Head reduction and normalization in a call-by-value lambda-calculus

Giulio Guerrieri

Laboratoire PPS, UMR 7126, Université Paris Diderot, Sorbonne Paris Cité
F-75205 Paris, France
giulio.guerrieri@pps.univ-paris-diderot.fr

Abstract

Recently, a standardization theorem has been proven for a variant of Plotkin's call-by-value lambda-calculus extended by means of two commutation rules (sigma-reductions): this result was based on a partitioning between head and internal reductions. We study the head normalization for this call-by-value calculus with sigma-reductions and we relate it to the weak evaluation of original Plotkin's call-by-value lambda-calculus. We give also a (non-deterministic) normalization strategy for the call-by-value lambda-calculus with sigma-reductions.

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1 Introduction

The call-by-value λ-calculus (λ_v-calculus or λ_v for short) and the operational machine for its evaluation has been introduced by Plotkin [15] inspired by Landin's seminal work [9] on the programming language ISWIM and the SECD machine. The λ_v-calculus is a paradigmatic language able to capture two features of many functional programming languages: call-by-value parameter passing policy (parameters are evaluated before being passed) and weak evaluation (the body of a function is evaluated only when parameters are supplied).

The syntax of λ_v is the same as that of the ordinary (i.e. call-by-name) λ-calculus (λ for short), but the reduction rule for λ_v, called β_v, is a restriction of the β-rule for λ: β_v allows the contraction of a redex $(\lambda x.M)N$ only in case the argument N is a value, i.e. a variable or an abstraction. Unfortunately, the semantic analysis of the λ_v-calculus has turned out to be more elaborate than that of ordinary λ-calculus. This is due essentially to the "weakness" of (full) β_v-reduction, a fact widely recognized: indeed, there are many proposals of alternative call-by-value λ-calculi extending Plotkin's one [11, 10, 8, 2, 1]. To have an example of the "weakness" of the rewriting rules of λ_v, it is sufficient to consider that it is impossible to have an internal operational characterization (i.e. one that uses the β_v-reduction) of the semantically meaningful notions of call-by-value solvability and potential valuability, as shown in [13, 14, 2].

In this paper we will study the λ_v^σ-calculus (λ_v^σ for short), a call-by-value extension of λ_v recently proposed in [4]: it keeps the λ_v (and λ) syntax and it adds to the β_v-reduction two commutation rules, called σ_1 and σ_3, which unblock "hidden" β_v-redexes that are concealed by the "hyper-sequential structure" of terms. The λ_v^σ-calculus enjoy some basic properties we expect from a calculus, namely confluence (see [4]) and standardization (see [7]). Moreover, λ_v^σ provides elegant characterizations of many semantic properties, e.g. solvability and potential

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valuability (see [4]), and it is conservative with respect to Plotkin's λ_v: in particular, [7] shows that the notions of solvability and potential valuability for λ_v^σ coincide with those for λ_v.

The v-reduction (i.e. the reduction for λ_v^σ) can be partitioned into head v-reduction and internal v-reduction; the head v-reduction is in turn decomposed into head β_v- and head σ-reduction. The head β_v-reduction is just the deterministic weak evaluation strategy for Plotkin's λ_v-calculus. According to a sequentialization theorem proven in [7, Theorem 22], any v-reduction sequence can be sequentialized in an initial head β_v-reduction sequence followed by a head σ-reduction sequence followed by an internal v-reduction sequence. Similar well-known results hold for λ and λ_v, and starting from them one can define a normalization strategy for λ and λ_v, i.e. a deterministic reduction strategy that reaches a normal form if and only if one exists: for example the leftmost reduction, see [19, Theorem 2.8] and [3, Theorem 13.2.2].

Is there a normalization strategy for λ_v^σ? Theorem 24, one of the main results of this paper, proves that, starting from the sequentialization theorem mentioned above, a normalization strategy can be defined for λ_v^σ, based on the notions of head β_v- and head σ-reductions.

A first difference appears here between λ_v^σ and λ_v (or λ): the normalization strategy for λ_v^σ is not deterministic. Indeed, while the head β_v-reduction (or the call-by-name head reduction) is deterministic (i.e. a partial function), the head v-reduction is non-deterministic and, still worse, non-confluent and there are terms having several head v-normal forms: this might appear disappointing. So, three natural questions arise:

- With respect to head v-reduction, do normalization and strong normalization coincide?[1]
- Can we relate the termination of head β_v-reduction and head v-reduction?
- Can we characterize the terms having a unique head v-normal form?

Our Theorem 21 gives a positive answer to the first two questions. Observe that the lack of any form of confluence for head v-reduction requires a more complex reasoning, passing through a syntactic characterization of head β_v- and head v-normal forms. Theorem 21 not only shows that the head v-reduction and the head β_v-reduction are deeply related (and hence, again, λ_v^σ is conservative with respect to λ_v) but also that both enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary λ-calculus.

Our Proposition 27 gives a partial answer to the third question above: it shows that in some cases (of interest) a head v-normalizable term has a unique head v-normal form; in particular, every closed head v-normalizable term has a unique head v-normal form.

So, λ_v^σ appears as an extension of Plotkin's λ_v-calculus that enjoys many meaningful conservation properties with respect to λ_v and therefore it is a useful tool for theoretical and semantic investigations about λ_v and call-by-value setting. See also conclusions in Section 6 for further and more precise motivations for this paper and future work.

Related work. The λ_v^σ-calculus has been recently introduced in [4] and further investigated in [7]. It is an extension of Plotkin's λ_v-calculus inspired by the call-by-value translation of λ-terms into linear logic proof-nets [6]. Other variants of λ_v have been introduced in the literature for modeling the call-by-value computation. We would like to cite here at least the contributions of Moggi [11], Felleisen and Sabry [18], Maraist et al. [10], Herbelin and Zimmerman [8], Accattoli and Paolini [2] (the latter is inspired by the call-by-value translation of λ-terms into linear logic proof-nets, see [1]). All these proposals are based on the introduction of new constructs to the syntax of λ_v, so the comparison between them is

[1] The answer is trivially positive in the case of call-by-name head normalization (for λ) and head β_v-normalization, since these reductions are deterministic.

not easy with respect to syntactical properties (some detailed comparison is given in [2]). We point out that the calculi introduced in [11, 18, 10, 8] present some variants of our σ_1 and/or σ_3 rules, often in a setting with explicit substitutions. Regnier [16, 17] used the rule σ_1 (but not σ_3) in ordinary (i.e. call-by-name) λ-calculus.

The head v-reduction investigated here has been introduced in [7]. Some results of this paper are inspired by the Takahashi's results [19] on the ordinary (i.e. call-by-name) λ-calculus, partially adapted by Crary [5] for λ_v.

Outline. In Section 2 we introduce the syntax and the reduction rules of the λ_v^σ-calculus. In Section 3 we define the head v-reduction and the internal v-reduction, and we recall some results already proven in [7] concerning them. Section 4 is devoted to proving the first main result of our paper: Theorem 21, which studies the normalization for the head v-reduction and relates it to the weak evaluation strategy for Plotkin's λ_v-calculus. In Section 5 we show that the head v-reduction can be used to define a normalization strategy for the λ_v^σ-calculus (Theorem 24), and moreover in some cases the head v-normal form (if any) of a term is unique (Proposition 27). In Section 6 we summarize the findings and suggest future work.

2 The call-by-value lambda calculus with sigma-rules

In this section we present λ_v^σ, a call-by-value λ-calculus introduced in [4] that adds two σ-reduction rules to pure (i.e. without constants) call-by-value λ-calculus defined by Plotkin in [15].

The syntax of terms of λ_v^σ [4] is the same as the one of ordinary λ-calculus and Plotkin's call-by-value λ-calculus λ_v [15] (without constants). Given a countable set $\mathcal{V}$ of *variables* (denoted by $x, y, z, \dots$), the sets Λ of *terms* and Λ_v of *values* are defined by mutual induction:

$$(\Lambda_v) \qquad V, U ::= x \mid \lambda x.M \qquad \textit{values}$$
$$(\Lambda) \qquad M, N, L ::= V \mid MN \qquad \textit{terms}$$

Clearly, $\Lambda_v \subsetneq \Lambda$. All terms are considered up to α-conversion. The set of free variables of a term M is denoted by $\mathsf{fv}(M)$. Given $V_1, \dots, V_n \in \Lambda_v$ and pairwise distinct variables $x_1, \dots, x_n$, $M\{V_1/x_1, \dots, V_n/x_n\}$ denotes the term obtained by the *capture-avoiding simultaneous substitution* of V_i for each free occurrence of x_i in the term M (for all $1 \leq i \leq n$). Note that, for all $V, V_1, \dots, V_n \in \Lambda_v$ and pairwise distinct variables $x_1, \dots, x_n$, $V\{V_1/x_1, \dots, V_n/x_n\} \in \Lambda_v$.

