# Optimal Morphs of Convex Drawings* 

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#### Abstract

We give an algorithm to compute a morph between any two convex drawings of the same plane graph. The morph preserves the convexity of the drawing at any time instant and moves each vertex along a piecewise linear curve with linear complexity. The linear bound is asymptotically optimal in the worst case.


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## 1 Introduction

Convex drawings of plane graphs are a classical topic of investigation in geometric graph theory. A characterization [25] of the plane graphs that admit convex drawings and a lineartime algorithm [10] to test whether a graph admits a convex drawing are known. Convex drawings in small area $[5,8,11]$, orthogonal convex drawings [18, 19, 25] , and convex drawings satisfying further geometric constraints $[16,17]$ have also been studied. It is intuitive, but far from trivial to prove, that the space of the convex drawings of any $n$-vertex plane graph $G$ is connected; i.e., the points in $\mathbb{R}^{2 n}$, each corresponding to the two-dimensional coordinates of a convex drawing of $G$, form a connected set. Expressed in yet another way, there exists a convex morph between any two convex drawings $\Gamma_{s}$ and $\Gamma_{t}$ of the same plane graph $G$, that is, a continuous deformation from $\Gamma_{s}$ to $\Gamma_{t}$ so that the intermediate drawing of $G$ is convex at any instant of the deformation. The main result of this paper is the existence of a convex morph between any two convex drawings of the same plane graph such that each vertex moves along a piecewise linear curve with linear complexity during the deformation.

The existence of a convex morph between any two convex drawings of the same plane graph was first proved by Thomassen [24] more than 30 years ago. His result confirmed a conjecture of Grünbaum and Shepard [15] and improved upon a result of Cairns [9], stating that there exists a continuous deformation, called a morph, between any two straight-line planar drawings of the same plane graph such that any intermediate straight-line drawing is planar. More recently, motivated by applications in computer graphics, animation, and modeling, a number of algorithms for morphing graph drawings have been designed [12, 13, 14, 21, 22]. These algorithms aim to construct morphs that preserve the topology of the given drawings at any time, while guaranteeing that the trajectories of the vertices are "nice" curves.

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Straight-line segments are undoubtedly the most readable and appealing curves for the vertex trajectories. However, linear morphs - morphs in which the vertices move along straight lines - do not always exist [12]. A natural way to overcome this problem is to allow vertices to move along piecewise linear curves. Since trajectories of large complexity would have a dramatically detrimental impact on the readability of the morph, an important goal is to minimize the complexity of these curves. This problem is formalized as follows. Let $\Gamma_{s}$ and $\Gamma_{t}$ be two planar straight-line drawings of a plane graph $G$. Find a sequence $\Gamma_{s}=\Gamma_{1}, \ldots, \Gamma_{k}=\Gamma_{t}$ of planar straight-line drawings of $G$ such that, for $1 \leq i \leq k-1$, the linear morph transforming $\Gamma_{i}$ into $\Gamma_{i+1}$, called a morphing step, is planar and $k$ is small.

The first polynomial upper bound for this problem was recently obtained by Alamdari et al. [1]. The authors proved that a morph between any two planar straight-line drawings of the same $n$-vertex connected plane graph exists with $O\left(n^{4}\right)$ morphing steps. The $O\left(n^{4}\right)$ bound was later improved to $O\left(n^{2}\right)$ [4] and then to a worst-case optimal $O(n)$ bound by Angelini et al. [3]. The algorithm of Angelini et al. [3] can be extended to work for disconnected graphs at the expense of an increase in the number of steps to $O\left(n^{1.5}\right)$ [2].

In this paper we give an algorithm to construct a convex morph between any two convex drawings of the same $n$-vertex plane graph with $O(n)$ morphing steps. Our algorithm preserves the convexity of the drawing at any time instant and in fact preserves strict convexity, if the given drawings are strictly-convex. The linear bound is tight in the worst case, as can be shown by adapting the lower bound construction of Angelini et al. [3]. We remark that Thomassen's algorithm [24] constructs convex morphs with an exponential number of steps. To the best of our knowledge, no other algorithm is known to construct a convex morph between any two convex drawings of the same plane graph.

The outline of our algorithm is simple. Let $\Gamma_{s}$ and $\Gamma_{t}$ be two convex drawings of the same convex graph $G$, that is, a plane graph that admits a convex drawing. Determine a connected subgraph $G^{\prime}$ of $G$ such that removing $G^{\prime}$ from $G$ results in a smaller convex graph $G^{\prime \prime}$. Then $G^{\prime}$ lies inside one face $f$ of $G^{\prime \prime}$. Morph $\Gamma_{s}$ into a drawing $\Gamma_{s}^{\prime}$ of $G$ and morph $\Gamma_{t}$ into a drawing $\Gamma_{t}^{\prime}$ of $G$ such that the cycle of $G$ corresponding to $f$ is delimited by a convex polygon in $\Gamma_{s}^{\prime}$ and in $\Gamma_{t}^{\prime}$. These morphs consist of one morphing step each. Remove $G^{\prime}$ from $\Gamma_{s}^{\prime}$ and $\Gamma_{t}^{\prime}$ to obtain two convex drawings $\Gamma_{s}^{\prime \prime}$ and $\Gamma_{t}^{\prime \prime}$ of $G^{\prime \prime}$. Finally, recursively compute a morph between $\Gamma_{s}^{\prime \prime}$ and $\Gamma_{t}^{\prime \prime}$. Since $f$ remains convex throughout the whole morph from $\Gamma_{s}^{\prime \prime}$ to $\Gamma_{t}^{\prime \prime}$, a morph of $G$ from $\Gamma_{s}^{\prime}$ to $\Gamma_{t}^{\prime}$ can be obtained from the morph of $G^{\prime \prime}$ from $\Gamma_{s}^{\prime \prime}$ to $\Gamma_{t}^{\prime \prime}$ by suitably drawing $G^{\prime}$ inside $f$ at each intermediate step of such a morph. The final morph from $\Gamma_{s}$ to $\Gamma_{t}$ consists of the morph from $\Gamma_{s}$ to $\Gamma_{s}^{\prime}$ followed by the morph from $\Gamma_{s}^{\prime}$ to $\Gamma_{t}^{\prime}$, and then the reverse of the morph from $\Gamma_{t}$ to $\Gamma_{t}^{\prime}$. Our algorithm has two main ingredients.

The first ingredient is a structural decomposition of convex graphs that generalizes a well-known structural decomposition of triconnected planar graphs due to Barnette and Grünbaum [6]. The latter states that any subdivision of a triconnected planar graph contains a path whose removal results in a subdivision of a smaller triconnected planar graph. For convex graphs we can prove a similar theorem which states, roughly speaking, that any convex graph contains a path, or three paths incident to the same vertex, whose removal results in a smaller convex graph. Our approach is thus based on removing a subgraph from the input graph. This differs from the recent papers on morphing graph drawings $[1,3,4]$, where the basic operation is to contract (i.e. move arbitrarily close) a vertex to a neighbor. One of the difficulties of the previous approach was to determine a trajectory for a contracted vertex inside the moving polygon of its neighbors. By removing a subgraph and forcing the newly formed face to be convex, we avoid this difficulty.

The second ingredient is a relationship between unidirectional morphs and level planar drawings of hierarchical graphs, which allows us to compute the above mentioned morphs
between $\Gamma_{s}$ and $\Gamma_{s}^{\prime}$ and between $\Gamma_{t}$ and $\Gamma_{t}^{\prime}$ with one morphing step. This relationship was first observed by Angelini et al. [3]. However, in order to use it in our setting, we need to prove that every strictly-convex graph admits a strictly-convex level planar drawing; this strengthens a result of Hong and Nagamochi [16] and might be of independent interest.

We leave open the question whether any two straight-line drawings of the same plane graph $G$ can be morphed so that every intermediate drawing has polynomial size (e.g., the ratio between the length of any two edges is polynomial in the size of $G$ during the entire morph). In order to solve this problem positively, our approach seems to be better than previous ones; intuitively, subgraph removals are more suitable than vertex contractions for a morphing algorithm that doesn't blow up the size of the intermediate drawings. Nevertheless, we haven't yet been able to prove that polynomial-size morphs always exist.

