of presentation is easy to generalize from monoids to categories (a monoid being the particular case of a category with one object): a presentation of category consists in generators for objects and morphisms, together with rules relating morphisms in the free category generated by the generators. Starting from this observation, people have considered a wild generalization of the notion of presentation, in order to present n-categories (computads [14, 13] or polygraphs [5]), thus providing us with a notion of *higher-dimensional rewriting system*. While we will not, in this article, consider much more than presentations of categories, the motivation for this work really comes from a limitation in presentations of 2-categories (and higher-dimensional categories) that we would like to overcome. We shall explain it on a simple example of a monoidal category (which, again, is a particular case of a 2-category with only one 0-cell).

Consider the well-known *simplicial category* Δ whose objects are integers $n \in \mathbb{N}$ and morphisms $f : m \to n$ are increasing functions $f : [m] \to [n]$ where $[m] = \{0, \ldots, m-1\}$. This category is monoidal, with tensor product being given by addition on objects ($m \otimes n = m + n$) and by "juxtaposition" on morphisms, and it is well known that it admits the following presentation as a monoidal category [12, 10]: its objects are generated by one object a, its morphisms are generated by $\mu : a \otimes a \to a$ and $\eta : I \to a$, and the relations are $\alpha : \mu \circ (\mu \otimes \mathrm{id}_a) = \mu \circ (\mathrm{id}_a \otimes \mu)$, $\lambda : \mu \circ (\eta \otimes \mathrm{id}_a) = \mathrm{id}_a$ and $\rho : \mu \circ (\mathrm{id}_a \otimes \eta) = \mathrm{id}_a$. This means that every morphism of Δ can be obtained as a composite of η and μ, and that two such formal composites represent the same morphism precisely when they can be related by the congruence generated by the above relations. As we can see on this example, a presentation P of a monoidal category consists in generators for objects (here $P_1 = \{a\}$), generators for morphisms ($P_2 = \{\eta, \mu\}$) and relations between composite of morphisms ($P_3 = \{\alpha, \lambda, \rho\}$). Notice that such a presentation *does not allow for relations between objects*, and thus is restricted to presenting monoidal categories whose underlying monoid of objects is free.

This limitation can be better understood by trying to present the monoidal category $\Delta \times \Delta$ with tensor product extending componentwise the one of Δ: the underlying monoid of objects is $\mathbb{N} \times \mathbb{N}$, which is not free. If we try to construct a presentation for this monoidal category, seen as consisting of "two copies" of the above category Δ, we are lead to consider a presentation containing "two copies" of the previous presentation: we consider a presentation P with $P_1 = \{a, b\}$ as object generators (where a and b respectively correspond to the objects $(1, 0)$ and $(0, 1)$), with $P_2 = \{\mu_a, \eta_a, \mu_b, \eta_b\}$ as morphism generators (with $\mu_a : a \otimes a \to a$, $\mu_b : b \otimes b \to b$, etc.), and with $P_3 = \{\alpha_a, \lambda_a, \rho_a, \alpha_b, \lambda_b, \rho_b\}$ as relations. If we stop here adding relations, the presented category has $\{a, b\}^*$ as underlying monoid of objects, which is not right: recalling the above presentation for $\mathbb{N} \times \mathbb{N}$, we should moreover add a relation $\gamma : ba = ab$. However, such a relation between objects is not allowed in the usual notion of presentation (only relations between morphisms are usually considered). In order to provide a meaning to it, three constructions are available: restrict P to some canonical representatives of objects modulo the equivalence generated by γ (typically the words of the form $a^m b^n$), quotient by γ the monoidal category $\|P\|$ presented by P, or formally invert the morphism γ in $\|P\|$. We show that under reasonable assumptions on the presentation, all three constructions coincide, thus providing one with a notion of *coherent* presentation modulo. As a fist step toward this situation, we study here the simpler case of presentations of categories and introduce a notion of presentation modulo for those, leaving the case of 2-categories for future work.

We begin by recalling the notion of presentation of a category (Section 2.1), then we extend it to work modulo a relation on objects (Section 2.2), and consider the quotient and localization wrt to the relation (Section 2.3). In order to compare those constructions, we consider equational rewriting systems equipped with a notion of residuation (Section 3.1) and satisfying a particular "cylinder" property (Section 3.2). We then show that, under suitable coherence conditions, the category of normal forms is isomorphic to the quotient (Section 4.1) and equivalent with the localization (Section 4.2), and illustrate our results on an example (Section 4.3). We finally discuss a possible extension of this work to presentations of 2-categories (Section 5) and conclude (Section 6).

2 Presentations of categories modulo a rewriting system

2.1 Presentations of categories

Recall that a *graph* (P_0, s_0, t_0, P_1) consists of two sets P_0 and P_1, of *vertices* and *edges* respectively, together with two functions $s_0, t_0 : P_1 \to P_0$ associating to an edge its *source* and *target* respectively. Such a graph generates a category with P_0 as objects and the set P_1^* of (directed) paths as morphisms. If we denote by $i_1 : P_1 \to P_1^*$ the coercion of edges to paths of length 1, and $s_0^*, t_0^* : P_1^* \to P_0$ the functions associating to a path its source and target respectively, we thus obtain a diagram as on the left below:

(1)

in **Set** which is commuting in the sense that $s_0^* \circ i_1 = s_0$ and $t_0^* \circ i_1 = t_0$.

▶ **Definition 1.** A *presentation* $P = (P_0, s_0, t_0, P_1, s_1, t_1, P_2)$, as pictured on the right of (1), consists in a graph (P_0, s_0, t_0, P_1) as above, the elements of P_0 (resp. P_1) being called *object* (resp. *morphism*) *generators*, together with a set P_2 of *relations* (or *2-generators*) and two functions $s_1, t_1 : P_2 \to P_1^*$ such that $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$. The category $\|P\|$ *presented* by P is the category obtained from the category generated by the graph (P_0, s_0, t_0, P_1) by quotienting morphisms by the smallest congruence wrt composition identifying any two morphisms f and g such that there exists $\alpha \in P_2$ satisfying $s_1(\alpha) = f$ and $t_1(\alpha) = g$.

In the following, we often simply write (P_0, P_1, P_2) for a presentation as above, leaving the source and target maps implicit. We write $f : x \to y$ for an edge $f \in P_1$ with $s_0(f) = x$ and $t_0(f) = y$, and $\alpha : f \Rightarrow g$ for a relation with f as source and g as target. We sometimes write $\alpha : f \Leftrightarrow g$ to indicate that $\alpha : f \Rightarrow g$ or $\alpha : g \Rightarrow f$ is an element of P_2, and we denote by $\overset{*}{\Leftrightarrow}$ the smallest congruence such that $f \overset{*}{\Leftrightarrow} g$ whenever there exists $\alpha : f \Rightarrow g$ in P_2.

▶ **Example 2.** The monoid $\mathbb{N}/2\mathbb{N}$ (seen as a category with only one object) admits the presentation P with $P_0 = \{x\}$, $P_1 = \{f : x \to x\}$ and $P_2 = \{\varepsilon : f \circ f \Rightarrow \mathrm{id}_x\}$.

▶ Remark. A presentation P generates a 2-category with invertible 2-cells (also called a (2,1)-category), whose set of 2-cells is denoted P_2^*, and the category presented by P is obtained from this 2-category by identifying 1-cells where there is a 2-cell in between [5, 10]. We write $\alpha : f \overset{*}{\Leftrightarrow} g$ for such a 2-cell.

▶ **Lemma 3.** *Any category $\mathcal{C}$ admits a presentation $P^{\mathcal{C}}$, called its* standard presentation, *with $P_0^{\mathcal{C}}$ being the set of objects of $\mathcal{C}$, $P_1^{\mathcal{C}}$ being the set of morphisms of $\mathcal{C}$ and $P_2^{\mathcal{C}}$ being the set of pairs $(f_2 \circ f_1, g) \in P_1^{\mathcal{C}*} \times P_1^{\mathcal{C}*}$ with $f_1, f_2, g \in P_1$ such that $s_0(f_1) = s_0(g)$, $t_0(f_2) = t_0(g)$ and $f_2 \circ f_1 = g$ in $\mathcal{C}$ (with projections as source and target functions).*

By previous lemma, every category admits at least one presentation. In general, it actually admits many presentations. It can be shown that two presentations present the same category if and only if they are related by Tietze transformations: those transformations generate all the operations one can do on a presentation without modifying the presented category [15, 8]. For instance, Knuth-Bendix completions are a particular case of those [9].