Contexts (with exactly one hole $(\!|\cdot|\!)$), denoted by $\mathtt{C}$, are defined as usual via the grammar:

$$\mathtt{C} ::= (\!|\cdot|\!) \mid \lambda x.\mathtt{C} \mid \mathtt{C}M \mid M\mathtt{C}\,.$$

We use $\mathtt{C}(\!|M|\!)$ for the term obtained by the capture-allowing substitution of the term M for the hole $(\!|\cdot|\!)$ in the context $\mathtt{C}$.

▶ **Notation.** From now on, we set $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

The reduction rules of λ_v^σ consist of Plotkin's β_v-reduction rule, introduced in [15], and two simple commutation rules called σ_1 and σ_3, studied in [4, 7].

▶ **Definition 1** (Reduction rules). *We define the following binary relations on Λ (for any $M, N, L \in \Lambda$ and any $V \in \Lambda_v$):*

$$(\lambda x.M)V \mapsto_{\beta_v} M\{V/x\}$$
$$(\lambda x.M)NL \mapsto_{\sigma_1} (\lambda x.ML)N \qquad \textit{with } x \notin \mathsf{fv}(L)$$
$$V((\lambda x.L)N) \mapsto_{\sigma_3} (\lambda x.VL)N \qquad \textit{with } x \notin \mathsf{fv}(V).$$

We set $\mapsto_\sigma = \mapsto_{\sigma_1} \cup \mapsto_{\sigma_3}$ *and* $\mapsto_{\mathsf{v}} = \mapsto_{\beta_v} \cup \mapsto_\sigma$.

For any $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathsf{v}\}$*, if* $M \mapsto_{\mathsf{r}} M'$ *then* M *is a* r-redex *and* M' *is its* r-contractum*. In this sense, a term of the shape* $(\lambda x.M)N$ *(for any* $M, N \in \Lambda$*) is a* β-redex*.*

The side conditions on $\mapsto_{\sigma_1}$ and $\mapsto_{\sigma_3}$ in Definition 1 can be always fulfilled by α-renaming.

Obviously, any β_v-redex is a β-redex but the converse does not hold: $(\lambda x.z)(yI)$ is a β-redex but not a β_v-redex.

▶ **Example 2.** Redexes of different kind may overlap: for example, the term $\Delta I \Delta$ is a σ_1-redex and it contains the β_v-redex ΔI; the term $\Delta(I\Delta)(xI)$ is a σ_1-redex and it contains the σ_3-redex $\Delta(I\Delta)$, which contains in turn the β_v-redex $I\Delta$.

▶ **Notation.** Let R be a binary relation on Λ. We denote by R^* (resp. R^+; $\mathsf{R}^=$) the reflexive-transitive (resp. transitive; reflexive) closure of R.

▶ **Definition 3** (Reductions)**.** *Let* $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathsf{v}\}$*.*

The r-reduction $\to_{\mathsf{r}}$ *is the contextual closure of* $\mapsto_{\mathsf{r}}$*, i.e.* $M \to_{\mathsf{r}} M'$ *iff there is a context* $\mathtt{C}$ *and* $N, N' \in \Lambda$ *such that* $M = \mathtt{C}(\!|N|\!)$*,* $M' = \mathtt{C}(\!|N'|\!)$ *and* $N \mapsto_{\mathsf{r}} N'$*.*

The r-equivalence $\simeq_{\mathsf{r}}$ *is the reflexive-transitive and symmetric closure of* $\to_{\mathsf{r}}$*.*

Let M *be a term:* M *is* r-normal *if there is no term* N *such that* $M \to_{\mathsf{r}} N$*;* M *is* r-normalizable *if there is a* r*-normal term* N *such that* $M \to_{\mathsf{r}}^* N$*;* M *is* strongly r-normalizable *if there is no sequence* $(N_i)_{i \in \mathbb{N}}$ *of terms such that* $M = N_0$ *and* $N_i \to_{\mathsf{r}} N_{i+1}$ *for any* $i \in \mathbb{N}$*.*

Obviously, $\to_\sigma = \to_{\sigma_1} \cup \to_{\sigma_3} \subsetneq \to_{\mathsf{v}}$ and $\to_{\beta_v} \subsetneq \to_{\mathsf{v}}$ and $\to_{\mathsf{v}} = \to_{\beta_v} \cup \to_\sigma$.

▶ **Remark 4.** For any $\mathsf{r} \in \{\beta_v, \sigma_1, \sigma_3, \sigma, \mathsf{v}\}$ (resp. $\mathsf{r} \in \{\sigma_1, \sigma_3, \sigma\}$), values are closed under r-reduction (resp. r-expansion): for any $V \in \Lambda_v$, if $V \to_{\mathsf{r}} M$ (resp. $M \to_{\mathsf{r}} V$) then $M \in \Lambda_v$; more precisely, $V = \lambda x.N$ and $M = \lambda x.N'$ for some $N, N' \in \Lambda$ with $N \to_{\mathsf{r}} N'$ (resp. $N' \to_{\mathsf{r}} N$).

For any $\mathsf{r} \in \{\beta_v, \mathsf{v}\}$, values are not closed under r-expansion: $I\Delta \to_{\beta_v} \Delta \in \Lambda_v$ but $I\Delta \notin \Lambda_v$.

▶ **Proposition 5** (See [4])**.** *The* σ*-reduction is confluent and strongly normalizing. The* v*-reduction is confluent.*

The λ_v^σ*-calculus*, λ_v^σ for short, is the set Λ of terms endowed with the v-reduction $\to_{\mathsf{v}}$. The set Λ endowed with the β_v-reduction $\to_{\beta_v}$ is the λ_v-calculus (λ_v for short), i.e. the Plotkin's call-by-value λ-calculus [15] (without constants), which is thus a sub-calculus of λ_v^σ.

▶ **Example 6.** $M = (\lambda y.\Delta)(xI)\Delta \to_{\sigma_1} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} \dots$ and $N = \Delta((\lambda y.\Delta)(xI)) \to_{\sigma_3} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} (\lambda y.\Delta\Delta)(xI) \to_{\beta_v} \dots$ are the only possible v-reduction paths from M and N respectively: M and N are not v-normalizable, and $M \simeq_{\mathsf{v}} N$. Meanwhile, M and N are β_v-normal and different, hence $M \not\simeq_{\beta_v} N$ (by confluence of $\to_{\beta_v}$, see [15]).

Informally, σ-rules unblock β_v-redexes which are hidden by the "hyper-sequential structure" of terms. This approach is alternative to the one in [2, 1] where hidden β_v-redexes are reduced using rules acting at a distance (through explicit substitutions). It can be shown that the call-by-value λ-calculus with explicit substitution introduced in [2] can be embedded in λ_v^σ.

It is well-known that the β_v-reduction can be simulated by linear logic cut-elimination via the call-by-value translation $(\cdot)^v$ of λ-terms into proof-nets, called by Girard [6, pp. 81-82] "boring" and defined by $(A \Rightarrow B)^v = !A^v \multimap !B^v$ (see also [1]). The images under $(\cdot)^v$ of a σ-redex and its σ-contractum are equal modulo some non-structural cut-elimination steps.

3 Head and internal reductions

In this section we introduce the definitions of head v-reduction (which is decomposed in head β_v- and head σ-reductions) and internal v-reduction, then we recall some results proven in [7].

▶ **Notation.** From now on, we always assume that $V, V' \in \Lambda_v$.

Note that the generic form of a term is $VM_1 \dots M_m$ for some $m \in \mathbb{N}$ (in particular, values are obtained when $m = 0$). The sequentialization result is based on a partitioning of v-reduction between head v-reduction and internal v-reduction.

▶ **Definition 7** (Head β_v-reduction). *The* head β_v-reduction $\xrightarrow{h}_{\beta_v}$ *is the binary relation on* Λ *defined inductively by the following rules ($m \in \mathbb{N}$ in both rules):*

$$\frac{}{(\lambda x.M)VM_1 \dots M_m \xrightarrow{h}_{\beta_v} M\{V/x\}M_1 \dots M_m}\,\beta_v \qquad \frac{N \xrightarrow{h}_{\beta_v} N'}{VNM_1 \dots M_m \xrightarrow{h}_{\beta_v} VN'M_1 \dots M_m}\,\mathit{right}$$

The head β_v-reduction $\xrightarrow{h}_{\beta_v}$ is exactly the (pure) "left reduction" defined in [15, p. 136] for λ_v and called "(weak) evaluation" in [18, 5]. If $N \xrightarrow{h}_{\beta_v} N'$ then N' is obtained from N by reducing the leftmost-outermost β_v-redex, not in the scope of a λ: thus, the head β_v-reduction is deterministic (i.e. it is a partial function from Λ to Λ) and does not reduce values.