## 2 Definitions and Preliminaries

In this section we give some definitions and preliminaries.

Drawings and Embeddings. A straight-line planar drawing $\Gamma$ of a graph maps vertices to points in the plane and edges to internally disjoint straight-line segments. Drawing $\Gamma$ partitions the plane into topologically connected regions, called faces. The bounded faces are internal and the unbounded face is the outer face. A vertex (an edge) is external if it is incident to the outer face and internal otherwise. A vertex $x$ is convex, flat, or concave in an incident face $f$ in $\Gamma$, if the angle at $x$ in $f$ is smaller than, equal to, or larger than $\pi$ radians, respectively. Drawing $\Gamma$ is convex (strictly-convex) if for each vertex $v$ and each face $f$ vertex $v$ is incident to, $v$ is either convex or flat (is convex) in $f$, if $f$ is internal, and $v$ is either concave or flat (is concave) in $f$, if $f$ is the outer face. A planar drawing determines a clockwise ordering of the edges incident to each vertex. Two planar drawings of a connected planar graph are equivalent if they determine the same clockwise orderings and have the same outer face. A plane embedding is an equivalence class of planar drawings. A graph with a plane embedding is a plane graph. A convex (strictly-convex) graph is a plane graph that admits a convex (resp. strictly-convex) drawing with the given plane embedding.

Subgraphs and Connectivity. A subgraph $G^{\prime}$ of a plane graph $G$ is regarded as a plane graph whose plane embedding is obtained from $G$ by removing all the vertices and edges not in $G^{\prime}$. We denote by $G-e($ by $G-S)$ the plane graph obtained from $G$ by removing an edge $e$ of $G$ (resp. a set $S$ of vertices and their incident edges).

We denote by $\operatorname{deg}(G, v)$ the degree of a vertex $v$ in a graph $G$. A graph $G$ is biconnected (triconnected) if removing any vertex (resp. any two vertices) leaves $G$ connected. A separation pair in a graph $G$ is a pair of vertices whose removal disconnects $G$. A biconnected plane graph $G$ is internally triconnected if introducing a new vertex in the outer face of $G$ and connecting it to all the vertices incident to the outer face of $G$ results in a triconnected graph. Thus, internally triconnected plane graphs form a super-class of triconnected plane graphs. A split component of a graph $G$ with respect to a separation pair $\{u, v\}$ is either an edge $(u, v)$ or a maximal subgraph $G^{\prime}$ of $G$ that does not contain edge $(u, v)$, that contains vertices $u$ and $v$, and such that $\{u, v\}$ is not a separation pair of $G^{\prime}$; we say that $\{u, v\}$ determines the split components with respect to $\{u, v\}$. For an internally triconnected plane graph $G$, every separation pair $\{u, v\}$ determines two or three split components; further, in the latter case, one of them is an edge $(u, v)$ not incident to the outer face of $G$.

A subdivision $G^{\prime}$ of a graph $G$ is a graph obtained from $G$ by replacing each edge ( $u, v$ ) with a path between $u$ and $v$; the internal vertices of this path are called subdivision vertices. Given a subgraph $H$ of $G$, the subgraph $H^{\prime}$ of $G^{\prime}$ corresponding to $H$ is obtained from $H$ by replacing each edge $(u, v)$ with a path with the same number of vertices as in $G^{\prime}$.

Convex Graphs. Convex graphs have been thoroughly studied, both combinatorially and algorithmically. Most of the known results about convex graphs are stated in the following setting. The input consists of a plane graph $G$ and a convex polygon $P$ representing the cycle $C$ delimiting the outer face of $G$. The problem asks whether $G$ admits a convex drawing in which $C$ is represented by $P$. The known characterizations for this setting imply characterizations and recognition algorithms for the class of convex graphs (with no constraint on the representation of the cycle delimiting the outer face). Quite surprisingly, the literature seems to lack explicit statements of the characterizations in this unconstrained setting. Here we present two theorems, whose proofs can be easily derived from known results [10, 16, 25].

- Theorem 1. A plane graph is convex if and only if it is a subdivision of an internally triconnected plane graph.
- Theorem 2. A plane graph is strictly-convex if and only if it is a subdivision of an internally triconnected plane graph and every degree-2 vertex is external.

Monotonicity. A straight arc $\boldsymbol{x y}$ is a straight line segment directed from a point $x$ to a point $y ; \boldsymbol{x} \boldsymbol{y}$ is monotone with respect to an oriented straight line $\boldsymbol{d}$ if the projection of $x$ on $\boldsymbol{d}$ precedes the projection of $y$ on $\boldsymbol{d}$ according to the orientation of $\boldsymbol{d}$. A path $\left(u_{1}, \ldots, u_{n}\right)$ is $\boldsymbol{d}$-monotone if $\boldsymbol{u}_{\boldsymbol{i}} \boldsymbol{u}_{\boldsymbol{i}+\boldsymbol{1}}$ is monotone with respect to $\boldsymbol{d}$, for $i=1, \ldots, n-1$; a polygon $Q$ is $\boldsymbol{d}$-monotone if it contains two vertices $s$ and $t$ such that the two paths between $s$ and $t$ in $Q$ are both $\boldsymbol{d}$-monotone. A path $P$ (a polygon $Q$ ) is monotone if there exists an oriented straight line $\boldsymbol{d}$ such that $P($ resp. $Q)$ is $\boldsymbol{d}$-monotone. We have the following.

- Lemma 3 (Angelini et al. [3]). Let $Q$ be a convex polygon and $\boldsymbol{d}$ be an oriented straight line not orthogonal to any straight line through two vertices of $Q$. Then $Q$ is $\boldsymbol{d}$-monotone.
- Lemma 4. Let $Q_{1}$ and $Q_{2}$ be strictly-convex polygons sharing an edge $e$ and lying on opposite sides of the line through $e$. Let $P_{i}$ be the path obtained from $Q_{i}$ by removing edge $e$, for $i=1,2$. The polygon $Q$ composed of $P_{1}$ and $P_{2}$ is monotone.

Proof sketch. $Q$ is monotone with respect to a line $\boldsymbol{l}$ orthogonal to $e$ - unless $Q$ contains edges parallel to $e$, in which case a slight perturbation of $\boldsymbol{l}$ suffices.

Morphing. A linear morph $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ between two straight-line planar drawings $\Gamma_{1}$ and $\Gamma_{2}$ of a plane graph $G$ moves each vertex at constant speed along a straight line from its position in $\Gamma_{1}$ to its position in $\Gamma_{2}$. A linear morph is planar if no crossing or overlap occurs between any two edges or vertices during the transformation. A linear morph is convex (strictly-convex) if it is planar and each face is delimited by a convex (resp. strictly-convex) polygon at any time instant of the morph. A convex linear morph is called a morphing step. A unidirectional linear morph [7] is a linear morph in which the straight-line trajectories of the vertices are parallel. A convex morph (a strictly-convex morph) $\left\langle\Gamma_{s}, \ldots, \Gamma_{t}\right\rangle$ between two convex drawings $\Gamma_{s}$ and $\Gamma_{t}$ of a plane graph $G$ is a finite sequence of convex (resp. strictly-convex) linear morphs that transforms $\Gamma_{s}$ into $\Gamma_{t}$. A unidirectional (strictly-) convex morph is such that each of its morphing steps is unidirectional.

## 3 Decompositions of Convex Graphs

Our morphing algorithm relies on a lemma stating that, roughly speaking, any convex graph has a "simple" subgraph whose removal results in a smaller convex graph. A similar result is known for a restricted graph class, namely the subdivisions of triconnected planar graphs.

On the way to proving that every triconnected planar graph is the skeleton of a convex polytope in $\mathbb{R}^{3}$, Barnette and Grünbaum [6] proved that every subdivision of a triconnected planar graph $G$ can be decomposed as follows (see also [20]). Starting from $G$, repeatedly remove a path whose internal vertices have degree two in the current graph, until a subdivision of $K_{4}$ is obtained. Barnette and Grünbaum proved that there is such a decomposition in which every intermediate graph is a subdivision of a simple triconnected plane graph.