▶ **Definition 4.** Given a presentation P, a *Tietze transformation* consists in
- adding (resp. removing) a generator $f \in P_1$ and a relation $\alpha : f \Rightarrow g \in P_2$ with $g \in (P_1 \setminus \{f\})^*$,
- adding (resp. removing) a relation $\alpha : f \Rightarrow g \in P_2$ such that f and g are equivalent wrt the congruence generated by the relations in $P_2 \setminus \{\alpha\}$.

▶ **Proposition 5.** *Two presentations P and P' are related by a finite sequence of Tietze transformations if and only if they present the same category, i.e. $\|P\| \cong \|P'\|$.*

2.2 Presentations modulo

In a presentation P of a category, relations are generated by elements of P_2: the morphisms of the free category on the underlying graph will be quotiented by those in order to obtain the presented category. We now extend this notion in order to also allow for quotienting objects in the process of constructing the presented category.

▶ **Definition 6.** A *presentation modulo* $(P, \tilde{P}_1)$ consists of a presentation $P = (P_0, P_1, P_2)$ together with a set $\tilde{P}_1 \subseteq P_1$, whose elements are called *equational generators*.

The morphisms of $\|P\|$ generated by the equational generators are called *equational morphisms*. Intuitively, the category presented by a presentation modulo should be the "quotient category" $\|P\| / \tilde{P}_1$, as explained in next section, where objects equivalent under $\tilde{P}_1$ (i.e. related by equational morphisms) are identified. We believe that the reason why presentations modulo of categories were not introduced before is that they are unnecessary, in the sense that we can always convert a presentation modulo into a regular presentation, see Lemma 10 below. However, the techniques developed here extend in the case of 2-categories (this will be developed in a subsequent article) and moreover, our framework already enables us to easily obtain interesting results on presented categories, see Section 4.3.

▶ **Definition 7.** Given a presentation modulo $(P, \tilde{P}_1)$, we define the presentation $P/\tilde{P}_1$ as the presentation (P'_0, P'_1, P'_2) where
- $P'_0 = P_0/\cong_1$ where $\cong_1$ is the smallest equivalence such that $x \cong_1 y$ whenever there exists a generator $f : x \to y$ in $\tilde{P}_1$, and we denote $[x]$ the equivalence class of $x \in P_0$,
- the elements of P'_1 are $f : [x] \to [y]$ for $f : x \to y$ in P_1,
- the elements of P'_2 are of the form $\alpha : f \to g$ for $\alpha : f \to g$ in P_2, or $\alpha_f : f \to \mathrm{id}_{[x]}$ for $f : x \to y$ in $\tilde{P}_1$.

We will sometimes need to consider presentations modulo with "arrows reversed":

▶ **Definition 8.** Given a presentation modulo $(P, \tilde{P}_1)$, the presentation modulo $(P^{\mathrm{op}}, \tilde{P}_1^{\mathrm{op}})$ is given by $P^{\mathrm{op}} = (P_0, P_1^{\mathrm{op}}, P_2^{\mathrm{op}})$ where $P_1^{\mathrm{op}} = \{f^{\mathrm{op}} : y \to x \mid f : x \to y \in P_1\}$ and where $P_2^{\mathrm{op}} = \{\alpha^{\mathrm{op}} : f^{\mathrm{op}} \Rightarrow g^{\mathrm{op}} \mid \alpha : f \Rightarrow g\}$ with $f^{\mathrm{op}} = f_1^{\mathrm{op}} \circ ... \circ f_k^{\mathrm{op}}$ for $f = f_k \circ \ldots \circ f_1$ and where $\tilde{P}_1^{\mathrm{op}}$ is the subset of P_1^{op} corresponding to $\tilde{P}_1$.

2.3 Quotient and localization of a presentation modulo

As explained above, we want to quotient our presentations modulo by equational morphisms, in order for the equational morphisms to induce equalities in the presented category. Given a category $\mathcal{C}$ and a set Σ of morphisms, there are essentially two canonical ways to "get rid" of the morphisms of Σ in $\mathcal{C}$: we can either force them to be identities, or to be isomorphisms, giving rise to the two following notions of quotient and localization of a category. These are standard construction in category theory and we recall them below.

▶ **Definition 9.** The *quotient* of a category $\mathcal{C}$ by a set Σ of morphisms of $\mathcal{C}$ is a category $\mathcal{C}/\Sigma$ together with a *quotient functor* $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ sending the elements of Σ to identities, such that for every functor $F : \mathcal{C} \to \mathcal{D}$ sending the elements of Σ to identities, there exists a unique functor $\tilde{F}$ such that $\tilde{F} \circ Q = F$.

Such a quotient category always exists for general reasons [2] and is unique up to isomorphism. Given a presentation modulo $(P, \tilde{P}_1)$, the category presented by the associated (non-modulo) presentation $P/\tilde{P}_1$ described in Definition 7, corresponds to considering the category presented by the (non-modulo) presentation P and quotient it by $\tilde{P}_1$.

▶ **Lemma 10.** *Suppose given a presentation modulo $(P, \tilde{P}_1)$, the categories $\|P\| / \tilde{P}_1$ and $\|P/\tilde{P}_1\|$ are isomorphic.*

A second, slightly different construction, consists in turning elements of Σ into isomorphisms (instead of identities):

▶ **Definition 11.** The *localization* of a category $\mathcal{C}$ by a set Σ of morphisms is the category $\mathcal{C}[\Sigma^{-1}]$ together with a *localization functor* $L : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ sending the elements of Σ to isomorphisms, such that for every functor $F : \mathcal{C} \to \mathcal{D}$ sending the elements of Σ to isomorphisms, there exists a unique functor $\tilde{F}$ such that $\tilde{F} \circ L = F$.

In the case where the category is presented, its localization admits the following presentation.

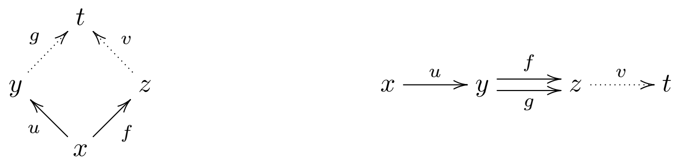
▶ **Lemma 12.** *Given a presentation $P = (P_0, P_1, P_2)$ and a subset Σ of P_1, the category presented by $P' = (P_0, P_1', P_2')$ where $P_1' = P_1 \uplus \left\{\overline{f} : y \to x \;\middle|\; f : x \to y \in \Sigma\right\}$ and where $P_2' = P_2 \uplus \left\{\overline{f} \circ f \Rightarrow \mathrm{id}, f \circ \overline{f} \Rightarrow \mathrm{id} \;\middle|\; f \in \Sigma\right\}$ is a localization of the category $\|P\|$ by Σ.*

▶ **Example 13.** Let us consider the category $\mathcal{C} = x \overset{f}{\underset{g}{\rightrightarrows}} y$ with two objects and two non-trivial morphisms. Its localization by $\Sigma = \{f, g\}$ is equivalent to the category with one object and $\mathbb{Z}$ as set of morphisms (with addition as composition), whereas its quotient by Σ is the category with one object and only identity as morphism. Notice that they are not equivalent.

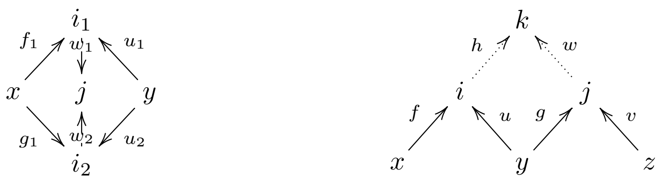
The description of the localization of a category provided by the universal property is often difficult to work with. When the set Σ has nice properties, the localization admits a much more tractable description [7, 4].