▶ **Definition 8** (Head σ- and head v-reductions). *The* head σ-reduction $\xrightarrow{h}_{\sigma}$ *is the binary relation on* Λ *defined inductively by the following rules ($m \in \mathbb{N}$ in all the rules, $x \notin \mathsf{fv}(L)$ in the rule σ_1, $x \notin \mathsf{fv}(V)$ in the rule σ_3):*

$$\frac{}{(\lambda x.M)NLM_1 \dots M_m \xrightarrow{h}_{\sigma} (\lambda x.ML)NM_1 \dots M_m}\,\sigma_1 \qquad \frac{N \xrightarrow{h}_{\sigma} N'}{VNM_1 \dots M_m \xrightarrow{h}_{\sigma} VN'M_1 \dots M_m}\,\mathit{right}$$

$$\frac{}{V((\lambda x.L)N)M_1 \dots M_m \xrightarrow{h}_{\sigma} (\lambda x.VL)NM_1 \dots M_m}\,\sigma_3$$

The head v-reduction *is* $\xrightarrow{h}_{\mathsf{v}} = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_{\sigma}$.

Let $\mathsf{r} \in \{\beta_v, \sigma, \mathsf{v}\}$ *and* $N \in \Lambda$*:* N *is* head r-normal *if there is no* $N' \in \Lambda$ *such that* $N \xrightarrow{h}_{\mathsf{r}} N'$*;* N *is* head r-normalizable *if there is a* r*-normal term* N' *such that* $N \xrightarrow{h}_{\mathsf{r}}^{*} N'$*;* N *is* strongly head r-normalizable *if there is no* $(N_i)_{i \in \mathbb{N}}$ *such that* $N = N_0$ *and* $N_i \xrightarrow{h}_{\mathsf{r}} N_{i+1}$ *for any* $i \in \mathbb{N}$.

Notice that $\mapsto_{\beta_v} \subsetneq \xrightarrow{h}_{\beta_v} \subsetneq \to_{\beta_v}$ and $\mapsto_{\sigma} \subsetneq \xrightarrow{h}_{\sigma} \subsetneq \to_{\sigma}$ and $\mapsto_{\mathsf{v}} \subsetneq \xrightarrow{h}_{\mathsf{v}} \subsetneq \to_{\mathsf{v}}$.

Informally, if $N \xrightarrow{h}_{\sigma} N'$ then N' is obtained from N by reducing "one of the leftmost" σ_1- or σ_3-redexes, not in the scope of a λ: in general, a term may contain several head σ_1- and σ_3-redexes. Indeed, differently from $\xrightarrow{h}_{\beta_v}$, the head σ-reduction $\xrightarrow{h}_{\sigma}$ is not deterministic, for example the leftmost-outermost σ_1- and σ_3-redexes may overlap: if $M = (\lambda y.y')(\Delta(xI))I$ then $M \xrightarrow{h}_{\sigma} (\lambda y.y'I)(\Delta(xI)) = N_1$ by applying the rule σ_1 and $M \xrightarrow{h}_{\sigma} (\lambda z.(\lambda y.y')(zz))(xI)I = N_2$ by applying the rule σ_3. Note that N_1 contains only a head σ_3-redex and $N_1 \xrightarrow{h}_{\sigma} (\lambda z.(\lambda y.y'I)(zz))(xI) = N$ which is head v-normal; meanwhile N_2 contains only a head σ_1-redex and $N_2 \xrightarrow{h}_{\sigma} (\lambda z.(\lambda y.y')(zz)I)(xI) = N'$ which is head v-normal: $N \neq N'$, so the head σ- and head v-reductions are not (locally) confluent and a term may have several head v-normal forms (this example does not contradict the confluence of σ-reduction because $N' \to_{\sigma} N$ but by performing an internal v-reduction step, see next Definition 9).

The head v-reduction $\xrightarrow{h}_{\mathsf{v}}$ is non-deterministic not only because the head σ-reduction $\xrightarrow{h}_{\sigma}$ is non-deterministic, but also because the leftmost-outermost β_v-redex of a term may overlap with "one of its leftmost" σ_1- or σ_3-redexes, as seen in Example 2.

▶ **Definition 9** (Internal v-reduction). *The* internal v-reduction $\overset{int}{\to}_{\mathsf{v}}$ *is the binary relation on* Λ *defined inductively by the following rules:*

$$\frac{(m \in \mathbb{N}) \qquad N \to_{\mathsf{v}} N'}{(\lambda x.N)M_1 \dots M_m \overset{int}{\to}_{\mathsf{v}} (\lambda x.N')M_1 \dots M_m}\ \lambda \qquad \frac{(m \in \mathbb{N}) \qquad N \overset{int}{\to}_{\mathsf{v}} N'}{VNM_1 \dots M_m \overset{int}{\to}_{\mathsf{v}} VN'M_1 \dots M_m}\ \mathit{right}$$

$$\frac{(m \in \mathbb{N}^+) \qquad M_i \to_{\mathsf{v}} M_i' \quad \text{for some } 1 \leq i \leq m}{VNM_1 \dots M_i \dots M_m \overset{int}{\to}_{\mathsf{v}} VNM_1 \dots M_i' \dots M_m}\ @\ .$$

The following fact collects many minor properties which can be easily proved by inspection of the rules of Definitions 7-9.

▶ **Fact 10.**

1. *The head* β_v*-reduction* $\overset{h}{\to}_{\beta_v}$ *does not reduce a value (in particular, does not reduce under* λ*'s), i.e., for any* $M \in \Lambda$ *and any* $V \in \Lambda_v$*, one has* $V \overset{h}{\not\to}_{\beta_v} M$*.*
2. *The head* σ*-reduction* $\overset{h}{\to}_{\sigma}$ *does neither reduce a value nor reduce to a value, i.e., for any* $M \in \Lambda$ *and any* $V \in \Lambda_v$*, one has* $V \overset{h}{\not\to}_{\sigma} M$ *and* $M \overset{h}{\not\to}_{\sigma} V$*.*
3. *Values are closed under* $\overset{int}{\to}_{\mathsf{v}}$*-expansion, i.e., for all* $M \in \Lambda$ *and* $V \in \Lambda_v$*, if* $M \overset{int}{\to}_{\mathsf{v}} V$ *then* $M \in \Lambda_v$*; more precisely,* $M = \lambda x.N$ *and* $V = \lambda x.N'$ *for some* $N, N' \in \Lambda$ *where* $N \to_{\mathsf{v}} N'$*.*
4. *If* $\mathsf{R} \in \{\overset{h}{\to}_{\beta_v}, \overset{h}{\to}_{\sigma}, \overset{h}{\to}_{\mathsf{v}}, \overset{int}{\to}_{\mathsf{v}}\}$ *and* $M \mathrel{\mathsf{R}} M'$*, then* $MN \mathrel{\mathsf{R}} M'N$ *for any* $N \in \Lambda$*.*

Clearly, $\overset{int}{\to}_{\mathsf{v}} \subsetneq \to_{\mathsf{v}}$. Next Proposition 11 (whose proof uses Fact 10.4) relates $\overset{int}{\to}_{\mathsf{v}}$ and $\overset{h}{\to}_{\mathsf{v}}$.

▶ **Proposition 11.** *One has* $\overset{int}{\to}_{\mathsf{v}} = \to_{\mathsf{v}} \smallsetminus \overset{h}{\to}_{\mathsf{v}}$*.*

Proof.

$\subseteq$: The proof that $\overset{int}{\to}_{\mathsf{v}} \subseteq \to_{\mathsf{v}}$ is trivial. The proof that $M \overset{int}{\to}_{\mathsf{v}} M'$ implies $M \overset{h}{\not\to}_{\mathsf{v}} M'$ is by induction on the derivation of $M \overset{int}{\to}_{\mathsf{v}} M'$. Let us consider its last rule r. If $\mathsf{r} \in \{\lambda, @\}$, then it is evident that there is no last rule to derive $M \overset{h}{\to}_{\mathsf{v}} M'$. If $\mathsf{r} = \mathit{right}$ then $M = VNM_1 \dots M_m$ and $M' = VN'M_1 \dots M_m$ with $m \in \mathbb{N}$ and $N \overset{int}{\to}_{\mathsf{v}} N'$; by induction hypothesis, $N \overset{h}{\not\to}_{\mathsf{v}} N'$ and hence there is no last rule to derive $M \overset{h}{\to}_{\mathsf{v}} M'$.