We now present a lemma that generalizes Barnette and Grünbaum's decomposition technique so that it applies to convex (not necessarily triconnected) graphs.

- Lemma 5. Let $G$ be a convex graph. There exists a sequence $G_{1}, \ldots, G_{\ell}$ of graphs such that: (i) $G_{1}=G$; (ii) $G_{\ell}$ is the simple cycle $C$ delimiting the outer face of $G$; (iii) for each $1 \leq i \leq \ell$, graph $G_{i}$ is a subgraph of $G$ and is a subdivision of a simple internally triconnected plane graph $H_{i}$; and (iv) for each $1 \leq i<\ell$, graph $G_{i+1}$ is obtained either:
- by deleting the edges and the internal vertices of a path $\left(u_{1}, u_{2}, \ldots, u_{k}\right)$ with $k \geq 2$ from $G_{i}$, where $u_{2}, \ldots, u_{k-1}$ are degree-2 internal vertices of $G_{i}$; or
- by deleting a degree-3 internal vertex $u$ of $G_{i}$ as well as the edges and the internal vertices of three paths $P_{1}, P_{2}$, and $P_{3}$ connecting $u$ with three vertices of the cycle $C$ delimiting the outer face of $G$, where $P_{1}, P_{2}$, and $P_{3}$ are vertex-disjoint except at $u$ and the internal vertices of $P_{1}, P_{2}$, and $P_{3}$ are degree- 2 internal vertices of $G_{i}$.

Proof. Set $G_{1}=G$. Suppose that a sequence $G_{1}, \ldots, G_{i}$ has been determined. If $G_{i}=G_{\ell}$ is the cycle delimiting the outer face of $G$, then we are done. Otherwise, we distinguish two cases, based on whether $G_{i}$ is a subdivision of a triconnected plane graph or not.

Suppose first that $G_{i}$ is a subdivision of a triconnected plane graph $H_{i}$. We construct graphs $G_{i}, \ldots, G_{\ell}$ one by one, in reverse order. Throughout the construction, we maintain the following invariant for every $\ell \geq j>i$. Suppose that $H_{j}$ contains an internal edge ( $u, v$ ) that is also an edge of $H_{i}$. Then there exists no path in $H_{i}$ that connects $u$ and $v$, that is different from edge $(u, v)$, and all of whose internal vertices are not in $H_{j}$.

Let $G_{\ell}$ be the cycle $C$ delimiting the outer face of $G_{i}$. Next, we determine $G_{\ell-1}$ (see Fig. 1(a)). Let $C_{i}$ be the cycle delimiting the outer face of $H_{i}$. Since $H_{i}$ is triconnected and has at least four vertices, there exist three paths that connect an internal vertex $v$ of $H_{i}$ with vertices of $C_{i}$, that share no vertices other than $v$, and whose internal vertices are not in $C_{i}$ (see Theorem 5.1 in [23]). Among all the triples of paths with these properties, choose a triple $\left(P_{x}, P_{y}, P_{z}\right)$ involving the largest number of vertices of $H_{i}$. Paths $P_{x}, P_{y}$, and $P_{z}$ and cycle $C_{i}$ form a graph $G_{\ell-1}^{H}$ that is a subdivision of $K_{4}$. The subgraph $G_{\ell-1}$ of $G_{i}$ corresponding to $G_{\ell-1}^{H}$ is hence a subdivision of $K_{4}$ in which $v$ is the only degree-3 internal vertex. The invariant is satisfied since $P_{x}, P_{y}$, and $P_{z}$ involve the largest number of vertices of $H_{i}$. Further, $G_{\ell}$ is obtained from $G_{\ell-1}$ by deleting a degree-3 internal vertex $v$ of $G_{\ell-1}$ as well as the edges and the internal vertices of $P_{x}, P_{y}$, and $P_{z}$, as required by the lemma.

Next, assume that a sequence $G_{\ell}, \ldots, G_{j}$ has been determined, for some $j \leq \ell-1$. If $G_{j}=G_{i}$, then we are done. Otherwise, $G_{j-1}$ is obtained by adding a path $P$ to $G_{j}$. The choice of $P$ distinguishes two cases (as in the proof of Theorem 2 in [6]).

In Case (A), a vertex $z$ exists such that $\operatorname{deg}\left(G_{j}, z\right)=2$ and $\operatorname{deg}\left(G_{i}, z\right) \geq 3$. Then, consider the unique path $P_{x y}$ in $G_{j}$ that contains $z$ as an internal vertex, whose internal vertices have degree two in $G_{j}$, and whose end-points $x$ and $y$ have degree at least three in $G_{j}$. Note

(a)

(b)

(c)

(d)

Figure 1 Illustration for the proof of Lemma 5 if $G_{i}$ is a subdivision of a triconnected plane graph $H_{i}$. White vertices belong to $H_{j}, G_{j}, H_{i}$, and $G_{i}$; grey vertices belong to $G_{j}, H_{i}$, and $G_{i}$, and not to $H_{j}$; black vertices belong to $G_{j}$ and $G_{i}$, and not to $H_{j}$ and $H_{i}$. (a) Graph $G_{\ell-1}$. (b)-(d) Graph $G_{j}$ and path $P$, together forming graph $G_{j-1} ;(\mathrm{b})$ and (c) illustrate Case (A) with $u_{k}$ having degree two and greater than two in $G_{j}$, respectively, while (d) depicts Case (B).
that $(x, y)$ is an edge of $H_{j}$. Since $\{x, y\}$ is not a separation pair in $H_{i}$, there exists a path $P=\left(u_{1}, u_{2}, \ldots, u_{k}\right)$ in $G_{i}$ such that $u_{1}$ is an internal vertex of $P_{x y}$, vertex $u_{h}$ does not belong to $G_{j}$, for every $2 \leq h \leq k-1$, and $u_{k}$ is a vertex of $G_{j}$ not in $P_{x y}$. Choose the path with these properties involving the largest number of vertices of $H_{i}$. Observe that $u_{k}$ might have degree two (as in Fig. 1(b)) or greater than two (as in Fig. 1(c)) in $G_{j}$.

In Case (B), there exists no vertex $z$ such that $\operatorname{deg}\left(G_{j}, z\right)=2$ and $\operatorname{deg}\left(G_{i}, z\right) \geq 3$ (see Fig. $1(\mathrm{~d}))$. Since $G_{j}$ is different from $G_{i}$, there exists a path $P=\left(u_{1}, u_{2}, \ldots, u_{k}\right)$ in $G_{i}$ such that $u_{1}$ and $u_{k}$ belong to $H_{j}$, and $u_{2}, \ldots, u_{k-1}$ do not belong to $G_{j}$. Also, a path $P$ satisfying these properties exists such that $u_{1}$ is an internal vertex of $H_{i}$ (otherwise $H_{i}$ would contain a separation pair composed of two external vertices). Choose a path $P$ involving the largest number of vertices of $H_{i}$, subject to the constraint that $u_{1}$ is an internal vertex of $H_{i}$.

In both cases, path $P$ has to be embedded inside a face $f$ of $G_{j}$, according to the plane embedding of $G_{i}$. Since $G_{j}$ contains the cycle delimiting the outer face of $G_{i}$, we have that $f$ is an internal face of $G_{j}$. Graph $G_{j-1}$ is obtained by inserting $P$ in $f$. Since $P$ and $G_{j}$ are subgraphs of $G_{i}$, graph $G_{j-1}$ is a subgraph of $G_{i}$. Also, it satisfies the invariant since $P$ is chosen as a path involving the largest number of vertices of $H_{i}$. It remains to prove that $G_{j-1}$ is a subdivision of a simple triconnected plane graph $H_{j-1}$. Let $H_{j-1}$ be the graph obtained from $G_{j-1}$ by replacing each maximal path whose internal vertices have degree two with a single edge. Thus, $G_{j-1}$ is a subdivision of $H_{j-1}$.

- Claim 1. Graph $H_{j-1}$ is plane, simple, and triconnected.