▶ **Definition 14.** A set Σ of morphisms of a category $\mathcal{C}$ is a *left calculus of fractions* when

1. the set Σ is closed under composition : for f and g composable morphisms in Σ, $g \circ f$ is in σ.
2. Σ contains the identities id_x for x in P_0.
3. for every pair of coinitial morphisms $u : x \to y$ in Σ and $f : x \to z$ in $\mathcal{C}$, there exists a pair of cofinal morphisms $v : z \to t$ in Σ and $g : y \to t$ in $\mathcal{C}$ such that $v \circ f = g \circ u$.
4. for every morphism $u : x \to y$ in Σ and pair of parallel morphisms $f, g : y \to z$ such that $f \circ u = g \circ u$ there exists a morphism $v : z \to t$ in Σ such that $v \circ f = v \circ g$.



▶ **Theorem 15.** *When Σ is a left calculus of fractions for a category $\mathcal{C}$, the localization $\mathcal{C}[\Sigma^{-1}]$ can be described as the* category of fractions *whose objects are the objects of $\mathcal{C}$ and morphisms from x to y are equivalence classes of pairs of cofinal morphisms (f, u) with $f : x \to i \in \mathcal{C}$ and $u : y \to i \in \Sigma$ under the equivalence relation identifying two such pairs (f_1, u_1) and (f_2, u_2) where there exists two morphisms $w_1, w_2 \in \Sigma$ such that $w_1 \circ u_1 = w_2 \circ u_2$ and $w_1 \circ f_1 = w_2 \circ f_2$, as shown on the left:*



identity on an object x is the equivalence class of $(\mathrm{id}_x, \mathrm{id}_x)$ and composition of two morphisms $(f, u) : x \to y$ and $(g, v) : y \to z$ is the equivalence class of $(h \circ f, w \circ v) : x \to z$ where the morphisms h and w are provided by property 1 of Definition 14.

In such a situation, the following property often enables one to show that there is a full and faithful embedding of the category into its localization [4]:

▶ **Proposition 16.** *Given a left calculus of fractions Σ for a category $\mathcal{C}$, if all the morphisms of Σ are mono then the inclusion functor $F : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ is faithful, where F is the identity on objects and sends a morphism $f : x \to y$ to (f, id_y).*

Given a presentation modulo, when the (abstract) rewriting system on objects given by the equational generators is convergent, normal forms for objects provide canonical representatives of objects modulo equational generators, and therefore we are actually provided with three possible and equally reasonable constructions for the category presented by a presentation modulo $(P, \tilde{P}_1)$:

1. the full subcategory on $\|P\|$ whose objects are normal forms wrt $\tilde{P}_1$,
2. the quotient category $\|P\| / \tilde{P}_1$,
3. the localization $\|P\| [\tilde{P}_1^{-1}]$.

The aim of this article is to provide reasonable assumptions on the presentation modulo ensuring that the two first categories are isomorphic, and equivalent to the third one. We introduce them gradually.

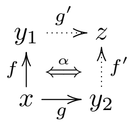
3 Confluence properties

In this section, we introduce a series of local conditions that our presentations modulo should satisfy in order for the constructions recalled above to coincide. These can be seen as a generalization of classical local confluence properties in our context in which rewriting rules correspond to equational generators only, and in which we keep track of 2-cells witnessing local confluence.

3.1 Residuation

We begin by extending to our setting the notion of residual, which is often associated to a confluent rewriting system in order to "keep track" of rewriting steps once others have been performed [11, 3, 6].

▶ **Assumption 1.** We suppose fixed a presentation modulo $(P, \tilde{P}_1)$ such that

1. for every pair of distinct coinitial generators $f : x \to y_1$ in $\tilde{P}_1$ and $g : x \to y_2$ in P_1, there exist a pair of cofinal morphisms $g' : y_1 \to z$ in P_1^* and $f' : y_2 \to z$ in $\tilde{P}_1^*$ and a relation $\alpha : g' \circ f \Leftrightarrow f' \circ g$ in P_2, as shown on the right,


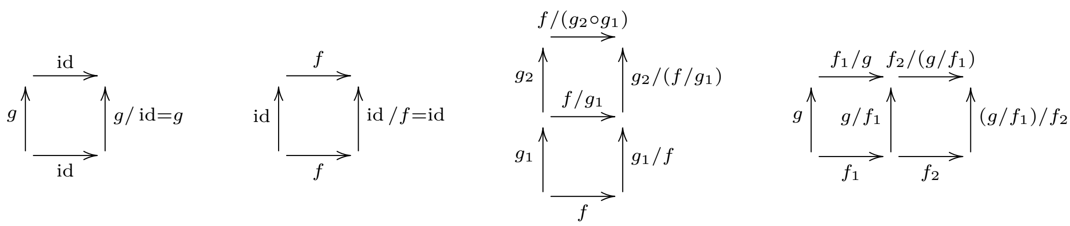
2. there is no infinite path with generators in $\tilde{P}_1$.

These assumptions ensure in particular that the (abstract) rewriting system on vertices with $\tilde{P}_1$ as set of rules is convergent. Given a vertex $x \in P_0$, we write $\hat{x}$ for the associated normal form. For every pair of distinct morphisms (f, g), as in the first assumption, we suppose fixed an arbitrary choice of a particular triple (g', α, f') associated to it, and write g/f for g', f/g for f' and $\rho_{f,g}$ for α. The morphism g/f (resp. f/g) is as the *residual* of g after f (resp. f after g): intuitively, g/f corresponds to what remains of g once f has been performed. It is natural to extend this definition to paths as follows:

▶ **Definition 17.** Given two coinitial paths $f : x \to y$ and $g : x \to z$ and P_1^* such that either f or g is in $\tilde{P}_1^*$, we define the *residual* g/f of g after f as above when f and g are distinct generators, and by induction with $f/f = \mathrm{id}_y$ and

$$g/\mathrm{id}_x = g \qquad \mathrm{id}_x/f = \mathrm{id}_y \qquad (g_2 \circ g_1)/f = (g_2/(f/g_1)) \circ (g_1/f) \qquad g/(f_2 \circ f_1) = (g/f_1)/f_2$$

(by convention the residual g/f is not defined when neither f nor g belongs to $\tilde{P}_1^*$). Graphically,



It can be checked that residuation is well-defined on the morphisms of the free category P_1^* in the sense that it is compatible with associativity and identities, and moreover it does not depend on the order in which rules are applied, see Lemma 20. In order for the definition to be well-founded, and thus always defined, we will make the following additional assumption.

▶ **Assumption 2.** There is a weight function $\omega_1 : P_1 \to \mathbb{N}$, and we still write $\omega_1 : P_1^* \to \mathbb{N}$ for its extension as morphism of category to the category corresponding to the additive monoid $(\mathbb{N}, +)$, such that for every generator $g \in P_1$ and $f \in \tilde{P}_1$, we have $\omega_1(g/f) < \omega_1(g)$.

▶ Remark. In order to simplify the presentation, we did not present the most general axiomatization for the weight function. An important point is that it induces a well-founded ordering on elements of P_1^* and satisfies properties similar to monomial orderings:

- it is compatible with composition: if $\omega_1(g) < \omega_1(g')$ then $\omega_1(h \circ g \circ f) < \omega_1(h \circ g' \circ f)$,
- identities are minimal elements: $\omega_1(\mathrm{id}) < \omega_1(f)$ for every $f \neq \mathrm{id}$; in particular, we have $\omega_1(g) < \omega_1(h \circ g \circ f)$ for $f, h \neq \mathrm{id}$.

In order to study confluence of the rewriting system provided by equational morphisms, through the use of residuals, we first introduce the following category, which allows us to consider, at the same time, both residuals g/f and f/g of two coinitial morphisms f and g.

▶ **Definition 18.** The *zig-zag presentation* associated to the presentation modulo $(P, \tilde{P}_1)$ is the presentation $Z = (Z_0, Z_1, Z_2)$ with $Z_0 = P_0$, $Z_1 = P_1 \uplus \overline{\tilde{P}_1}$ (generators in $\overline{\tilde{P}_1}$ are of the form $\overline{f} : B \to A$ for some generator $f : A \to B$ in $\tilde{P}_1$) and relations in Z_2 are of the form $g \circ \overline{f} \Rightarrow \overline{(f/g)} \circ (g/f)$ or $f \circ \overline{f} \Rightarrow \mathrm{id}_y$ for some pair of distinct coinitial generators $f : x \to y \in \tilde{P}_1$ and $g : x \to z \in P_1$.