$\supseteq$: We show that $M \to_{\mathsf{v}} M'$ and $M \overset{h}{\not\to}_{\mathsf{v}} M'$ implies $M \overset{int}{\to}_{\mathsf{v}} M'$, for all $M, M' \in \Lambda$. Since $M \to_{\mathsf{v}} M'$, there exist a context $\mathtt{C}$ and terms N and N' such that $M = \mathtt{C}(\!|N|\!)$, $M' = \mathtt{C}(\!|N'|\!)$ and $N \mapsto_{\beta_v} N'$. We proceed by induction on $\mathtt{C}$.

If $\mathtt{C} = (\!|\cdot|\!)$ then $M = N \mapsto_{\beta_v} N' = M'$ and thus $M \overset{h}{\to}_{\mathsf{v}} M'$ since $\mapsto_{\beta_v} \subseteq \overset{h}{\to}_{\mathsf{v}}$, which contradicts the hypothesis.

If $\mathtt{C} = \lambda x.\mathtt{C}'$ for some context $\mathtt{C}'$, then $M \overset{int}{\to}_{\mathsf{v}} M'$ by applying the rule λ for $\overset{int}{\to}_{\mathsf{v}}$, since $\mathtt{C}'(\!|N|\!) \to_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$.

If $\mathtt{C} = \mathtt{C}'L$ for some context $\mathtt{C}'$ and term L, then $\mathtt{C}'(\!|N|\!) \to_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$ and $\mathtt{C}'(\!|N'|\!) \overset{h}{\not\to}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$ (by Fact 10.4, since $\mathtt{C}'(\!|N|\!)L \overset{h}{\not\to}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)L$). By induction hypothesis, $\mathtt{C}'(\!|N|\!) \overset{int}{\to}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$, then $M = \mathtt{C}'(\!|N|\!)L \overset{int}{\to}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)L = M'$ by Fact 10.4.

If $\mathtt{C} = V\mathtt{C}'$ for some context $\mathtt{C}'$ and value V, then $\mathtt{C}'(\!|N|\!) \to_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$. There are two cases:

- either $\mathtt{C}'(\!|N'|\!) \overset{h}{\to}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$, hence $M = V\mathtt{C}'(\!|N|\!) \overset{h}{\to}_{\mathsf{v}} V\mathtt{C}'(\!|N'|\!) = M'$ by the rule *right* for $\overset{h}{\to}_{\beta_v}$ or $\overset{h}{\to}_{\sigma}$, which contradicts the hypothesis;
- or $\mathtt{C}'(\!|N'|\!) \overset{h}{\not\to}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$, hence $\mathtt{C}'(\!|N'|\!) \overset{int}{\to}_{\mathsf{v}} \mathtt{C}'(\!|N'|\!)$ by induction hypothesis, thus $M = V\mathtt{C}'(\!|N|\!) \overset{int}{\to}_{\mathsf{v}} V\mathtt{C}'(\!|N'|\!) = M'$ by applying the rule *right* for $\overset{int}{\to}_{\mathsf{v}}$.

Finally, if $\mathtt{C} = L\mathtt{C}'$ for some context $\mathtt{C}'$ and term $L \notin \Lambda_v$, then $L = VN_0 \dots N_n$ for some $n \in \mathbb{N}$, thus $M = VN_0 \dots N_n\mathtt{C}'(\!|N|\!) \overset{int}{\to}_{\mathsf{v}} VN_0 \dots N_n\mathtt{C}'(\!|N'|\!) = M'$ by the rule @ for $\overset{int}{\to}_{\mathsf{v}}$. ◀

We end this section by recalling three results proven in [7] concerning head v-reduction and internal v-reduction: they will be used to prove the main results in Sections 4-5.

The following lemma (proven in [7, Lemma 14]) shows that a head σ-reduction step can be postponed after a head β_v-reduction step, and hence every head v-reduction sequence can be rearranged into a head β_v-reduction sequence followed by a head σ-reduction sequence.

▶ **Lemma 12** (Commutation of head β_v- and head σ-reductions, see [7])**.**
1. *If $M \xrightarrow{h}_{\sigma} L \xrightarrow{h}_{\beta_v} N$ then there exists $L' \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}{}^{=}_{\sigma} N$.*
2. *If $M \xrightarrow{h}{}^{*}_{\mathsf{v}} M'$ then there exists $N \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} N \xrightarrow{h}{}^{*}_{\sigma} M'$.*

Next Lemma 13 (proven in [7, Corollary 21]) says that internal v-reduction can be shifted after head v-reductions.[2]

▶ **Lemma 13** (Postponement, see [7])**.** *If $M \xrightarrow{int}_{\mathsf{v}} L$ and $L \xrightarrow{h}_{\beta_v} N$ (resp. $L \xrightarrow{h}_{\sigma} N$), then there exist $L', L'' \in \Lambda$ such that $M \xrightarrow{h}{}^{+}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} L'' \xrightarrow{int}{}^{*}_{\mathsf{v}} N$ (resp. $M \xrightarrow{h}{}^{*}_{\beta_v} L' \xrightarrow{h}{}^{*}_{\sigma} L'' \xrightarrow{int}{}^{*}_{\mathsf{v}} N$).*

Next Theorem 14 is one of the main result proven in [7, Theorem 22] by adapting Takahashi's method [19, 5]: any v-reduction sequence can be sequentialized into a head β_v-reduction sequence followed by a head σ-reduction sequence, followed by an internal v-reduction sequence. In ordinary λ-calculus, the well-known result corresponding to our Theorem 14 states that a β-reduction sequence can be factorized in a head reduction sequence followed by an internal reduction sequence (see for example [19, Corollary 2.6]).

▶ **Theorem 14** (Sequentialization, see [7])**.** *If $M \to^{*}_{\mathsf{v}} M'$ then there exist $L, N \in \Lambda$ such that $M \xrightarrow{h}{}^{*}_{\beta_v} L \xrightarrow{h}{}^{*}_{\sigma} N \xrightarrow{int}{}^{*}_{\mathsf{v}} M'$.*

The sequentialization of Theorem 14 imposes no order between head σ-reductions. Indeed, the example in [7, p. 10] shows that it is impossible to sequentialize them by giving way to head σ_1- or head σ_3-redexes: a head σ_1-reduction step can create a head σ_3-redex, and vice versa.

In [7, Definition 27 and Corollary 29] it has also been proven that the v-equivalence (and in particular the σ-equivalence) is contained in the call-by-value observational equivalence.

4 Head normalization

In this section we prove the first main result of our paper: Theorem 21, which studies the normalization for head v-reduction and relates it to the head β_v-reduction (i.e. the weak evaluation strategy for Plotkin's λ_v-calculus). Let us start with a preliminary remark.

▶ **Remark 15.** According to Facts 10.1-2, every $V \in \Lambda_v$ is head β_v- and head σ-normal, and hence is head v-normal. The converse does not hold: xI is head v-normal but $xI \notin \Lambda_v$.

First, we give a syntactic characterization of head v- and head β_v-normal forms.

▶ **Definition 16.** *We define the subsets Λ_a, Λ_b and Λ_c (whose elements are denoted by A, B and C respectively) of Λ as follows (for any variable x, any $V \in \Lambda_v$ and any $N \in \Lambda$):*

$$(\Lambda_a) \quad A ::= xV \mid xA \mid AN \qquad (\Lambda_b) \quad B ::= (\lambda x.N)A \qquad (\Lambda_c) \quad C ::= xV \mid VC \mid CN$$

[2] In [7, Corollary 21] there is a more informative statement of our Lemma 13, involving a notion of internal parallel reduction $\overset{int}{\Rightarrow}$. Our Lemma 13 follows immediately from [7, Corollary 21] since $\xrightarrow{int}_{\mathsf{v}} \subseteq \overset{int}{\Rightarrow} \subseteq \xrightarrow{int}{}^{*}_{\mathsf{v}}$.

Notice that $\Lambda_a \cup \Lambda_b \subsetneq \Lambda_c$ and $M, N \in \Lambda_c \smallsetminus (\Lambda_a \cup \Lambda_b)$ where $M = (\lambda y.\Delta)(xI)\Delta$ and $N = \Delta((\lambda y.\Delta)(xI))$ (as in Example 6). Moreover, $\Lambda_v \cap \Lambda_a = \Lambda_v \cap \Lambda_b = \Lambda_v \cap \Lambda_c = \Lambda_a \cap \Lambda_b = \emptyset$ and all terms in $\Lambda_a \cup \Lambda_b \cup \Lambda_c$ are not closed. All terms in Λ_b are β-redexes that are not β_v-redexes; all terms in Λ_a have a free "head variable" and are neither a value nor a β-redex.