Proof sketch. First, $H_{j-1}$ is a plane graph since $G_{j-1}$ is a plane graph. Second, in Case (A) $H_{j-1}$ is simple because $H_{j}$ is simple and $u_{1}$ does not belong to $H_{j}$; further, it can be proved that $H_{j-1}$ contains no separation pair (hence it is triconnected) because $H_{j}$ contains three internally disjoint paths between any pair of vertices and because each of $u_{1}$ and $u_{k}$ contains three internally disjoint paths to vertices of $H_{j}$. Third, in Case (B) $H_{j-1}$ is simple because of the invariant and it is triconnected because $H_{j}$ is triconnected.

We now turn to the case in which $G_{i}$ is not a subdivision of a triconnected plane graph. In this case $G_{i}$ is a subdivision of a simple internally triconnected plane graph $H_{i}$ with minimum degree three and containing some separation pairs. Recall that $H_{i}$ has either two or three split components with respect to any separation pair $\{u, v\}$.

Suppose that a separation pair $\{u, v\}$ exists in $H_{i}$ determining three split components. Since $H_{i}$ is internally triconnected, one of these split components is an internal edge $(u, v)$ of $H_{i}$ corresponding to a path $P=\left(u=u_{1}, \ldots, u_{k}=v\right)$ in $G_{i}$, where $u_{2}, \ldots, u_{k-1}$ are degree- 2 internal vertices of $G_{i}$. Let $G_{i+1}=G_{i}-\left\{u_{2}, \ldots, u_{k-1}\right\}$ and let $H_{i+1}=H_{i}-(u, v)$. Note that


Figure 2 Illustration for the proof of Lemma 5 if $G_{i}$ is not a subdivision of a triconnected plane graph $H_{i}$. The faces of $D_{1}, \ldots, D_{m}$ not incident to $Q$ are colored gray in $G_{i}, \ldots, G_{i+m-1}$. The faces of $M_{1}, \ldots, M_{m}$ not incident to $(u, v)$ are colored gray in $H_{i}, \ldots, H_{i+m-1}$. (a) Graph $G_{i}$. (b) Graph $H_{i}$ and separation pair $\{u, v\}$. (c) Graph $L$. (d) Graph $D=D_{1}$. (e) Graph $M=M_{1}$. (f) Graph $G_{i+1}$. (g) Graph $H_{i+1}$. (h) Graph $G_{i+2}$. (i) Graph $H_{i+2}$. (j) Graph $G_{i+3}$. (k) Graph $H_{i+3}$.
$G_{i+1}$ is a subdivision of $H_{i+1}$. Then $H_{i+1}$ is an internally triconnected simple plane graph, given that $H_{i}$ is an internally triconnected simple plane graph with three split components with respect to $\{u, v\}$.

Suppose next that every separation pair of $H_{i}$ determines two split components, as in Fig. 2(a). Let $\{u, v\}$ be a separation pair of $H_{i}$ determining two split components $A$ and $B$ such that $A$ does not contain any separation pair of $H_{i}$ different from $\{u, v\}$, as in Fig. 2(b), (e.g., let $\{u, v\}$ be a separation pair such that the number of vertices in $A$ is minimum among all separation pairs). Let $L$ be the subgraph of $H_{i}$ composed of $A$ and of the path $Q$ between $u$ and $v$ that delimits the outer face of $H_{i}$ and that belongs to $B$; see Fig. 2(c). Let $D$ be the subgraph of $G_{i}$ corresponding to $L$; see Fig. 2(d). The graph $M$ obtained from $L$ by replacing $Q$ with an edge $(u, v)$, shown in Fig. 2(e), is triconnected, given that the vertex set of $A$ does not contain any separation pair of $H_{i}$ different from $\{u, v\}$. Thus, $D$ is a subdivision of a simple triconnected plane graph $M$.

By means of the same algorithm described in the case in which $G_{i}$ is a subdivision of a triconnected plane graph, we determine a sequence $D_{1}, \ldots, D_{m}$ of subdivisions of triconnected plane graphs $M_{1}, \ldots, M_{m}$, where $D_{1}=D, M_{1}=M$, and $M_{m}=K_{3}$. Further, we define a sequence $H_{i+1}, \ldots, H_{i+m-1}$ of graphs where, for each $2 \leq j \leq m-1$, graph $H_{i+j-1}$ is obtained from $H_{i}$ by replacing $M$ with $M_{j}$ (see Figs. 2(b), 2(g), and 2(i)), and where $H_{i+m-1}$ is obtained from $H_{i}$ by replacing $M$ with an edge ( $u, v$ ) (see Fig. 2(k)). Analogously, we define a sequence $G_{i+1}, \ldots, G_{i+m-1}$ of graphs where, for each $2 \leq j \leq m$, graph $G_{i+j-1}$ is obtained from $G_{i}$ by replacing $D$ with $D_{j}$ (see Figs. 2(a), 2(f), 2(h), and 2(j)). Then, for each $2 \leq j \leq m$, graph $G_{i+j-1}$ is a subdivision of $H_{i+j-1}$. Further, for each $1 \leq j \leq m-2$, graph $G_{i+j}$ is obtained from $G_{i+j-1}$ by deleting the edges and the internal vertices of a path $\left(u_{1}, \ldots, u_{k}\right)$ with $k \geq 2$, where $u_{2}, \ldots, u_{k-1}$ are degree- 2 internal vertices of $G_{i+j-1}$. Moreover, graph $G_{i+m-1}$ is obtained by deleting from $G_{i+m-2}$ a degree-3 internal vertex $x$ as well as the edges and the internal vertices of three paths $P_{1}, P_{2}$, and $P_{3}$, as required by the lemma. Finally, since $M_{2}, \ldots, M_{m}$ are simple triconnected plane graphs, $H_{i+1}, \ldots, H_{i+m-1}$ are simple internally triconnected plane graphs.

Note that $H_{i+m-1}$ is obtained from $H_{i}$ by replacing $A$ with edge $(u, v)$, hence $\{u, v\}$ is not a separation pair in $H_{i+m-1}$. Thus, the repetition of the described transformations over different separation pairs $\{u, v\}$ eventually leads to a graph $G_{x}$ that is the subdivision of a simple triconnected plane graph $H_{x}$; then a sequence $G_{x}, \ldots, G_{\ell}$ of subdivisions of triconnected plane graphs such that $G_{\ell}$ is a subdivision of $K_{3}$ is determined as above.

## 4 Convex Drawings of Hierarchical Convex Graphs

A hierarchical graph is a tuple $(G, \boldsymbol{d}, L, \gamma)$ where $G$ is a graph, $\boldsymbol{d}$ is an oriented straight line in the plane, $L$ is a set of parallel lines orthogonal to $\boldsymbol{d}$, and $\gamma$ is a function that maps each vertex of $G$ to a line in $L$ so that adjacent vertices are mapped to distinct lines. The lines in $L$ are ordered as they are encountered when traversing $\boldsymbol{d}$ according to its orientation (we write $l_{1}<l_{2}$ if a line $l_{1}$ precedes a line $l_{2}$ in $L$ ). Furthermore, each line $l_{i} \in L$ is oriented so that $\boldsymbol{d}$ cuts $l_{i}$ from the right to the left of $l_{i}$; a point $a$ precedes a point $b$ on $l_{i}$ if $a$ is encountered before $b$ when traversing $l_{i}$ according to its orientation. For the sake of readability, we will often write $G$ instead of $(G, \boldsymbol{d}, L, \gamma)$ to denote a hierarchical graph. A level drawing of a hierarchical graph $G$ maps each vertex $v$ to a point on the line $\gamma(v)$ and each edge $(u, v)$ of $G$ with $\gamma(u)<\gamma(v)$ to an $\operatorname{arc} \boldsymbol{u} \boldsymbol{v}$ monotone with respect to $\boldsymbol{d}$. A hierarchical graph $G$ with a prescribed plane embedding is a hierarchical plane graph if there is a level planar drawing $\Gamma$ of $G$ that respects the prescribed plane embedding. A path $\left(u_{1}, \ldots, u_{k}\right)$ in $G$ is monotone if $\gamma\left(u_{i}\right)<\gamma\left(u_{i+1}\right)$, for $1 \leq i \leq k-1$. An st-face in a hierarchical plane graph $G$ is a face delimited by two monotone paths connecting two vertices $s$ and $t$, where $s$ is the source and $t$ is the sink of the face. Furthermore, $G$ is a hierarchical-st plane graph if every face of $G$ is an st-face; note that a face $f$ of $G$ is an st-face if and only if the polygon delimiting $f$ in a straight-line level planar drawing of $G$ is $\boldsymbol{d}$-monotone.