▶ **Lemma 19.** *The rewriting system on morphisms in Z_1^* with Z_2 as rules is convergent. Given two coinitial morphisms $f : x \to y$ in $\tilde{P}_1^*$ and $g : x \to z$ in P_1^*, the normal form of $g \circ \overline{f}$ is $\overline{(f/g)} \circ (g/f)$.*

Proof. We extend the weight function of Assumption 2 to morphisms in Z_1^* by setting $\omega_1(\overline{f}) = 0$ for $\overline{f}$ in $\overline{\tilde{P}_1}$. This ensures that the rewriting system on morphisms in Z_1^* with Z_2 as rules is terminating. Moreover, because the left members of rules are of the form $g \circ \overline{f}$ with $g \in P_1$ and $\overline{f} \in \overline{\tilde{P}_1}$, there are no critical pairs, which means that the rewriting system is locally confluent and thus convergent by Newman's lemma. Given two coinitial morphisms $f : x \to y$ in $\tilde{P}_1^*$ and $g : x \to z$ in P_1^*, we prove by recurrence on $\omega_1(g \circ \overline{f})$ that the normal form of $g \circ \overline{f}$ is $\overline{(f/g)} \circ (g/f)$. ◀

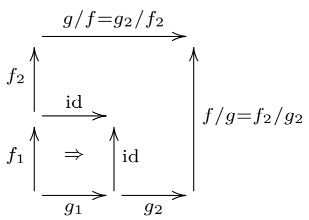
As a direct corollary of the convergence of the rewriting system, one can show that Definition 17 makes sense:

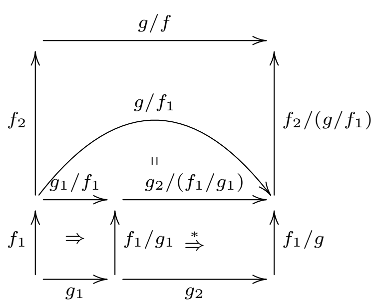
▶ **Lemma 20.** *The residuation operation does not depend on the order in which equalities of Definition 17 are applied.*

Moreover, a "global" version of the residuation property (Assumption 1) holds:

▶ **Proposition 21.** *Given two coinitial morphisms $f : x \to y$ in $\tilde{P}_1^*$ and $g : x \to z$ in P_1^*, there exists a relation $\alpha : (g/f) \circ f \overset{*}{\Leftrightarrow} (f/g) \circ g$.*

Proof. By Lemma 19, there exists a rewriting path $\beta : g \circ \overline{f} \Rightarrow \overline{(f/g)} \circ (g/f)$ in Z_2^*. By induction on its length, we can construct a relation $\alpha : (g/f) \circ f \overset{*}{\Leftrightarrow} (f/g) \circ g$ in the following way. The case where β is empty is immediate, otherwise we have $f = f_2 \circ f_1$ and $g = g_2 \circ g_1$ where f_2 is in $\tilde{P}_1^*$ (resp. g_2 in P_1^*) and f_1 is a generator in $\tilde{P}_1$ (resp. g_1 in P_1). We distinguish two cases depending on the form of the first rule of β:





If $f_1 = g_1$, i.e. the first step of β corresponds to rewriting $g_2 \circ g_1 \circ \overline{f_1} \circ \overline{f_2}$ to $g_2 \circ \overline{f_2}$ by applying the rewriting rule $f_1 \circ \overline{f_1} \Rightarrow$ id of Z_2, then by induction hypothesis, there exists a relation $\hat{\alpha}' : (g_2/f_2) \circ f_2 \overset{*}{\Leftrightarrow} (f_2/g_2) \circ g_2$. Besides, $f_2/g_2 = f/g$ and $g_2/f_2 = g/f$ which means that there exists a relation $(g/f) \circ f \overset{*}{\Leftrightarrow} (f/g) \circ g$. Otherwise $f_1 \neq g_1$, and $g_2 \circ g_1 \circ \overline{f_1} \circ \overline{f_2}$ rewrites to $g_2 \circ \overline{(f_1/g_1)} \circ (g_1/f_1) \circ \overline{f_2}$ by applying the rewriting rule $g_1 \circ \overline{f_1} \Rightarrow \overline{(f_1/g_1)} \circ (g_1/f_1)$ of Z_2. By definition of the relations in Z_2, there exists a relation $(g_1/f_1) \circ f_1 \Leftrightarrow (f_1/g_1) \circ g_1$ in P_2. Moreover, by Lemma 19, the morphism $g_2 \circ \overline{(f_1/g_1)}$ in Z_1^* rewrites to $\overline{(f_1/g)} \circ (g_2/(f_1/g_1))$, and therefore by induction hypothesis, there exists a relation $(g_2/(f_1/g_1)) \circ (f_1/g_1) \overset{*}{\Leftrightarrow} ((f_1/g_1)/g_2) \circ g_2$ in P_2^*. This means that there is a relation in P_2^*

$$(g/f_1) \circ f_1 = (g_2/(f_1/g_1)) \circ (g_1/f_1) \circ f_1 \overset{*}{\Leftrightarrow} ((f_1/g_1)/g_2) \circ g_2 \circ g_1 = (f_1/g) \circ g$$

Similarly, by lemma 19, $(g/f_1) \circ \overline{f_2}$ rewrites to $\overline{(f_2/(g/f_1))} \circ (g/f)$ by rules in Z_2, which means that there exists a relation $(g/f) \circ f_2 \overset{*}{\Leftrightarrow} (f_2/(g/f_1)) \circ (g/f_1)$ in P_2^* and therefore, there exists a relation in P_2^*:

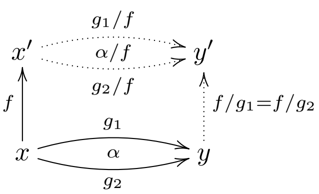
$$(g/f) \circ f = (g/f) \circ f_2 \circ f_1 \overset{*}{\Leftrightarrow} (f_2/(g/f_1)) \circ (f_1/g) \circ g = (f/g) \circ g$$

from which we conclude, as indicated in the above diagram. ◀

3.2 The cylinder property

In previous section, we have studied residuation, which enables one to recover a residual g/f of a morphism g after a coinitial equational morphism f. We now strengthen our hypothesis in order to ensure that if two morphisms are equal (wrt the equivalence generated by P_2^*) then their residuals after a same morphism are equal, i.e. equality is compatible with residuation.

▶ **Assumption 3.** The presentation $(P, \tilde{P}_1)$ satisfies the *cylinder property*: for every triple of coinitial morphisms $f : x \to x'$ in $\tilde{P}_1$ (resp. in P_1) and $g_1, g_2 : x \to y$ in P_1^* (resp. in $\tilde{P}_1^*$) such that there exists a relation $\alpha : g_1 \Leftrightarrow g_2$, we have $f/g_1 = f/g_2$ and there exists a 2-cell $g_1/f \overset{*}{\Leftrightarrow} g_2/f$. We write α/f for an arbitrary choice of such a 2-cell.



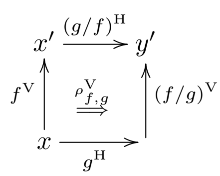
As in previous section, we would like to extend this "local" property (f and α are supposed to be generators) to a "global" one (where f and α can be composite of cells):

▶ **Proposition 22** (Global cylinder property)**.** *Given coinitial morphisms $f : x \to x'$ in $\tilde{P}_1^*$ (resp. in P_1^*) and $g_1, g_2 : x \to y$ in P_1^* (resp. in $\tilde{P}_1^*$) such that there exists a composite relation $\alpha : g_1 \overset{*}{\Leftrightarrow} g_2$, we have $f/g_1 = f/g_2$ and there exists a 2-cell $g_1/f \overset{*}{\Leftrightarrow} g_2/f$.*

The proof of previous proposition requires generalizing, in dimension 2, the termination condition (Assumption 2) and the construction of the zig-zag presentation (Definition 18).