▶ **Proposition 17** (Characterization of head β_v-normal forms)**.** *Let M be a term.*

1. *M is head β_v-normal and is not a λ-value if and only if $M \in \Lambda_c$.*
2. *M is head β_v-normal if and only if $M \in \Lambda_v \cup \Lambda_c$.*

Proof. Statement (2) is an immediate consequence of statement (1) and Remark 15.

$\Rightarrow$: We prove the left-to-right direction of statement (1), by induction on $M \in \Lambda$.
The case where $M \in \Lambda_v$ is impossible by hypothesis.
If $M = M_1M_2$ (for some $M_1, M_2 \in \Lambda$) is head β_v-normal then M is not a λ-value and M_1 and M_2 are head β_v-normal, moreover either $M_1 \neq \lambda x.N$ (for any $N \in \Lambda$) or $M_2 \notin \Lambda_v$ (otherwise M would be a head β_v-redex). Therefore, there are only three cases:
- either $M_1 \notin \Lambda_v$, thus $M_1 \in \Lambda_c$ by induction hypothesis, and hence $M \in \Lambda_c$;
- or $M_1 \in \Lambda_v$ and $M_2 \notin \Lambda_v$, so $M_2 \in \Lambda_c$ by induction hypothesis, and thus $M \in \Lambda_c$;
- or M_1 is a variable and $M_2 \in \Lambda_v$, hence $M \in \Lambda_c$ (this is the base case).

$\Leftarrow$: The right-to-left direction of statement (1) can easily be proved by induction on $M \in \Lambda_c$. ◀

A consequence of Proposition 17 is that all closed head β_v-normal forms are abstractions.

▶ **Proposition 18** (Characterization of head v-normal forms)**.** *Let $M \in \Lambda$.*

1. *M is head v-normal and is neither a λ-value nor a β-redex if and only if $M \in \Lambda_a$.*
2. *M is head v-normal and is a β-redex if and only if $M \in \Lambda_b$.*
3. *M is head v-normal if and only if $M \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$.*

Proof. Statement (3) is an immediate consequence of statements (1)-(2) and Remark 15.

$\Rightarrow$: We prove simultaneously the left-to-right direction of statements (1) and (2), by induction on $M \in \Lambda$. The case where $M \in \Lambda_v$ is impossible by hypothesis.
If $M = M_1M_2$ (for some $M_1, M_2 \in \Lambda$) is head v-normal then M is not a λ-value and M_1 and M_2 are head v-normal, moreover M_1 is not a β-redex (otherwise M would be a head σ_1-redex), and either $M_1 \neq \lambda x.N$ (for any $N \in \Lambda$) or $M_2 \notin \Lambda_v$ (otherwise M would be a head β_v-redex), and either $M_1 \notin \Lambda_v$ or M_2 is not a β-redex (otherwise M would be a head σ_3-redex). There are only three cases:
- either M_1 is a variable and M_2 is not a β-redex, so M is not a β-redex; if $M_2 \in \Lambda_v$ then $M \in \Lambda_a$ (this is the base case); otherwise $M_2 \in \Lambda_a$ by induction hypothesis, so $M \in \Lambda_a$;
- or $M_1 \notin \Lambda_v$, thus M is not a β-redex and $M_1 \in \Lambda_a$ by induction hypothesis, so $M \in \Lambda_a$;
- or $M_1 = \lambda x.N$ for some $N \in \Lambda$ and M_2 is neither a λ-value nor a β-redex, so M is a β-redex, furthermore $M_2 \in \Lambda_a$ by induction hypothesis, and thus $M \in \Lambda_b$.

$\Leftarrow$: The right-to-left direction of statement (1) can easily be proved by induction on $M \in \Lambda_a$. Let us prove the right-to-left direction of statement (2): if $M \in \Lambda_b$ then $M = (\lambda x.N)A$ for some $N \in \Lambda$ and $A \in \Lambda_a$, thus M is a β-redex. For any $M' \in \Lambda$, the last rule of the derivation of $M \xrightarrow{h}_{\mathsf{v}} M'$ might be neither σ_1 nor σ_3 (because A is not a β-redex by statement 1) nor β_v (because $A \notin \Lambda_v$ by statement 1 again) nor *right* (because A is head v-normal, by statement 1 again). Therefore, M is head v-normal. ◀

As a consequence of Proposition 18, all closed head v-normal forms are abstractions.

The sets of terms Λ_a, Λ_b and Λ_c of Definition 16 enjoy the closure properties summarized in Lemma 19 below. Together with the syntactic characterizations of head β_v-normal forms

(Proposition 17) and head v-normal forms (Proposition 18), these closure properties allow one to reason about head v-reduction in spite of its non-confluence: they will be used to prove our main results, Theorems 21 and 24 and Proposition 27.

▶ **Lemma 19** (Closure properties).

1. *The set* Λ_a *is closed under* v*-internal reduction and expansion, i.e., for any* $N' \in \Lambda$ *and* $N \in \Lambda_a$*, if* $N' \xrightarrow{int}_{\mathsf{v}} N$ *or* $N \xrightarrow{int}_{\mathsf{v}} N'$ *then* $N' \in \Lambda_a$*.*
2. *The set* Λ_b *is closed under* v*-internal reduction and expansion, i.e., for any* $N' \in \Lambda$ *and* $N \in \Lambda_b$*, if* $N' \xrightarrow{int}_{\mathsf{v}} N$ *or* $N \xrightarrow{int}_{\mathsf{v}} N'$ *then* $N' \in \Lambda_b$*.*
3. *Head* v*-normal forms are closed under* v*-internal reduction and expansion, i.e., for any* $N, N' \in \Lambda$ *where* N *is head* v*-normal, if* $N' \xrightarrow{int}_{\mathsf{v}} N$ *or* $N \xrightarrow{int}_{\mathsf{v}} N'$ *then* N' *is head* v*-normal.*
4. *Head* β_v*-normal forms are closed under head* σ*-reduction and expansion, i.e., for any* $N, N' \in \Lambda$ *where* N *is head* β_v*-normal, if* $N' \xrightarrow{h}_{\sigma} N$ *or* $N \xrightarrow{h}_{\sigma} N'$ *then* N' *is head* β_v*-normal.*

Proof.

1. We show that if $N \in \Lambda_a$ and $N' \xrightarrow{int}_{\mathsf{v}} N$ (resp. $N \xrightarrow{int}_{\mathsf{v}} N'$) then $N' \in \Lambda_a$, by induction on the derivation of $N' \xrightarrow{int}_{\mathsf{v}} N$ (resp. $N \xrightarrow{int}_{\mathsf{v}} N'$). Let us consider its last rule r.
   Since $N \in \Lambda_a$ (see Definition 16), $N = xLN_1 \dots N_n$ for some $n \in \mathbb{N}$, some variable x, some $L \in \Lambda_v \cup \Lambda_a$ and some $N_1, \dots, N_n \in \Lambda$, thus $\mathsf{r} \neq \lambda$ and hence either $\mathsf{r} = \mathit{right}$ or $\mathsf{r} = @$.
   If $\mathsf{r} = \mathit{right}$ then $N' = xL'N_1 \dots N_n$ where $L' \xrightarrow{int}_{\mathsf{v}} L$ (resp. $L \xrightarrow{int}_{\mathsf{v}} L'$). Since $L \in \Lambda_v \cup \Lambda_a$, there are two cases:
   - either $L \in \Lambda_a$ and then $L' \in \Lambda_a$ by induction hypothesis, so $N' = xL'N_1 \dots N_n \in \Lambda_a$;
   - or $L \in \Lambda_v$ and then $L' \in \Lambda_v$ by Fact 10.3 (resp. Remark 4, since $\xrightarrow{int}_{\mathsf{v}} \subseteq \to_{\mathsf{v}}$), therefore $N' = xL'N_1' \dots N_n' \in \Lambda_a$.