In this section we give an algorithm to construct strictly-convex level planar drawings of hierarchical-st strictly-convex graphs, that are hierarchical-st plane graphs ( $G, \boldsymbol{d}, L, \gamma$ ) such that $G$ is a strictly-convex graph. We have the following.

- Theorem 6. Every hierarchical-st strictly-convex graph admits a drawing which is simultaneously strictly-convex and level planar.

Proof. Let $(G, \boldsymbol{d}, L, \gamma)$ be a hierarchical-st strictly-convex graph, in the following simply denoted by $G$, and let $C$ be the cycle delimiting the outer face $f$ of $G$. Construct a strictlyconvex level planar drawing $P_{C}$ of $C$ in which the clockwise order of the vertices along $P_{C}$ is the same as prescribed in $G$. Hong and Nagamochi [16] showed an algorithm to construct a (non-strictly) convex level planar drawing $\Gamma$ of $G$ in which $C$ is represented by $P_{C}$. We show how to modify $\Gamma$ into a strictly-convex level planar drawing of $G$.

We give some definitions. Let $s$ and $t$ be the vertices of $G$ such that $\gamma(s)<\gamma(u)<\gamma(t)$, for every vertex $u \neq s, t$ of $G$. Given a vertex $v$ of $G$, the leftmost (rightmost) top neighbor of $v$ is the neighbor $x$ of $v$ with $\gamma(x)>\gamma(v)$ such that for the neighbor $y$ of $v$ counter-clockwise (clockwise) following $x$ we have that either $\gamma(y)<\gamma(v)$, or $\gamma(y)>\gamma(v)$ and both $x$ and $y$ are incident to $f$ (this only happens when $v=s$ ). The leftmost and the rightmost bottom neighbor of $v$ are defined analogously. Also, the leftmost (rightmost) top path of $v$ is the monotone path $P$ from $v$ to $t$ obtained by initializing $P=(v)$ and by repeatedly adding the leftmost (resp. rightmost) top neighbor of the last vertex. The leftmost and rightmost bottom path of $v$ are defined analogously. Let $v$ be a vertex of $G$ that is flat in a face $g$ of $\Gamma$; $v$ is an internal vertex of $G$, since $P_{C}$ is strictly-convex. Let $x$ and $y$ be the neighbors of $v$ in $g$; then either $\gamma(x)<\gamma(v)<\gamma(y)$ or $\gamma(y)<\gamma(v)<\gamma(x)$. Assume the former. If $g$ lies to the left of path $(x, v, y)$ when traversing it from $x$ to $y$, then we say that $v$ is a left-flat vertex in $\Gamma$, otherwise $v$ is a right-flat vertex. By Theorem 2 and since $v$ is an internal vertex of $G$, we have $\operatorname{deg}(G, v) \geq 3$, hence $v$ cannot be both a left-flat and a right-flat vertex in $\Gamma$. A left-flat (right-flat) path in $\Gamma$ is a maximal path whose internal vertices are all left-flat (resp. right-flat) vertices and are all flat in the same face (see Fig. 3(a)). Let $Q=(x, \ldots, y)$ be a left-flat path in $\Gamma$; the elongation $E_{Q}$ of $Q$ is the monotone path between $s$ and $t$ obtained by concatenating the rightmost bottom path of $x, Q$, and the rightmost top path of $y$. Let


Figure 3 (a) A left-flat path $Q$ (red thick line), its elongation $E(Q)$ (red and black thick lines), graphs $G_{r}(Q)$ (gray) and $G_{l}(Q)$ (yellow). (b) Drawing $\Gamma$. (c) Drawing $\Gamma^{\prime}$.
$G_{l}(Q)\left(G_{r}(Q)\right)$ be the subgraph of $G$ whose outer face is delimited by the cycle composed of $E_{Q}$ and of the leftmost (resp. rightmost) top path of $s$. For a right-flat path $Q$ in $\Gamma$, the elongation $E_{Q}$ of $Q$, and graphs $G_{l}(Q)$ and $G_{r}(Q)$ are defined analogously.

In order to modify $\Gamma$ into a strictly-convex level planar drawing of $G$, we proceed by induction on the number $a(\Gamma)$ of flat angles in $\Gamma$. If $a(\Gamma)=0$, then $\Gamma$ is strictly-convex and there is nothing to be done. If $a(\Gamma) \geq 1$, then there exists a path $Q$ that is either a left-flat path or a right-flat path in $\Gamma$. Assume the former, the other case is symmetric. Also, assume w.l.o.g. up to a rotation of the axes, that the lines in $L$ are horizontal.

Ideally, we would like to move the internal vertices of $Q$ to the right, so that the polygon delimiting the face on which the internal vertices of $Q$ are flat becomes strictly-convex. There is one obstacle to such a modification, though: An internal vertex of $Q$ might be the first or the last vertex of a left-flat path $Q^{\prime}$; thus, moving that vertex to the right would cause the polygon delimiting the face on which the internal vertices of $Q^{\prime}$ are flat to become concave (in Fig. 3(a) moving $u_{i}$ to the right causes an angle incident to $w_{i}$ to become concave). We now argue that there is a left-flat path $Q^{*}$ such that $G_{r}\left(Q^{*}\right)$ contains no internal left-flat path; then we modify $\Gamma$ by moving the internal vertices of $Q^{*}$ to the right.

Let $Q^{*}=(x, \ldots, y)$ be a left-flat path such that the number of internal vertices of $G_{r}\left(Q^{*}\right)$ is minimum. Suppose, for a contradiction, that $G_{r}\left(Q^{*}\right)$ contains an internal left-flat path $Q^{\prime}$. Then $G_{r}\left(Q^{\prime}\right)$ has less internal vertices than $G_{r}\left(Q^{*}\right)$, since $G_{r}\left(Q^{\prime}\right)$ is a subgraph of $G_{r}\left(Q^{*}\right)$ and the internal vertices of $Q^{\prime}$ are internal vertices of $G_{r}\left(Q^{*}\right)$ and external vertices of $G_{r}\left(Q^{\prime}\right)$. This contradiction proves that $G_{r}\left(Q^{*}\right)$ does not contain any internal left-flat path.

We construct a convex drawing $\Gamma^{\prime}$ of $G$ with $a\left(\Gamma^{\prime}\right)<a(\Gamma)$. Initialize $\Gamma^{\prime}=\Gamma$ and remove the internal vertices of $Q^{*}$. Let $\epsilon>0$ be to be determined later. Consider segment $\overline{x y}$, its mid-point $z$, and a point $p$ in the half-plane to the right of $\overline{x y}$ such that segment $\overline{z p}$ is orthogonal to $\overline{x y}$ and has length $\epsilon$. Let $a$ be the arc of circumference between $x$ and $y$ passing through $p$. Place each internal vertex $v$ of $Q^{*}$ at the intersection point of $\gamma(v)$ with $a$, which exists since $Q^{*}$ is monotone. Denote by $\Gamma^{\prime}$ the resulting drawing. We have the following.

- Claim 2. The following statements hold, provided that $\epsilon$ is sufficiently small: (i) $\Gamma^{\prime}$ is convex; (ii) every vertex that is flat in an incident face in $\Gamma^{\prime}$ is flat in the same face in $\Gamma$; and (iii) every internal vertex of $Q^{*}$ is convex in every incident face in $\Gamma^{\prime}$.

Proof sketch. Moving the internal vertices of $Q^{*}$ from $\overline{x y}$ to $a$ results in these vertices being convex in the unique face $g$ of $G_{l}\left(Q^{*}\right)$ they are all incident to in $\Gamma^{\prime}$. Further, the difference between the size of any angle in $\Gamma^{\prime}$ and the size of the corresponding angle in $\Gamma$ tends to 0 as $\epsilon \rightarrow 0$; in particular, angles that are flat in $\Gamma$ either have the same or smaller size in $\Gamma^{\prime}$ (see $w$ in Figs. 3(b)-(c)), given that $G_{r}\left(Q^{*}\right)$ does not contain any internal left-flat path.