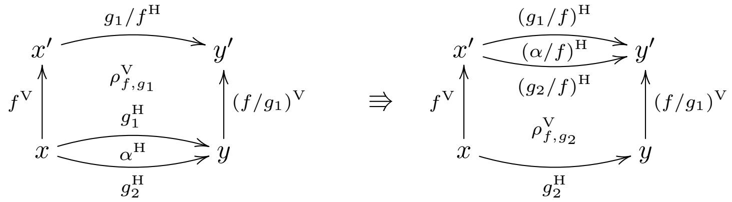
▶ **Definition 23.** The *2-zig-zag presentation* associated to $(P, \tilde{P}_1)$ is $Y = (Y_0, Y_1, Y_2)$ with $Y_0 = P_0$, $Y_1 = P_1^{\mathrm{H}} \uplus P_1^{\mathrm{V}}$ (where the morphisms of P_1^{H} are called *horizontal* of the form $f^{\mathrm{H}} : A \to B$ for some morphism $f : A \to B$ in P_1 and similarly for the morphisms in P_1^{V} which are called *vertical*), and the 2-cells in $Y_2 = Y_2^{\mathrm{H}} \uplus Y_2^{\mathrm{V}}$ are either

- horizontal 2-cells: $Y_2^{\mathrm{H}} = P_2^{\mathrm{H}} \uplus \overline{P_2}^{\mathrm{H}}$ (i.e. relations in P_2 taken forward or backward, and decorated by H)
- vertical 2-cells: given two generators $f : x \to y$ and $g : x \to z$ in P_1 such that f or g belongs to $\tilde{P}_1$, we have a relation $\rho_{f,g}^{\mathrm{V}} : (g/f)^{\mathrm{H}} \circ f^{\mathrm{V}} \Rightarrow (f/g)^{\mathrm{V}} \circ g^{\mathrm{H}}$ in Y_2^{V}.



We consider the following rewriting system on the 2-cells in Y_2^*: for every 2-cell $\alpha : g_1 \Leftrightarrow g_2 : x \to y$ in P_2, for every coinitial 1-cell $f : x \to x'$ in P_1 such that either f or both g_1 and g_2 belong to $\tilde{P}_1^*$, there is a rewriting rule

$$((f/g_1)^{\mathrm{V}} \circ \alpha^{\mathrm{H}}) \bullet \rho^{\mathrm{V}}_{f,g_1} \quad \Rrightarrow \quad \rho^{\mathrm{V}}_{f,g_2} \bullet ((\alpha/f)^{\mathrm{H}} \circ f^{\mathrm{V}})$$



(2)

where $\circ$ (resp. $\bullet$) denotes horizontal (resp. vertical) composition in a 2-category.

In order to ensure the termination of the rewriting system, we suppose the following.

▶ **Assumption 4.** There is a weight function $\omega_2 : P_2^{\mathrm{H}} \to \mathbb{N}$ such that for every $\alpha : g_1 \Rightarrow g_2$ in Y_2^* and f in P_1 such that α/f exists we have $\omega_2(\alpha/f) < \omega_2(\alpha)$. We still write $\omega_2 : \left(P_2^{\mathrm{H}} \uplus \overline{P_2}^{\mathrm{H}}\right)^* \to \mathbb{N}$ for the function such that $\omega_2(\overline{\alpha}) = \omega_2(\alpha)$ and both horizontal and vertical compositions are sent to addition ($\mathbb{N}$ being a commutative additive monoid, this definition is compatible with axioms of 2-categories, such as associativity or exchange law).

▶ **Corollary 24.** *The rewriting system* (2) *is convergent.*

Proposition 22 follows easily, by a reasoning similar to Proposition 21.

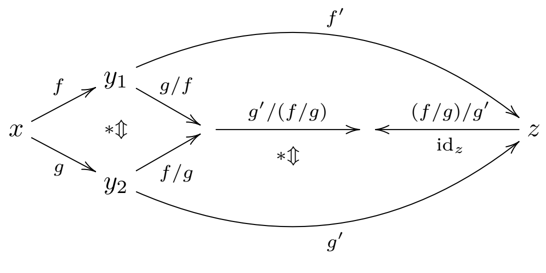
The cylinder property has many interesting consequences for the residuation operation, as we now investigate.

▶ **Proposition 25.** *In the category* $\|P\|$*, every equational morphism is epi.*

Proof. Suppose given $f : x \to y$ in $\tilde{P}_1^*$, and $g_1, g_2 : y \to z$ in P_1^* such that $g_1 \circ f \overset{*}{\Leftrightarrow} g_2 \circ f$. By Proposition 22, we have $g_1 = (g_1 \circ f)/f \overset{*}{\Leftrightarrow} (g_2 \circ f)/f = g_2$. ◀

▶ **Proposition 26.** *In the category* $\|P\|$*, every morphism* g *admits a pushout along a coinitial equational morphism* f *given by* g/f*.*

Proof. Suppose given $f : x \to y_1$ in $\tilde{P}_1^*$ and $g : x \to y_2$ in P_1^*. By Proposition 21, we have $(g/f) \circ f \overset{*}{\Leftrightarrow} (f/g) \circ g$ and we now show that $(g/f, f/g)$ forms a universal cocone. Suppose given $f' : y_1 \to z$ and $g' : y_2 \to z$ such that $f' \circ f \overset{*}{\Leftrightarrow} g' \circ g$.



We have $(g'/(f/g)) \circ (f/g) \overset{*}{\Leftrightarrow} ((f/g)/g') \circ g'$, where residuals exist because f/g is in $\tilde{P}_1^*$. Moreover, by applying Proposition 22 to morphism f and 2-cell $f' \circ f \overset{*}{\Leftrightarrow} g' \circ g$, we have $(f/g)/g' = f/(g' \circ g) \overset{*}{\Leftrightarrow} f/(f' \circ f) = \mathrm{id}_z$. Finally, we have $f' \circ f \overset{*}{\Leftrightarrow} g' \circ g \overset{*}{\Leftrightarrow} (g'/(f/g)) \circ (g/f) \circ f$, and by Proposition 25, we have $f' \overset{*}{\Leftrightarrow} (g'/(f/g)) \circ (g/f)$. From which we conclude. ◀

4 Comparing presented categories

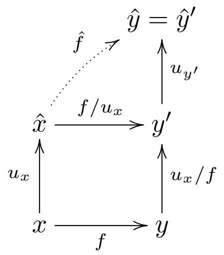
4.1 The category of normal forms

We show here that with our hypothesis on the rewriting system, the quotient category $\|P\|/\tilde{P}_1$ can be recovered as the following subcategory of $\|P\|$, whose objects are those which are in normal form for $\tilde{P}_1$.

▶ **Definition 27.** The *category of normal forms* $\|P\|{\downarrow}\tilde{P}_1$ is the full subcategory of $\|P\|$ whose objects are the normal forms of elements of P_0 wrt rules in $\tilde{P}_1$. We write $I : \|P\|{\downarrow}\tilde{P}_1 \to \|P\|$ for the inclusion functor.

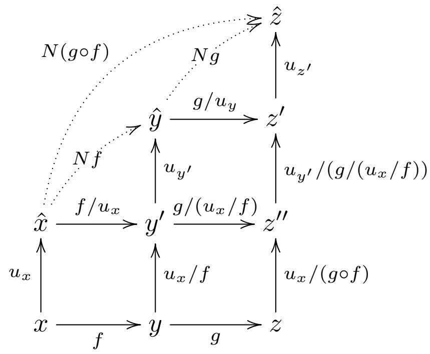
▶ **Theorem 28.** *The category $\|P\|{\downarrow}\tilde{P}_1$ is (isomorphic to) the quotient category $\|P\|/\tilde{P}_1$.*

Proof. Recall that for every object $x \in \|P\|$, the associated normal form wrt rules in $\tilde{P}_1$ is denoted by $\hat{x}$, and we write $u_x : x \to \hat{x}$ for any equational morphism from x to its normal form. In particular, we always have $u_{\hat{x}} = \mathrm{id}_{\hat{x}}$. We define a functor $N : \|P\| \to \|P\|{\downarrow}\tilde{P}_1$ as the functor associating to each object x its normal form $\hat{x}$ under $\tilde{P}_1$, and to each morphism $f : x \to y$, the morphism $\hat{f} : \hat{x} \to \hat{y}$ where $\hat{f} = u_{y'} \circ (f/u_x)$ with y' being the target of f/u_x:



Notice that this definition depends on a choice of a representative in P_1^* for f, and in $\tilde{P}_1^*$ for u_x and $u_{y'}$, in the equivalence classes of morphisms modulo the relations in P_2. The global cylinder property shown in Proposition 22 ensures that the definition is independent of the choice of such representatives. Given two composable morphisms $f : x \to y$ and $g : y \to z$ we have

$$
\begin{aligned}
Ng \circ Nf &= u_{z'} \circ (g/u_y) \circ u_{y'} \circ (f/u_x) \\
&= u_{z'} \circ (g/(u_{y'} \circ (u_x/f))) \circ u_{y'} \circ (f/u_x) \\
&= u_{z'} \circ (g/(u_x/f))/u_{y'} \circ u_{y'} \circ (f/u_x) \\
&= u_{z'} \circ u_{y'}/(g/(u_x/f)) \circ g/(u_x/f) \circ (f/u_x) \\
&= u_{z''} \circ ((g \circ f)/u_x) \\
&= N(g \circ f)
\end{aligned}
$$



The image of an equational morphism $u : x \to y$ under the functor N is an identity. Namely, we have $Nu = \hat{u} = u_z \circ (u/u_x)$: since u/u_x is an equational morphism (since it is the residual of an equational morphism) whose source is a normal form, necessarily $u/u_x = \mathrm{id}_{\hat{x}}$, $z = \hat{x}$ and $u_z = \mathrm{id}_{\hat{x}}$. In particular, N preserves identities.

Suppose given a functor $F : \|P\| \to \mathcal{C}$ sending the equational morphisms to identities. In order to obtain the result, we have to show that there exits a unique functor $G : \|P\|{\downarrow}\tilde{P}_1 \to \mathcal{C}$

such that $G \circ N = F$. Writing $I : \|P\| {\downarrow} \tilde{P}_1 \to \|P\|$ for the inclusion functor, it is easy to show I is a section of F, i.e. $N \circ I = \mathrm{Id}_{\|P\|{\downarrow}\tilde{P}_1}$. Since F sends equational morphisms to identities, it is easy to check that $G \circ N = F$: given an object x, we have

$$G \circ N(x) = G(\hat{x}) = F \circ I(\hat{x}) = F(\hat{x}) = F(x)$$

the last equality, being due to the fact that $F(u_x) = \mathrm{id}_{F(\hat{x})} = \mathrm{id}_{F(x)}$, and similarly for morphisms. Finally, we check the uniqueness of G. Suppose given another functor $G' : \|P\| {\downarrow} \tilde{P}_1 \to \mathcal{C}$ such that $G' \circ N = F = G \circ N$. We have $G' = G' \circ N \circ I = G \circ N \circ I = G$. ◀

4.2 Equivalence with localization

We now show that the two previous constructions (quotient and normal forms) also coincide with the third possible construction which consists in formally adding inverses for equational morphisms. First, notice that we can use the description of the localization $\|P\| [\tilde{P}_1^{-1}]$ as a category of fractions given in Theorem 15:

▶ **Lemma 29.** *The set of equational morphisms of $\|P\|$ is a left calculus of fractions.*

Proof. We have to show that the set of equational morphisms satisfies the four conditions of Definition 14: the two first (closure under composition and identities) are immediate, the third one follows from Proposition 21, and the last one is ensured by the fact that all equational morphisms are epi by Proposition 25. ◀

Our proof of the equivalence is based on the embedding of the presented category into the localization provided by Proposition 16. In order for the hypothesis of this proposition to hold, we first need to impose that the same properties hold for the opposite presentation as for the presentation itself:

▶ **Assumption 5.** The presentation modulo $(P^{\mathrm{op}}, \tilde{P}^{\mathrm{op}})$ satisfies Assumptions 1, 2, 3 and 4.

This implies that the duals of previously shown properties hold for $\|P\|$. For instance, by dual of Proposition 25, all equational morphisms are mono, from which follows, by Proposition 16:

▶ **Proposition 30.** *The canonical functor $\|P\| \to \|P\| [\tilde{P}_1^{-1}]$ is faithful.*

▶ Remark. This generalizes Dehornoy's theorem [6] stating that under conditions (which are generalized here), there is an embedding of a monoid into its envelopping groupoid: by localizing wrt all morphisms rather than simply a subset of them, we recover this result. Besides, our hypothesis on relations are weaker (for instance, we only require fixed a choice of residual instead that there is only one possible choice for those).

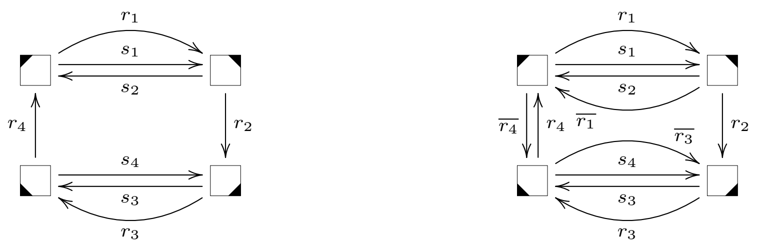
▶ **Definition 31.** A presentation modulo satisfying assumptions 1 to 5 is called *coherent.*

▶ **Theorem 32.** *Given a coherent presentation modulo $(P, \tilde{P}_1)$, the categories $\|P\| / \tilde{P}_1$ and $\|P\| [\tilde{P}_1^{-1}]$ are equivalent.*

Proof. Consider the functor $F : \|P\| {\downarrow} \tilde{P}_1 \to \|P\| [\tilde{P}_1^{-1}]$ defined as the composite of the inclusion functor $I : \|P\| {\downarrow} \tilde{P}_1 \to \|P\|$, see Definition 27, with the localization functor $L : \|P\| \to \|P\| [\tilde{P}_1^{-1}]$, see Definition 11. The functor F is faithful since it is the case for both I and L by Proposition 30. It is also full. Namely, by Theorem 15, given any two objects $\hat{x}$ and $\hat{y}$ of $\|P\| {\downarrow} \tilde{P}_1$, a morphism from $F(\hat{x}) = \hat{x}$ to $F(\hat{y}) = \hat{y}$ in $\|P\| [\tilde{P}_1^{-1}]$ is of the form (f, u) with $f : \hat{x} \to z$ and $u : \hat{y} \to z$ equational. Since $\hat{y}$ is a normal form, we necessarily have $u = \mathrm{id}_{\hat{y}}$ and thus $(f, u) = Ff$. Finally, given an object $y \in \|P\| [\tilde{P}_1^{-1}]$, there is a morphism $u : y \to \hat{y}$ in $\tilde{P}_1^*$ to its normal form which induces an isomorphism $y \cong \hat{y}$ in $\|P\| [\tilde{P}_1^{-1}]$. The functor F is thus an equivalence of categories. ◀

4.3 An example: the dihedral category $D_4^\bullet$

As an illustration of previous properties, we are going to study a presentation of a category which is a variant of the dihedral group. Recall that the *dihedral group* D_n is the group of isometries of the plane preserving a regular polygon with n faces. This group is generated by a rotation r of angle $2\pi/n$ and a reflection s, and can be described as the free group over the two generators r and s quotiented by the congruence generated by the three relations $s^2 = \mathrm{id}$, $r^n = \mathrm{id}$ and $rsr = s$. We consider here a variant of this group: the category $D_n^\bullet$ of isometries of the plane preserving a regular polygon with n faces together with a distinguished vertex (the category thus has n objects). For instance, the category $D_4^\bullet$ is pictured on the left below, the distinguished vertex of the square being pictured by a black triangle:



(3)

This category $D_4^\bullet$ admits a presentation P with 4 objects and 8 generating morphisms, as pictured on the left above, satisfying the 12 relations:

$$r_{i+3} \circ r_{i+2} \circ r_{i+1} \circ r_i = \mathrm{id} \qquad s_{j+1} \circ s_j = \mathrm{id} \qquad r_j \circ s_{j+1} \circ r_j = s_j$$
$$s_j \circ s_{j+1} = \mathrm{id} \qquad r_{j+3} \circ s_{j+2} \circ r_{j+1} = s_{j+1}$$

for $i \in \{1, \ldots, 4\}$ and $j \in \{1, 3\}$, where the indices are to be taken modulo 4 so that they lie in $\{1, \ldots, 4\}$.