   Finally, if $\mathsf{r} = @$ then $n \in \mathbb{N}^+$ and $N' = xLN_1 \dots N_i' \dots N_n$ for some $1 \leq i \leq n$ with $N_i' \to_{\mathsf{v}} N_i$ (resp. $N_i \to_{\mathsf{v}} N_i'$), hence $N' \in \Lambda_a$ because $xL \in \Lambda_a$.
2. We show that if $N \in \Lambda_b$ and $N' \xrightarrow{int}_{\mathsf{v}} N$ (resp. $N \xrightarrow{int}_{\mathsf{v}} N'$) then $N' \in \Lambda_b$, by induction on the derivation of $N' \xrightarrow{int}_{\mathsf{v}} N$ (resp. $N \xrightarrow{int}_{\mathsf{v}} N'$). Let us consider its last rule r. Since $N \in \Lambda_b$, then $N = (\lambda x.M)A$ for some $M \in \Lambda$ and $A \in \Lambda_a$, hence $\mathsf{r} \neq @$ because N has not the shape $VLM_1 \dots M_m$ for any $m \in \mathbb{N}^+$; therefore either $\mathsf{r} = \lambda$ or $\mathsf{r} = \mathit{right}$:
   - if $\mathsf{r} = \lambda$, then $N' = (\lambda x.M')A$ where $M' \to_{\mathsf{v}} M$ (resp. $M \to_{\mathsf{v}} M'$), hence $N' \in \Lambda_b$;
   - if $\mathsf{r} = \mathit{right}$, then $N' = (\lambda x.M)A'$ where $A' \xrightarrow{int}_{\mathsf{v}} A$ (resp. $A \xrightarrow{int}_{\mathsf{v}} A'$), thus $A' \in \Lambda_a$ by Lemma 19.1, hence $N' \in \Lambda_b$;
3. Thanks to Proposition 18.3, it is sufficient to show that if $N \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$ and $N' \xrightarrow{int}_{\mathsf{v}} N$ (resp. $N \xrightarrow{int}_{\mathsf{v}} N'$) then $N' \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$. If $N \in \Lambda_v$ then $N' \in \Lambda_v$ by Fact 10.3 (resp. Remark 4, since $\xrightarrow{int}_{\mathsf{v}} \subseteq \to_{\mathsf{v}}$). If $N \in \Lambda_a$ then $N' \in \Lambda_a$ by Lemma 19.1. Finally, if $N \in \Lambda_b$ then $N' \in \Lambda_b$ by Lemma 19.2.
4. By Proposition 17.2, $N \in \Lambda_v \cup \Lambda_c$. Since $M \xrightarrow{h}_{\sigma} N$ or $N \xrightarrow{h}_{\sigma} M$, $N \notin \Lambda_v$ by Fact 10.2. We prove by induction on $N \in \Lambda_c$ that $M \in \Lambda_c$. By Definition 16, there are only two cases:
   - either $N = xVN_1 \dots N_n$ for some $n \in \mathbb{N}$, variable x, $V \in \Lambda_v$ and $N_1, \dots, N_n \in \Lambda$, but this is impossible since the last rule of the derivation of $M \xrightarrow{h}_{\sigma} N$ or $N \xrightarrow{h}_{\sigma} M$ can be neither σ_1 nor σ_3 (because of the subterm xV) nor *right* (because of Fact 10.2);
   - or $N = VLN_1 \dots N_n$ for some $n \in \mathbb{N}$, $V \in \Lambda_v$, $L \in \Lambda_c$ and $N_1, \dots, N_n \in \Lambda$, and then there are three sub-cases, depending on the last rule r of the derivation of $M \xrightarrow{h}_{\sigma} N$ (resp. $N \xrightarrow{h}_{\sigma} M$):
     - if $\mathsf{r} = \sigma_1$ then $V = \lambda x.N'N_0$ (resp. $\lambda x.N'$) and $M = (\lambda x.N')LN_0 \dots N_n$ (resp. $M = (\lambda x.N'N_1)LN_2 \dots N_n$ with $n > 0$) for some $N', N_0 \in \Lambda$, hence $M \in \Lambda_c$;
     - if $\mathsf{r} = \sigma_3$ then $V = \lambda x.V'N'$ (resp. $L = (\lambda x.N')L'$) and $M = V'((\lambda x.N')L)N_1 \dots N_n$ (resp. $M = (\lambda x.VN')L'N_1 \dots N_n$) for some $V' \in \Lambda_v$ (resp. $L' \in \Lambda_c$) and $N' \in \Lambda$, thus $(\lambda x.N')L \in \Lambda_c$ (resp. $(\lambda x.VN')L' \in \Lambda_c$) and hence $M \in \Lambda_c$;

- if $\mathsf{r} = right$ then $M = VL'N_1 \dots N_n$ for some $L' \in \Lambda$ such that $L' \xrightarrow{h}_\sigma L$ (resp. $L \xrightarrow{h}_\sigma L'$), so $L' \in \Lambda_c$ by induction hypothesis, and hence $M \in \Lambda_c$.

◀

Lemma 19.4 is a formalization of the two following facts: (a) a head σ-reduction step may create a new β_v-redex but in this case it is not a head β_v-redex; (b) when $M \xrightarrow{h}_\sigma N$, the head β_v-redex of M (if any) has a residual in N which is the head β_v-redex of N.

▶ **Lemma 20.** *There exists no infinite head* v*-reduction sequence with finitely many head* β_v*-reduction steps.*

Proof. Suppose the opposite holds: then there would exist $m \in \mathbb{N}$ and an infinite sequence of terms $(M_i)_{i \in \mathbb{N}}$ such that $M_i \xrightarrow{h}_\mathsf{v} M_{i+1}$ for any $1 \leq i \leq m$, $M_m \xrightarrow{h}_{\beta_v} M_{m+1}$ and $M_i \xrightarrow{h}_\sigma M_{i+1}$ for any $i > m$ (since $\xrightarrow{h}_\mathsf{v} = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma$). But this is impossible because $\xrightarrow{h}_\sigma$ is strongly normalizing (by Proposition 5 and since $\xrightarrow{h}_\sigma \subseteq \to_\sigma$). Contradiction. ◀

Now we can state and prove our main result about head β_v- and head v-normalization.

▶ **Theorem 21** (Head normalization). *Let* $M \in \Lambda$. *The following are equivalent:*

1. *there exists a head* β_v*-normal form* N *such that* $M \simeq_{\beta_v} N$*;*
2. *there exists a head* v*-normal form* N *such that* $M \simeq_\mathsf{v} N$*;*
3. M *is head* v*-normalizable;*
4. M *is head* β_v*-normalizable;*
5. *there is no* v*-reduction sequence from* M *with infinitely many head* β_v*-reduction steps;*
6. M *is strongly head* v*-normalizable.*

Proof.

(1)⇒(2) By hypothesis, there exists a head β_v-normal $N \in \Lambda$ such that $M \simeq_{\beta_v} N$, thus $M \simeq_\mathsf{v} N$. Since $\xrightarrow{h}_\sigma$ is strongly normalizing (by Proposition 5 and because $\xrightarrow{h}_\sigma \subseteq \to_\sigma$), there exists a head σ-normal $N' \in \Lambda$ such that $N \xrightarrow{h}{}^*_\sigma N'$, therefore $M \simeq_\mathsf{v} N'$ since $\xrightarrow{h}_\sigma \subseteq \to_\mathsf{v}$. By Lemma 19.4, N' is also head β_v-normal and hence head v-normal.

(2)⇒(3) Since $M \simeq_\mathsf{v} N$, there is $L \in \Lambda$ such that $M \to^*_\mathsf{v} L$ and $N \to^*_\mathsf{v} L$, by confluence of $\to_\mathsf{v}$ (Proposition 5). By Theorem 14, there are $M_1, M_2, N_1, N_2 \in \Lambda$ such that $M \xrightarrow{h}{}^*_{\beta_v} M_1 \xrightarrow{h}{}^*_\sigma M_2 \xrightarrow{int}{}^*_\mathsf{v} L$ and $N \xrightarrow{h}{}^*_{\beta_v} N_1 \xrightarrow{h}{}^*_\sigma N_2 \xrightarrow{int}{}^*_\mathsf{v} L$. As N is head v-normal, $N = N_1 = N_2 \xrightarrow{int}{}^*_\mathsf{v} L$. By Lemma 19.3, L and M_2 are v-head normal. So, $M \xrightarrow{h}{}^*_\mathsf{v} M_2$ with M_2 head v-normal.

(3)⇒(4) By hypothesis, there is $N \in \Lambda$ head v-normal such that $M \xrightarrow{h}{}^*_\mathsf{v} N$. By Lemma 12.2, there is $L \in \Lambda$ such that $M \xrightarrow{h}{}^*_{\beta_v} L \xrightarrow{h}{}^*_\sigma N$. Since N is head v-normal and in particular head β_v-normal, L is head β_v-normal according to Lemma 19.4. So M is head β_v-normalizable.