Claim 2 implies that $\Gamma^{\prime}$ is convex and that $a\left(\Gamma^{\prime}\right)<a(\Gamma)$. The theorem follows.


Figure 4 (a) Drawings $\Gamma_{s}$ (black circles and black lines) and $\Gamma_{s}^{\prime}$ (white circles and blue lines), together with points $q_{1}, \ldots, q_{n}$. (b) Morph $\left\langle\Gamma_{s}^{\prime}, \ldots, \Gamma\right\rangle$ after two steps. (c) Drawing $\Gamma$. (d) Drawings $\Gamma_{t}$ (black circles and black lines) and $\Gamma_{t}^{\prime}$ (white circles and blue lines), together with points $q_{1}, \ldots, q_{n}$.

## 5 A Morphing Algorithm

In this section we give algorithms to morph convex drawings of plane graphs. We start with a lemma about unidirectional linear morphs. Two level planar drawings $\Gamma_{1}$ and $\Gamma_{2}$ of a hierarchical plane graph $(G, \boldsymbol{d}, L, \gamma)$ are left-to-right equivalent if, for any line $l_{i} \in L$, for any vertex or edge $x$ of $G$, and for any vertex or edge $y$ of $G$, we have that $x$ precedes (follows) $y$ on $l_{i}$ in $\Gamma_{1}$ if and only if $x$ precedes (resp. follows) $y$ on $l_{i}$ in $\Gamma_{2}$. We have the following.

- Lemma 7. The linear morph $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ between two left-to-right equivalent strictly-convex level planar drawings $\Gamma_{1}$ and $\Gamma_{2}$ of a hierarchical-st strictly-convex graph ( $G, \boldsymbol{d}, L, \gamma$ ) is strictly-convex and unidirectional.

Proof sketch. Morph $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ is planar and unidirectional [3]. Also, it is strictly-convex since an angle $\widehat{v u z}$ that is convex in $\Gamma_{1}$ and in $\Gamma_{2}$ stays convex during $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$; this descends from the planarity of $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ if $\gamma(u)<\gamma(v), \gamma(z)$ or $\gamma(v), \gamma(z)<\gamma(u)$ and from the fact that $u, v$, and $z$ are never aligned during $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ if $\gamma(z)<\gamma(u)<\gamma(v)$ (see [7]).

We now describe an algorithm to construct a strictly-convex morph between any two strictly-convex drawings $\Gamma_{s}$ and $\Gamma_{t}$ of a plane graph $G$ with $n$ vertices and $m$ internal faces. The algorithm works by induction on $m$ and consists of at most $2 n+2 m$ morphing steps.

In the base case we have $m=1$, hence $G$ is a cycle. We have the following.

- Claim 3. There exists a strictly-convex unidirectional morph with at most $2 n+2$ steps between any two strictly-convex drawings $\Gamma_{s}$ and $\Gamma_{t}$ of cycle $G$.

Proof Sketch. Let $v_{1}, \ldots, v_{n}$ be the vertices of $G$ as they appear clockwise around $G$. Let $\ell$ be a straight line not orthogonal to any line through two vertices of $G$ in $\Gamma_{s}$ and in $\Gamma_{t}$. Draw a circumference $\mathcal{C}$ enclosing both $\Gamma_{s}$ and $\Gamma_{t}$. Morph $\Gamma_{s}\left(\Gamma_{t}\right)$ into a drawing $\Gamma_{s}^{\prime}\left(\Gamma_{t}^{\prime}\right)$ such that all the vertices of $G$ are on $\mathcal{C}$ (see Fig. 4(a) and Fig. 4(d)) with a single strictly-convex morphing step which is unidirectional in the direction orthogonal to $\ell$ (each vertex moves in the direction that does not make it collide with the initial drawing of $G$ ).

Consider $n$ points $q_{1}, \ldots, q_{n}$ in this clockwise order on $\mathcal{C}$ both in $\Gamma_{s}^{\prime}$ and in $\Gamma_{t}^{\prime}$ such that the arc of $\mathcal{C}$ between $q_{1}$ and $q_{n}$ containing $q_{2}$ does not contain any vertex of $G$. Morph $\Gamma_{s}^{\prime}\left(\Gamma_{t}^{\prime}\right)$ into a drawing $\Gamma$ of $G$ in which $v_{i}$ is placed at $q_{i}$, for $1 \leq i \leq n$, as follows (see Figs. 4(a)-(c)). Let $v_{k}$ be the first vertex of $G$ encountered when clockwise traversing $\mathcal{C}$ from $q_{n}$. For $j=k-1, \ldots, 1, k, \ldots, n$, move $v_{j}$ to $p_{j}$. These morphs consist of $n$ unidirectional strictly-convex morphing steps each. Hence, $\left\langle\Gamma_{s}, \Gamma_{s}^{\prime}, \ldots, \Gamma, \ldots, \Gamma_{t}^{\prime}, \Gamma_{t}\right\rangle$ is a unidirectional strictly-convex morph between $\Gamma_{s}$ and $\Gamma_{t}$ with $2 n+2$ morphing steps.

In the inductive case we have $m>1$. Then we apply Lemma 5 to $G$ in order to obtain a graph $G^{\prime}$ with $m^{\prime}<m$ internal faces. We proceed as follows.

Assume first that, according to Lemma 5, a degree-3 internal vertex $u$ of $G$ as well as the edges and the internal vertices of paths $P_{1}, P_{2}$, and $P_{3}$ can be removed from $G$ resulting in a convex graph $G^{\prime}$, where: (i) $P_{1}, P_{2}$, and $P_{3}$ respectively connect $u$ with vertices $u_{1}, u_{2}$, and $u_{3}$ of the cycle $C$ delimiting the outer face $f$ of $G$; (ii) $P_{1}, P_{2}$, and $P_{3}$ are vertex-disjoint except at $u$; and (iii) the internal vertices of $P_{1}, P_{2}$, and $P_{3}$ are degree- 2 internal vertices of $G$. Graph $G$ has no degree-2 internal vertices, since it is strictly-convex (see Theorem 2), hence $P_{1}, P_{2}$, and $P_{3}$ are edges $\left(u, u_{1}\right),\left(u, u_{2}\right)$, and $\left(u, u_{3}\right)$, respectively.

Vertex $u$ lies in the interior of triangle $\Delta\left(u_{1}, u_{2}, u_{3}\right)$ both in $\Gamma_{s}$ and in $\Gamma_{t}, \operatorname{since} \operatorname{deg}(G, u)=$ 3 and the angles incident to $u$ are smaller than $\pi$ both in $\Gamma_{s}$ and in $\Gamma_{t}$. Hence, the position of $u$ is a convex combination of the positions of $u_{1}, u_{2}$, and $u_{3}$ both in $\Gamma_{s}$ and in $\Gamma_{t}$ (the coefficients of such convex combinations might be different in $\Gamma_{s}$ and in $\Gamma_{t}$ ). Further, no vertex other than $u$ and no edge other than those incident to $u$ lie in the interior of triangle $\Delta\left(u_{1}, u_{2}, u_{3}\right)$ in $\Gamma_{s}$ and $\Gamma_{t}$, since these drawings are strictly-convex. With a single unidirectional linear morph, move $u$ in $\Gamma_{s}$ to the point that is a convex combination of the positions of $u_{1}, u_{2}$, and $u_{3}$ with the same coefficients as in $\Gamma_{t}$. This morph is strictly-convex since $u$ stays inside $\Delta\left(u_{1}, u_{2}, u_{3}\right)$ at any time instant. Let $\Gamma_{s}^{\prime}$ be the resulting drawing of $G$.