The methodology introduced earlier can be used to show that by quotienting (resp. localizing) by $\Sigma = \{r_2, r_4\}$, we obtain a category which is isomorphic (resp. equivalent) to $D_2^\bullet$: intuitively, "forgetting" about those rotations quotients the square under symmetry wrt an horizontal axis. We thus consider the presentation modulo $(P, \tilde{P}_1)$ with $\tilde{P}_1 = \Sigma$. Unfortunately, this presentation does not satisfy the assumptions required to apply our results; for instance, there is no residual of r_2 after s_2. It is thus necessary to complete the presentation in order to have the confluence properties (namely, the residuation and cylinder properties). In rewriting theory, when a rewriting system is not confluent, one usually tries to complete it (typically using a Knuth-Bendix completion algorithm) in order for confluence to hold. Similarly, we can transform our presentation using a series of Tietze transformations (Definition 4 and Proposition 5) while preserving the same presented category, in order to obtain another presentation of the same category which satisfies the required assumptions.

We first consider the presentation P' obtained from P by adding the generator $\overline{r}_4 = r_3 \circ r_2 \circ r_1$ and its defining relation, as well as the derivable relations $r_4 \circ \overline{r}_4 = \mathrm{id}$ and $r_4 \circ \overline{r}_4 = \mathrm{id}$. We can now define r_2/s_2 as $\overline{r}_4$, if we consider $\overline{r}_4$ as an equational morphism. Fortunately, following lemma shows that we can quotient, or localize, by $\overline{r}_4$ instead of r_4, and we therefore define $\tilde{P}'_1 = \{r_2, \overline{r}_4\}$:

▶ **Lemma 33.** *Let P be a presentation of category such that there exist f and g in P_1 and two relations $f \circ g \Leftrightarrow \mathrm{id}$ and $g \circ f \Leftrightarrow \mathrm{id}$ in P_2. Let Σ be a subset of P_1 not containing f nor g. Then the quotients (resp. localizations) of $\|P\|$ by $\Sigma \uplus \{f\}$, $\Sigma \uplus \{f, g\}$, and $\Sigma \uplus \{g\}$ are isomorphic.*

In this way, we have transformed the presentation $(P, \tilde{P}_1)$ into a presentation $(P', \tilde{P}'_1)$ for which we can now define the residual r_2/s_2. Similarly, in order for all the required residual to be defined, we modify P' using Tietze transformations by adding generators $\overline{r}_1 = r_4 \circ r_3 \circ r_2$ and $\overline{r}_3 = r_2 \circ r_1 \circ r_4$ and modifying the set of relations. Finally, the presentation we end up with a presentation P'' which has 11 morphism generators $r_i, s_i, \overline{r}_k$, as shown on the right of (3), and 16 relations:

$$\begin{array}{lllll} s_{j+1} \circ s_j = \mathrm{id} & r_1 \circ s_2 \circ r_1 = s_1 & r_k \circ \overline{r}_k = \mathrm{id} & r_2 \circ r_1 = \overline{r}_3 \circ \overline{r}_4 & s_3 \circ r_2 = \overline{r}_4 \circ s_2 \\ s_j \circ s_{j+1} = \mathrm{id} & \overline{r}_3 \circ s_3 \circ \overline{r}_3 = s_4 & \overline{r}_k \circ r_k = \mathrm{id} & r_3 \circ r_2 = \overline{r}_4 \circ \overline{r}_1 & r_2 \circ s_1 = s_4 \circ \overline{r}_4 \end{array}$$

for $i \in \{1, \ldots, 4\}$, $j \in \{1, 3\}$ and $k \in \{1, 3, 4\}$, which is considered modulo $\tilde{P}''_1 = \{r_2, \overline{r}_4\}$. This presentation modulo is coherent. It satisfies convergence assumption 1, and residuals are defined by

$$r_2/s_2 = r_2/\overline{r}_1 = \overline{r}_4 \quad \overline{r}_4/s_1 = \overline{r}_4/r_1 = r_2 \quad s_1/\overline{r}_4 = s_4 \quad r_1/\overline{r}_4 = \overline{r}_3 \quad s_2/r_2 = s_3 \quad \overline{r}_1/r_2 = r_3$$

For termination assumption 2, we define ω_1 as equal to 1 on s_1, s_2, $\overline{r}_1$ and r_1 and 0 on other morphism generators. The cylinder assumption 3 follows from considering 5 diagrams. For termination assumption 4 we define ω_2 as 1 on relation generators such that the only morphism generators occurring in the source or the target are r_1, $\overline{r}_1$, s_1 or s_2, and as 0 otherwise. It can be checked similarly that $(P''^{\mathrm{op}}, (\tilde{P}''_1)^{\mathrm{op}})$ satisfies the assumptions. Therefore $\|P''\|{\downarrow}\{r_2, \overline{r}_4\}$ is isomorphic to $\|P''\| \,/\, \{r_2, \overline{r}_4\}$ by Theorem 28, and equivalent to $\|P''\| \,[\{r_2, \overline{r}_4\}^{-1}]$ by Theorem 32, the left-to-right part of the equivalence being an embedding by Proposition 30. An explicit (non-modulo) presentation for the quotient can be obtained by Lemma 10, and this presentation is Tietze equivalent to the canonical presentation of $D_2^\bullet$. We finally obtain the following result:

▶ **Theorem 34.** *The category $D_2^\bullet$ is isomorphic to the quotient $D_4^\bullet / \{r_2, r_4\}$, embeds fully and faithfully into the category $D_4^\bullet$, and is equivalent to the localization $D_4^\bullet[\{r_2, r_4\}^{-1}]$.*

▶ Remark. In this case, since r_2 and r_4 are already invertible in $\|P\|$, we moreover have $D_4^\bullet[\{r_2, r_4\}^{-1}] \cong D_4^\bullet$.

This illustrates the fact that, even though restricted for now to categories, the tools developed in this article enable one to obtain interesting results about presented categories.

5 Towards an extension to 2-categories

We would like to briefly mention how this work can be extended to presentations of 2-categories, and thus be able to handle examples such as the presentation of the monoidal (i.e. 2-)category $\Delta \times \Delta$ described in the introduction: it should admit a presentation modulo $(P, \tilde{P}_2)$ where $P = (P_0, P_1, P_2, P_3)$ is a presentation of a 2-category and $\tilde{P}_2 \subseteq P_2$ is a set of equational 2-generators, and in particular we should be able to show that the 2-category of normal forms $\|P\|{\downarrow}\tilde{P}_2$ is isomorphic to the quotient 2-category $\|P\| \,/\tilde{P}_2$ and equivalent to the localization $\|P\| \,[\tilde{P}_2^{-1}]$.

While we leave such an extension for future work, we would like to briefly mention some of the adjustments necessary to cover this case. Firstly, since the exchange law in a 2-category ensures that two disjoint rewrites commute, it is enough to impose the existence of suitable residuals for critical pairs only (this is, in our context, a variant of Newman's lemma), and similarly the cylinder property only has to be imposed for triples of coinitial rewriting rules forming a critical triple. Secondly, since in practice not all operations (residuation for instance in our example) are compatible with exchange law, one actually has to explicitly handle this law and work in the setting of sesquicategories. Thirdly, the precise notion of equivalence between 2-categories is subtle. For instance, the canonical "inclusion" functor $\|P\|{\downarrow}\tilde{P}_2 \hookrightarrow \|P\|$, exhibiting the restriction to 1-cells in normal form as a "sub-2-category" of $\|P\|$, is in fact a lax 2-functor: the 0-composition of two 1-cells in normal form is not necessarily a normal form, but always normalizes to one.

6 Conclusion

We have introduced a notion of presentation of a category modulo an "equational" rewriting system, and provided a series of reasonable coherence conditions ensuring that the equational rules are well-behaved wrt the generators. In particular, we show that, under those assumptions, all the three possible natural constructions for the presented category are equivalent. These assumptions are "local" in the sense that they are given directly on the presentations, and can thus be used in practice in order to perform computations, as illustrated in the article. In the future, we would like to investigate more applications, by studying classes of presentations (presentations of monoids and groups are well investigated, but there are fewer studied examples of presentations of categories), and also extend this work to presentations of 2-(and possibly higher-)categories.