(4)⇒(5) Lemma 12.1 says that if $N \xrightarrow{h}_\sigma L \xrightarrow{h}_{\beta_v} N'$ then there exists $L' \in \Lambda$ such that $N \xrightarrow{h}_{\beta_v} L' \xrightarrow{h}{}^=_\sigma N'$; Lemma 13 and Fact 10.3 show that if $N \xrightarrow{int}_\mathsf{v} L \xrightarrow{h}_{\beta_v} N'$ then there exist $L', L'' \in \Lambda$ such that $N \xrightarrow{h}{}^+_{\beta_v} L' \xrightarrow{h}{}^*_\sigma L'' \xrightarrow{int}{}^*_\mathsf{v} N'$. Since $\to_\mathsf{v} = \xrightarrow{h}_{\beta_v} \cup \xrightarrow{h}_\sigma \cup \xrightarrow{int}_\mathsf{v}$, this means that if there is an infinite v-reduction sequence from M with infinitely many head β_v-reduction steps, then for any $n \in \mathbb{N}$ there is a head β_v-reduction sequence from M whose length is at least n. Therefore, M is not head β_v-normalizable, since the head β_v-reduction is deterministic.

(5)⇒(6) If M is not strongly head v-normalizable then there exists an infinite head v-reduction sequence. By Lemma 20, this head v-reduction (and hence v-reduction, since $\xrightarrow{h}_\mathsf{v} \subseteq \to_\mathsf{v}$) sequence has infinitely many head β_v-reduction steps.

(6)⇒(1) As M is strongly head v-normalizable, in particular is head v-normalizable, hence there exists $N \in \Lambda$ head v-normal and in particular head β_v-normal such that $M \xrightarrow{h}_{\mathsf{v}}^{*} N$. By Lemma 12.2, there exists $L \in \Lambda$ such that $M \xrightarrow{h}_{\beta_v}^{*} L \xrightarrow{h}_{\sigma}^{*} N$. Therefore $M \simeq_{\beta_v} L$ since $\xrightarrow{h}_{\beta_v} \subseteq \to_{\beta_v}$. According to Lemma 19.4, L is head β_v-normal. ◀

In Theorem 21, the equivalence (3)⇔(6) means that (weak) normalization and strong normalization are equivalent for head v-reduction (for head β_v-reduction they are trivially equivalent since the head β_v-reduction is deterministic), therefore if one is interested in studying the termination of head v-reduction, no difficulty arises from its non-determinism. The equivalence (4)⇔(3) or (4)⇔(6) says that the weak evaluation process defined for Plotkin's λ_v-calculus (the head β_v-reduction) terminates if and only if the weak evaluation process defined for λ_v^σ (the head v-reduction) terminates: σ-rules play no role in deciding the termination of a head v-reduction sequence. The equivalence (3)⇔(2) (resp. (4)⇔(1)) is the version for λ_v^σ (resp. λ_v) of a well-known theorem for ordinary λ-calculus (see for example [3, Theorem 8.3.11]): in some sense, it claims that the head v-reduction (resp. head β_v-reduction) is complete with respect to the v-equivalence (resp. β_v-equivalence). The equivalence (5)⇔(2) (resp. (5)⇔(1)) can be seen as the version for λ_v^σ (resp. λ_v) of the Quasi-Head Reduction Theorem [19, Theorem 2.10] stated by Takahashi for ordinary λ-calculus.

5 Normalization strategy and other results

Theorems 14 and 21 strengthen the idea that, in spite of non-determinism and non-confluence of head v-reduction and non-sequentiability of head σ-reduction steps, the head v-reduction can be used to define a normalization strategy for the λ_v^σ-calculus, as proven in next Theorem 24, the second main result of our paper: given a term M, one starts the (unique) head β_v-head reduction sequence from M as long as a head β_v-normal form N is reached (recall that, according to Theorem 21, a term is (strongly) head v-normalizable if and only if it is head β_v-normalizable); then, one starts a head σ-reduction sequence from N (where head σ_1- and head σ_3-reduction steps can be performed in whatever order) as long as a head σ-normal form N' is reached (such a N' always exists because $\xrightarrow{h}_{\sigma}$ is strongly normalizing, and it is head v-normal by Lemma 19.4); finally, one performs the internal v-reduction steps starting from N' by iterating the head β_v-reduction sequences and then the head σ-reduction sequences as above on the subterms of N', from the left to the right. More precisely:

▶ **Definition 22** (Successors path). *Let $M \in \Lambda$.*
A successor of M *is a $M' \in \Lambda$ defined by induction on $M \in \Lambda$ as follows:*
- *if M is not head β_v-normal, then M' is such that $M \xrightarrow{h}_{\beta_v} M'$;*
- *if M is head β_v-normal but not head σ-normal, then M' is such that $M \xrightarrow{h}_{\sigma} M'$;*
- *if M is head v-normal then:*
  - *if M is a variable then $M' = M$,*
  - *if $M = \lambda x.N$ for some $N \in \Lambda$, then $M' = \lambda x.N'$ for some successor N' of N,*
  - *if $M = NL$ for some $N, L \in \Lambda$, then either N is not v-normal and $M' = N'L$ where N' is a successor of N, or N is v-normal and $M' = NL'$ where L' is a successor of L.*

A successors path of M *is an infinite sequence $(M_i)_{i\in\mathbb{N}}$ of terms such that $M_0 = M$ and M_{i+1} is a successor of M_i, for any $i \in \mathbb{N}$.*

Clearly, for every term M there is at least one successor M' of M; moreover, this successor M' is unique when M is not head β_v-normal, since the head β_v-reduction is deterministic, and $M = M'$ when M is v-normal.

▶ **Remark 23.** Let $M \in \Lambda$ and let $(M_i)_{i\in\mathbb{N}}$ be a successors path of M.
1. For every $i \in \mathbb{N}$, there exist $0 \leq j \leq k \leq i$ such that $M \overset{h}{\to}^*_{\beta_v} M_j \overset{h}{\to}^*_{\sigma} M_k \overset{int}{\to}^*_{\mathsf{v}} M_i$.
2. For every $i \in \mathbb{N}$, if M_i is v-normal then M_j is v-normal for any $j \geq i$.

A successors path of a term M is a call-by-value left-to-right v-evaluation strategy starting from M that can reduce under a λ only when a head v-normal from is reached. Due to the non-determinism of the head σ-reduction, a term M may have several successors paths. We cannot get rid of the non-determinism of the successors path of M because of the non-sequentiability of head σ-reductions, see p. 9 and [7, p. 10].

▶ **Theorem 24** (Normalization strategy)**.** *Let $M \in \Lambda$. Every successors path $(M_i)_{i\in\mathbb{N}}$ of M is a normalization strategy for M, i.e. if M is* v*-normalizable then there exists $j, k, \ell \in \mathbb{N}$ such that $j \leq k \leq \ell$, M_j is head β_v-normal, M_k is head* v*-normal and M_ℓ is* v*-normal.*

Proof. Let $(M_i)_{i\in\mathbb{N}}$ be a successors path of M and $N \in \Lambda$ be such that N is v-normal and $M \to^*_{\mathsf{v}} N$: we prove by induction on $N \in \Lambda$ that there exist $j, k, \ell \in \mathbb{N}$ such that M_j is head β_v-normal, M_k is head v-normal and M_ℓ is v-normal.

Since M is v-normalizable, then it is head β_v-normalizable (because $\overset{h}{\to}_{\beta_v} \subseteq \to_{\mathsf{v}}$), thus there exists $j \in \mathbb{N}$ such that M_j is head β_v-normal because $\overset{h}{\to}_{\beta_v}$ is deterministic. As $\overset{h}{\to}_{\sigma}$ is strongly normalizing (by Proposition 5, since $\overset{h}{\to}_{\sigma} \subseteq \to_{\sigma}$), there exists $k \in \mathbb{N}$ with $j \leq k$ such that M_k is head σ-normal. According to Lemma 19.4, M_k is also head β_v-normal, hence M_k is head v-normal. Certainly, $M_k = VN_1 \dots N_n$ for some $n \in \mathbb{N}$, $V \in \Lambda_v$ and $N_1, \dots, N_n \in \Lambda$. By confluence of $\to_{\mathsf{v}}$ (Proposition 5) and since N is v-normal and M_k is head v-normal, one has $M_k \overset{int}{\to}^*_{\mathsf{v}} N$ and hence $N = V'N'_1 \dots N'_n$ for some v-normal $V' \in \Lambda_v$ and some v-normal $N'_1, \dots, N'_n \in \Lambda$ such that $V \to^*_{\mathsf{v}} V'$ and $N_r \to^*_{\mathsf{v}} N'_r$ for any $1 \leq r \leq n$. By induction hypothesis, for every successors path $(V_i)_{i\in\mathbb{N}}$ of V and, for any $1 \leq r \leq n$, for every successors path $(L^r_i)_{i\in\mathbb{N}}$ of N^r there exist $p, p_1, \dots, p_n \in \mathbb{N}$ such that $V_p, L^1_{p_1}, \dots, L^n_{p_n}$ are v-normal: by confluence of $\to_{\mathsf{v}}$ (Proposition 5), $V_p = V'$ and $N'_r = L^r_{p_r}$ for any $1 \leq r \leq n$.