Let $Q_{1}, Q_{2}$, and $Q_{3}$ be the polygons delimiting the faces of $G$ incident to $u$ in $\Gamma_{s}$. Let $\Lambda_{s}^{\prime}$ be the drawing of $G^{\prime}$ obtained from $\Gamma_{s}^{\prime}$ by removing $u$ and its incident edges. We claim that $\Lambda_{s}^{\prime}$ is strictly-convex. Indeed, every internal face of $G^{\prime}$ different from the face $f_{u}$ that used to contain $u$ is also a face in $\Gamma_{s}^{\prime}$, hence it is delimited by a strictly-convex polygon. Further, every internal angle of the polygon delimiting $f_{u}$ is either an internal angle of $Q_{1}, Q_{2}$, or $Q_{3}$, hence it is smaller than $\pi$, since $\Gamma_{s}^{\prime}$ is strictly-convex, or is incident to $u_{1}, u_{2}$, or $u_{3}$; however, these vertices are concave in $f$, hence they are convex in $f_{u} \neq f$. Analogously, the drawing $\Lambda_{t}^{\prime}$ of $G^{\prime}$ obtained from $\Gamma_{t}$ by removing $u$ and its incident edges is strictly-convex.

Inductively construct a unidirectional convex morph $\left\langle\Lambda_{s}^{\prime}=\Lambda_{0}, \ldots, \Lambda_{\ell}=\Lambda_{t}^{\prime}\right\rangle$ with $\ell \leq 2(n-1)+2(m-2)$ morphing steps. For each $1 \leq j \leq \ell-1$, draw $u$ in $\Lambda_{j}$ at a point that is the convex combination of the positions of $u_{1}, u_{2}$, and $u_{3}$ with the same coefficients as in $\Gamma_{s}^{\prime}$ and in $\Gamma_{t}$; denote by $\Gamma_{j}$ the resulting drawing of $G$. Morph $\left\langle\Gamma_{s}^{\prime}=\Gamma_{0}, \ldots, \Gamma_{\ell}=\Gamma_{t}\right\rangle$ is strictly-convex and unidirectional. Namely, in every morphing step $\left\langle\Gamma_{j}, \Gamma_{j+1}\right\rangle$, vertex $u$ moves between two points that are convex combinations of the positions of $u_{1}, u_{2}$, and $u_{3}$ with the same coefficients, hence it moves parallel to each of $u_{1}, u_{2}$, and $u_{3}$ (from which $\left\langle\Gamma_{0}, \ldots, \Gamma_{\ell}\right\rangle$ is unidirectional) and it stays inside $\Delta\left(u_{1}, u_{2}, u_{3}\right)$ at any time instant of $\left\langle\Gamma_{j}, \Gamma_{j+1}\right\rangle$ (from which $\left\langle\Gamma_{0}, \ldots, \Gamma_{\ell}\right\rangle$ is strictly-convex). Thus, $\left\langle\Gamma_{s}, \Gamma_{s}^{\prime}=\Gamma_{0}, \ldots, \Gamma_{\ell}=\Gamma_{t}\right\rangle$ is a unidirectional strictly-convex morph between $\Gamma_{s}$ and $\Gamma_{t}$ with $\ell+1 \leq 2 n+2 m-5$ morphing steps.

Assume next that, according to Lemma 5, the edges and the internal vertices of a path $P$, whose internal vertices are degree-2 internal vertices of $G$, can be deleted from $G$ so that the resulting graph $G^{\prime}$ is convex. Graph $G$ has no degree-2 internal vertices, since it is strictly-convex (see Theorem 2), hence $P$ is an edge ( $u, v$ ). Removing (u,v) from $\Gamma_{s}$ (from $\Gamma_{t}$ ) results in a drawing $\Lambda_{s}$ (resp. $\Lambda_{t}$ ) of $G^{\prime}$ which is not, in general, convex, since vertices $u$ and $v$ might be concave in the face $f_{u v}$ of $G^{\prime}$ that used to contain $(u, v)$, as in Fig. 5. By Lemma 4, there exists an oriented straight line $\boldsymbol{d}_{\boldsymbol{s}}$ such that the polygon $Q_{u v}$ representing the cycle $C_{u v}$ delimiting $f_{u v}$ is $\boldsymbol{d}_{\boldsymbol{s}}$-monotone. By slightly perturbing the slope of $\boldsymbol{d}_{\boldsymbol{s}}$, we can assume that it is not orthogonal to any line through two vertices of $G^{\prime}$. Let $L_{s}^{\prime}$ be the set of parallel and distinct lines through vertices of $G^{\prime}$ and orthogonal to $\boldsymbol{d}_{\boldsymbol{s}}$. Let $\gamma_{s}^{\prime}$ be the function that maps each vertex of $G^{\prime}$ to the line in $L_{s}^{\prime}$ through it. We have the following.

- Lemma 8. $\left(G^{\prime}, \boldsymbol{d}_{s}, L_{s}^{\prime}, \gamma_{s}^{\prime}\right)$ is a hierarchical-st convex graph.


Figure 5 Drawings (a) $\Gamma_{s}$, (b) $\Lambda_{s}$, (c) $\Lambda_{s}^{\prime \prime}$, and (d) $\Gamma_{s}^{\prime}$.

Analogously, there exists an oriented straight line $\boldsymbol{d}_{\boldsymbol{t}}$ that leads to define a hierarchical-st convex graph $\left(G^{\prime}, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime}, \gamma_{t}^{\prime}\right)$ for which $\Lambda_{t}$ is a straight-line level planar drawing.

We now distinguish three cases, based on whether $\operatorname{deg}\left(G^{\prime}, u\right), \operatorname{deg}\left(G^{\prime}, v\right)>2$ (Case 1), $\operatorname{deg}\left(G^{\prime}, u\right)=2$ and $\operatorname{deg}\left(G^{\prime}, v\right)>2$ (Case 2), or $\operatorname{deg}\left(G^{\prime}, u\right)=\operatorname{deg}\left(G^{\prime}, v\right)=2$ (Case 3). The case in which $\operatorname{deg}\left(G^{\prime}, u\right)>2$ and $\operatorname{deg}\left(G^{\prime}, v\right)=2$ is symmetric to Case 2.

In Case 1 graph $G^{\prime}$ is strictly-convex, since it is convex and all its internal vertices have degree greater than two. By Theorem $6,\left(G^{\prime}, \boldsymbol{d}_{\boldsymbol{s}}, L_{s}^{\prime}, \gamma_{s}^{\prime}\right)$ and $\left(G^{\prime}, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime}, \gamma_{t}^{\prime}\right)$ admit strictlyconvex level planar drawings $\Lambda_{s}^{\prime}$ and $\Lambda_{t}^{\prime}$, respectively. Let $\Gamma_{s}^{\prime}\left(\Gamma_{t}^{\prime}\right)$ be the strictly-convex level planar drawing of $\left(G, \boldsymbol{d}_{\boldsymbol{s}}, L_{s}^{\prime}, \gamma_{s}^{\prime}\right)$ (resp. of $\left(G, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime}, \gamma_{t}^{\prime}\right)$ ) obtained by inserting edge $(u, v)$ as a straight-line segment in $\Lambda_{s}^{\prime}$ (resp. $\Lambda_{t}^{\prime}$ ). Drawings $\Gamma_{s}$ and $\Gamma_{s}^{\prime}\left(\Gamma_{t}\right.$ and $\left.\Gamma_{t}^{\prime}\right)$ are left-to-right equivalent. This is argued as follows. First, since $G$ is a plane graph, its outer face is delimited by the same cycle $C$ in both $\Gamma_{s}$ and $\Gamma_{s}^{\prime}$; further, the clockwise order of the vertices along $C$ is the same in $\Gamma_{s}$ and in $\Gamma_{s}^{\prime}$ (recall that Theorem 6 allows us to arbitrarily prescribe the strictly-convex polygon representing $C$ ). Consider any two vertices or edges $x$ and $y$ both intersecting a line $\ell$ in $L_{s}^{\prime}$; assume this line to be oriented in any way. Suppose, for a contradiction, that $x$ precedes $y$ on $\ell$ in $\Gamma_{s}$ and follows $y$ on $\ell$ in $\Gamma_{s}^{\prime}$. Since $\Gamma_{s}$ and $\Gamma_{s}^{\prime}$ are strictly-convex, there exists a $\boldsymbol{d}_{\boldsymbol{s}}$-monotone path $P_{x}\left(P_{y}\right)$ containing $x$ (resp. $y$ ) and connecting two vertices of $C$. Then $P_{x}$ and $P_{y}$ properly cross, contradicting the planarity of $\Gamma_{s}$ or of $\Gamma_{s}^{\prime}$, or they share a vertex which has a different clockwise order of its incident edges in the two drawings, contradicting the fact that $\Gamma_{s}$ and $\Gamma_{s}^{\prime}$ are drawings of the same plane graph. By Lemma 7, linear morphs $\left\langle\Gamma_{s}, \Gamma_{s}^{\prime}\right\rangle$ and $\left\langle\Gamma_{t}, \Gamma_{t}^{\prime}\right\rangle$ are strictly-convex and unidirectional.