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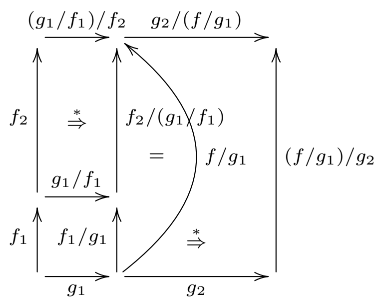
A Omitted proofs

Proof of Lemma 10. It is enough to show that $\|P/\tilde{P}_1\|$ is a quotient of $\|P\|$ by $\tilde{P}_1$. We define a quotient functor $Q : \|P\| \to \|P/\tilde{P}_1\|$ on generators by $Q(x) = [x]$ for $x \in P_0$ and $Q(f) = f$ for $f \in P_1$ (this extends to a functor since for every 2-generator $\alpha \in P_2$ there is a corresponding 2-generator in $P/\tilde{P}_1$). For every generator $f \in \tilde{P}_1$, we immediately have $Q(f) = \mathrm{id}$. Suppose given a functor $F : \|P\| \to \mathcal{C}$ sending equational morphisms to identities. We define a functor $\tilde{F} : \|P/\tilde{P}_1\| \to \mathcal{C}$ by $\tilde{F}[x] = Fx$ for an object $[x]$ of $\|P/\tilde{P}_1\|$ (this does not depend on the choice of the representative of class) and, given $f = f_k \circ \ldots \circ f_1$ in $\|P/\tilde{P}_1\|$ with $f_i \in P_1$, we define $\tilde{F}f = Ff_k \circ \ldots \circ Ff_1$ (it can be checked that this is also well-defined). The functor $\tilde{F}$ satisfies $F = \tilde{F} \circ Q$, and it is the only such functor since it has to send elements of $\tilde{P}_1$ to identities. ◀

Proof of Lemma 12. The localization functor L is defined by $Lx = x$ for $x \in P_0$, and $Lf = f$ for $f \in P_1^*$. This functor is well-defined since for any 2-generator $\alpha : f \Rightarrow g$ in P_2, we have that $Lf = f$ and $Lg = g$, and there is a relation $f \Rightarrow g$ in P_2' by definition. Besides, for any f in Σ, $Lf = f$ is an isomorphism since $\overline{f}$ is an inverse for f. Suppose given $F : \|P\| \to \mathcal{C}$ sending the elements of Σ to isomorphisms. We define a functor $\tilde{F} : \|P'\| \to \mathcal{C}$ on the generators by $\tilde{F}x = Fx$ for $x \in P_0$, $\tilde{F}f = Ff$ for $f \in P_1$ and $\tilde{F}\overline{f} = (Ff)^{-1}$. This functor is well-defined, since for any relation $\alpha : f \Rightarrow g$ in $P_2 \subset P_2'$, we have $\tilde{F}f = Ff = Fg = \tilde{F}g$ and $\tilde{F}(f \circ \overline{f}) = Ff \circ F\overline{f} = Ff \circ (Ff)^{-1} = \mathrm{id}$ and similarly $\tilde{F}(\overline{f} \circ f) = \mathrm{id}$. This functor satisfies $\tilde{F} \circ L = F$ and is the unique such functor. ◀

Proof of Lemma 19. We extend the weight function of Assumption 2 to morphisms in Z_1^* by setting $\omega_1(\overline{f}) = 0$ for $\overline{f}$ in $\overline{\tilde{P}_1}$. This ensures that the rewriting system on morphisms in Z_1^* with Z_2 as rules is terminating. Moreover, because the left members of rules are of the form $g \circ \overline{f}$ with $g \in P_1$ and $\overline{f} \in \overline{\tilde{P}_1}$, there are no critical pairs, which means that the rewriting system is locally confluent and thus convergent by Newman's lemma. Given two coinitial morphisms $f : x \to y$ in $\tilde{P}_1^*$ and $g : x \to z$ in P_1^*, we prove by recurrence on $\omega_1(g \circ \overline{f})$ that the normal form of $g \circ \overline{f}$ is $\overline{(f/g)} \circ (g/f)$. If either f or g is an identity, this result is direct. Otherwise, $f = f_2 \circ f_1$ and $g = g_2 \circ g_1$ where f_1, f_2, g_1 and g_2 are non identity-morphisms.



By induction, we have

$$g_1 \circ \overline{f_1} \overset{*}{\Rightarrow} \overline{(f_1/g_1)} \circ (g_1/f_1) \qquad \text{and} \qquad (g_1/f_1) \circ \overline{f_2} \overset{*}{\Rightarrow} \overline{(f_2/(g_1/f_1))} \circ ((g_1/f_1)/f_2)$$

since $\omega_1((g_1/f_1)\circ\overline{f_2}) < \omega_1\left(g_2\circ\left(\overline{(f_1/g_1)}\circ(g_1/f_1)\right)\circ\overline{f}_2\right) < \omega_1(g\circ\overline{f})$. And therefore,

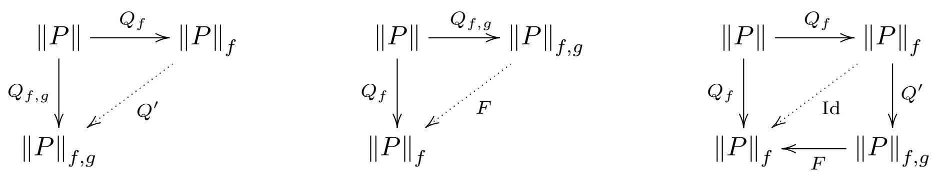
$$\begin{aligned} g\circ\overline{f} &\overset{*}{\Rrightarrow} g_2\circ\left(\overline{(f_1/g_1)}\circ(g_1/f_1)\right)\circ\overline{f}_2 \\ &\overset{*}{\Rrightarrow} g_2\circ\overline{(f_1/g_1)}\circ\left(\overline{(f_2/(g_1/f_1))}\circ((g_1/f_1)/f_2)\right) \\ &\overset{*}{\Rrightarrow} g_2\circ\overline{(f/g_1)}\circ(g_1/f) \end{aligned}$$

Similarly $\omega_1(g_2\circ\overline{(f/g_1)}) < \omega_1(g\circ\overline{f})$, therefore $\omega_1(g_2\circ\overline{(f/g_1)}) \overset{*}{\Rrightarrow} \overline{((f/g_1)/g_2)}\circ(g_2/(f/g_1))$, and we have

$$\begin{aligned} g\circ\overline{f} &\overset{*}{\Rrightarrow} g_2\circ\overline{(f/g_1)}\circ(g_1/f) \\ &\overset{*}{\Rrightarrow} \overline{((f/g_1)/g_2)}\circ(g_2/(f/g_1))\circ(g_1/f) \\ &\overset{*}{\Rrightarrow} \overline{(f/g)}\circ(g/f) \end{aligned}$$

from which we conclude. ◀

Proof of Lemma 33. The isomorphism of localizations follows from Lemma 12 and the usual proof that a morphism admits at most one inverse in a category. We now consider the case of quotient: we are going to show that the categories $\|P\|_{f,g} = \|P\|/(\Sigma\uplus\{f,g\}$ and $\|P\|_f = \|P\|/(\Sigma\uplus\{f\})$ are isomorphic. We write $Q_{f,g} : \|P\| \to \|P\|_{f,g}$ and $Q_f : \|P\| \to \|P\|_f$ for the quotient functors. By the universal property of Q_f, there exist a unique $Q' : \|P\|_f \to \|P\|_{f,g}$ such that $Q_{f,g} = Q'\circ Q_f$:



Moreover, since $Q_f(f) = \mathrm{id}$, we get that $Q_f(g) = Q_f(g)\circ Q_f(f) = Q_f(g\circ f) = \mathrm{id}$ and therefore, by the universal property of $Q_{f,g}$, there exists a unique functor $F : \|P\|_f \to \|P\|_{f,g}$ such that $Q_f = F\circ Q_{f,g}$. From these equalities, we get that $Q_f = F\circ Q'\circ Q_f$ and that $Q_{f,g} = Q'\circ F\circ Q_{f,g}$. By universal property of Q_f, the identity is the unique endofunctor of $\|P\|_f$ such that $\mathrm{id}\circ Q_f = Q_f$, and therefore $F\circ Q' = \mathrm{Id}$. Similarly, we have $Q'\circ I = \mathrm{Id}$, and therefore the categories $\|P\|_f$ and $\|P\|_{f,g}$ are isomorphic. ◀