Let us consider the infinite sequence of terms $s = (M = M_0, \; \dots\dots \;, M_k = VN_1 \dots N_n = V_0 N_1 \dots N_n, \; \dots\dots \;, V_p N_1 \dots N_n = V'L^1_0 N_2 \dots N_n, \; \dots\dots \;, V'L^1_{p_1} N_2 \dots N_n = V'N'_1 L^2_0 \dots N_n, \; \dots\dots \;, V'N'_1 N'_2 \dots N'_n = N, N, \dots\dots)$: this is a successors path of M and, for an opportune choice of the successors paths $(V_i)_{i\in\mathbb{N}}$, $(L^1_i)_{i\in\mathbb{N}}$, ..., $(L^n_i)_{i\in\mathbb{N}}$, one has that $s = (M_i)_{i\in\mathbb{N}}$, in particular there exists $\ell \in \mathbb{N}$ such that $j \leq k \leq \ell$ and $M_\ell = N$. ◀

In ordinary λ-calculus, the well-known theorem corresponding to our Theorem 24 is the Leftmost Reduction Theorem, see [19, Theorem 2.8] or [3, Theorem 13.2.2]. Differently from the leftmost reduction of ordinary λ-calculus, our normalization strategy is not deterministic, i.e., our Theorem 24 provides a family of normalization strategies.

Finally, we have shown at p. 7 that the head σ- and head v-reductions are not (locally) confluent and a term may have several head v-normal forms. Nevertheless, the characterization of head v-normal forms given by Proposition 18 allows us to claim that (see next Proposition 27) in some cases (of interest), more precisely when a term has a head v-normal form which is a value or an element of Λ_a, the head v-normal form is unique (Proposition 27.1): all terms having several head v-normal forms are such that all their head v-normal forms are in Λ_b. In particular, every head v-normalizable closed term has a unique head v-normal form, which is an abstraction and coincides with its head β_v-normal form (Proposition 27.2).

▶ **Remark 25.** By inspection on the rules of Definition 8, it easy to check that the head σ-reduction does not reduce to a term in Λ_a, i.e., for any $M \in \Lambda$ and $N \in \Lambda_a$, one has $M \overset{h}{\not\to}_{\sigma} N$.

Remark 25 does not hold if we replace $\overset{h}{\to}_{\sigma}$ with $\overset{h}{\to}_{\beta_v}$: for instance, $x(II) \overset{h}{\to}_{\beta_v} xI \in \Lambda_a$.

▶ **Fact 26.** *For every $N \in \Lambda_v \cup \Lambda_a$, one has $M \xrightarrow{h}^*_{\beta_v} N$ if and only if $M \xrightarrow{h}^*_{\mathsf{v}} N$.*

Proof. The left-to-right direction follows from $\xrightarrow{h}_{\beta_v} \subseteq \xrightarrow{h}_{\mathsf{v}}$. The right-to-left direction is a consequence of Lemma 12.2 and either Fact 10.2 (if $N \in \Lambda_v$) or Remark 25 (if $N \in \Lambda_a$). ◀

Fact 26 means that, given a head v-reduction sequence, the head σ-reduction plays no role not only in deciding its termination (as stated in Theorem 21), but also in reaching a particular value or term in Λ_a. Fact 26 will be used in the proof of Proposition 27.

▶ **Proposition 27** (Uniqueness of "some" head v-normal forms)**.** *Let $M \in \Lambda$ and $M \xrightarrow{h}^*_{\mathsf{v}} N$.*

1. *If $N \in \Lambda_v \cup \Lambda_a$ then, for every head v-normal $L \in \Lambda$, $M \xrightarrow{h}^*_{\mathsf{v}} L$ implies $N = L$.*
2. *If M is closed and N is head v-normal, then $M \xrightarrow{h}^*_{\beta_v} N$ and $N = \lambda x.N'$ for some $N' \in \Lambda$ such that $\mathsf{fv}(N') \subseteq \{x\}$; moreover, for any head v-normal $L \in \Lambda$, $M \xrightarrow{h}^*_{\mathsf{v}} L$ implies $N = L$.*

Proof.

1. Since $N \in \Lambda_v \cup \Lambda_a$, $M \xrightarrow{h}^*_{\mathsf{v}} N$ implies $M \xrightarrow{h}^*_{\beta_v} N$ by Fact 26. According to Proposition 18.3, N is head v-normal.
   Let $L \in \Lambda$ be head v-normal and such that $M \xrightarrow{h}^*_{\mathsf{v}} L$: by Proposition 18.3, $L \in \Lambda_v \cup \Lambda_a \cup \Lambda_b$. We claim that $L \notin \Lambda_b$. Otherwise, $L \in \Lambda_b$ and then, by confluence of $\to_{\mathsf{v}}$ there would exist $M' \in \Lambda$ such that $N \to^*_{\mathsf{v}} M'$ and $L \to^*_{\mathsf{v}} M'$. According to Proposition 11 and since N and L are head v-normal, $N \xrightarrow{int}^*_{\mathsf{v}} M'$ and $L \xrightarrow{int}^*_{\mathsf{v}} M'$. By Remark 4 (since $\xrightarrow{int}_{\mathsf{v}} \subseteq \to_{\mathsf{v}}$) and Lemma 19.1, $M' \in \Lambda_v \cup \Lambda_a$. By Lemma 19.2, $M' \in \Lambda_b$. But $\Lambda_v \cap \Lambda_b = \emptyset = \Lambda_a \cap \Lambda_b$: contradiction, therefore $L \notin \Lambda_b$.
   So, $L \in \Lambda_v \cup \Lambda_a$ and thus $M \xrightarrow{h}^*_{\beta_v} L$ by Fact 26, hence $N = L$ since $\xrightarrow{h}_{\beta_v}$ is deterministic.
2. Since M is closed, N is closed too. Hence, by Proposition 18.3, $N \in \Lambda_v$ (since the terms in $\Lambda_a \cup \Lambda_b$ are not closed) and N is not a variable, therefore $N = \lambda x.N'$ for some $N' \in \Lambda$ such that $\mathsf{fv}(N') \subseteq \{x\}$. By Fact 26, $M \xrightarrow{h}^*_{\beta_v} N$. According to Proposition 27.1, for every head v-normal $L \in \Lambda$, $M \xrightarrow{h}^*_{\mathsf{v}} L$ implies $N = L$.

◀

Recall that all head v-normal terms are head β_v-normal, since $\xrightarrow{h}_{\beta_v} \subseteq \xrightarrow{h}_{\mathsf{v}}$.

6 Conclusions and future work

In this paper, we have investigated the λ_v^σ-calculus introduced in [4], an extension of Plotkin's call-by-value λ-calculus λ_v [15] with the same syntax as λ_v (without constants) and ordinary (i.e. call-by-name) λ-calculus. The peculiarity of λ_v^σ is in its reduction rules: the v-reduction adds to Plotkin's β_v-reduction two commutation rules called σ_1 and σ_3 which unblock "hidden" β_v-redexes. We have studied the head v-reduction, a non-confluent sub-reduction of the v-reduction already introduced in [7]. We now summarize our main contributions:

1. Theorem 21 is about head v-normalization, it shows that:
   - for the head v-reduction, normalization coincides with strong normalization;
   - the head v-reduction is deeply related to Plotkin's deterministic weak evaluation strategy for λ_v (the former terminates if and only if the latter terminates);
   - both head v-reduction and weak evaluation strategy for λ_v enjoy good properties analogous to the ones of the (call-by-name) head reduction for ordinary λ-calculus.
2. Theorem 24 is about v-normalization: it proves that a top-down extension of the head v-normalization provides a family of normalization strategies for the (full) v-reduction.
3. Proposition 27 is about the uniqueness of the head v-normal form: it shows that, even if there are terms having several head v-normal forms, in some case of interest (for instance, closed terms) the head v-normal form, if any, is unique.

These results, together with the results proven in [4, 7], shows that λ_v^σ is a useful tool to study some theoretical and semantic properties of Plotkin's λ_v-calculus, for instance the notions of call-by-value solvability and potential valuability. This is hard (or impossible) to obtain directly in λ_v because of the "weakness" of Plotkin's β_v-reduction. In the case of ordinary (i.e. call-by-name) λ-calculus, head reduction and solvability are the starting point to investigate separability, semi-separability and Böhm's trees. Hence, it may reasonably be supposed that we have all the ingredients for tackling the question of separability, semi-separability and Böhm's trees in a call-by-value setting. In particular, one may reasonably hope to improve in λ_v^σ the separability theorem already proven by Paolini [12] for λ_v.

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