Inductively construct a unidirectional strictly-convex morph $\left\langle\Lambda_{s}^{\prime}=\Lambda_{0}, \Lambda_{1}, \ldots, \Lambda_{\ell}=\Lambda_{t}^{\prime}\right\rangle$ with $\ell \leq 2 n+2(m-1)$ morphing steps between $\Lambda_{s}^{\prime}$ and $\Lambda_{t}^{\prime}$. For each $0 \leq j \leq \ell$, draw edge $(u, v)$ in $\Lambda_{j}$ as a straight-line segment $\overline{u v}$; let $\Gamma_{j}$ be the resulting drawing of $G$. We have that morph $\left\langle\Gamma_{s}^{\prime}=\Gamma_{0}, \Gamma_{1}, \ldots, \Gamma_{\ell}=\Gamma_{t}^{\prime}\right\rangle$ is strictly-convex and unidirectional given that $\left\langle\Lambda_{0}, \Lambda_{1}, \ldots, \Lambda_{\ell}\right\rangle$ is strictly-convex and unidirectional and given that, at any time instant of $\left\langle\Lambda_{0}, \Lambda_{1}, \ldots, \Lambda_{\ell}\right\rangle$, segment $\overline{u v}$ splits the strictly-convex polygon delimiting $f_{u v}$ into two strictlyconvex polygons. Thus, $\left\langle\Gamma_{s}, \Gamma_{s}^{\prime}=\Gamma_{0}, \Gamma_{1}, \ldots, \Gamma_{\ell}=\Gamma_{t}^{\prime}, \Gamma_{t}\right\rangle$ is a unidirectional strictly-convex morph between $\Gamma_{s}$ and $\Gamma_{t}$ with $\ell+2 \leq 2 n+2 m$ morphing steps.

In Case 2 let $G^{\prime \prime}$ be the graph obtained from $G^{\prime}$ by replacing path $(x, u, y)$ with edge $(x, y)$, where $x$ and $y$ are the only neighbors of $u$ in $G^{\prime}$. Graph $G^{\prime \prime}$ is strictly-convex, since $G^{\prime}$ is convex and is a subdivision of $G^{\prime \prime}$, and since all the internal vertices of $G^{\prime \prime}$ have degree greater than two. Moreover, since $\left(G^{\prime}, \boldsymbol{d}_{\boldsymbol{s}}, L_{s}^{\prime}, \gamma_{s}^{\prime}\right)$ and $\left(G^{\prime}, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime}, \gamma_{t}^{\prime}\right)$ are hierarchical-st convex graphs, it follows that $\left(G^{\prime \prime}, \boldsymbol{d}_{s}, L_{s}^{\prime \prime}, \gamma_{s}^{\prime \prime}\right)$ and $\left(G^{\prime \prime}, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime \prime}, \gamma_{t}^{\prime \prime}\right)$ are hierarchical-st strictly-convex graphs, where $L_{s}^{\prime \prime}=L_{s}^{\prime} \backslash\left\{\gamma_{s}^{\prime}(u)\right\}, L_{t}^{\prime \prime}=L_{t}^{\prime} \backslash\left\{\gamma_{t}^{\prime}(u)\right\}, \gamma_{s}^{\prime \prime}(z)=\gamma_{s}^{\prime}(z)$ for each vertex $z$ in $G^{\prime \prime}$, and $\gamma_{t}^{\prime \prime}(z)=\gamma_{t}^{\prime}(z)$ for each vertex $z$ in $G^{\prime \prime}$. By Theorem $6,\left(G^{\prime \prime}, \boldsymbol{d}_{\boldsymbol{s}}, L_{s}^{\prime \prime}, \gamma_{s}^{\prime \prime}\right)$ and $\left(G^{\prime \prime}, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime \prime}, \gamma_{t}^{\prime \prime}\right)$ admit strictly-convex level planar drawings $\Lambda_{s}^{\prime \prime}$ and $\Lambda_{t}^{\prime \prime}$, respectively.

We modify $\Lambda_{s}^{\prime \prime}$ into a drawing $\Gamma_{s}^{\prime}$ of $\left(G, \boldsymbol{d}_{s}, L_{s}^{\prime}, \gamma_{s}^{\prime}\right)$, as in Fig. 5. Assume w.l.o.g. that
$\gamma_{s}^{\prime}(x)<\gamma_{s}^{\prime}(u)<\gamma_{s}^{\prime}(y)$. Let $w$ be the intersection point of $\gamma_{s}^{\prime}(u)$ and $\overline{x y}$ in $\Lambda_{s}^{\prime \prime}$ (where line $\gamma_{s}^{\prime}(u)$ is the same as in $\left.\Lambda_{s}\right)$. Let $C_{u v}^{\prime \prime}$ be the facial cycle of $G^{\prime \prime}$ such that the facial cycle $C_{u v}$ of $G^{\prime}$ is a subdivision of $C_{u v}^{\prime \prime}$. Insert $u$ in the interior of $C_{u v}^{\prime \prime}$, on $\gamma_{s}^{\prime}(u)$, at distance $\epsilon>0$ from $w$. Remove edge $(x, y)$ from $\Lambda_{s}^{\prime \prime}$ and insert edges $(u, v),(u, x)$, and $(u, y)$ as straight-line segments. Denote by $\Gamma_{s}^{\prime}$ the resulting drawing. We have the following.

- Claim 4. $\Gamma_{s}^{\prime}$ is a strictly-convex level planar drawing of $\left(G, \boldsymbol{d}_{s}, L_{s}^{\prime}, \gamma_{s}^{\prime}\right)$, provided that $\epsilon>0$ is sufficiently small.

A strictly-convex level planar drawing $\Gamma_{t}^{\prime}$ of $\left(G, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime}, \gamma_{t}^{\prime}\right)$ can be constructed analogously from $\Lambda_{t}^{\prime \prime}$. Drawings $\Gamma_{s}$ and $\Gamma_{s}^{\prime}\left(\Gamma_{t}\right.$ and $\left.\Gamma_{t}^{\prime}\right)$ are left-to-right equivalent, which can be proved as in Case 1. By Lemma 7, morphs $\left\langle\Gamma_{s}, \Gamma_{s}^{\prime}\right\rangle$ and $\left\langle\Gamma_{t}, \Gamma_{t}^{\prime}\right\rangle$ are strictly-convex and unidirectional.

Inductively construct a unidirectional strictly-convex morph $\left\langle\Lambda_{s}^{\prime \prime}=\Lambda_{0}, \Lambda_{1}, \ldots, \Lambda_{\ell}=\Lambda_{t}^{\prime \prime}\right\rangle$ with $\ell \leq 2(n-1)+2(m-1)$ morphing steps between $\Lambda_{s}^{\prime \prime}$ and $\Lambda_{t}^{\prime \prime}$. Let $0<\xi<1$ be sufficiently small so that the following holds true: For every $0 \leq j \leq \ell$, insert $u$ in $\Lambda_{j}$ at a point which is a convex combination of the positions of $x, y$, and $v$ with coefficients $\left(\frac{1-\xi}{2}, \frac{1-\xi}{2}, \xi\right)$, remove edge $(x, y)$, and insert edges $(u, x),(u, y)$, and $(u, v)$ as straight-line segments; then the resulting drawing $\Gamma_{j}$ of $G$ is strictly-convex. Such a $\xi>0$ exists. Namely, placing $v$ as a convex combination of the positions of $x, y$, and $v$ results in angles incident to $u$ and $v$ that are all convex. Moreover, as $\xi \rightarrow 0$, the point at which $u$ is placed approaches segment $\overline{x y}$, hence the size of any angle incident to $x$ or $y$ approaches the size of an angle incident to $x$ or $y$ in $\Lambda_{j}$, and the latter is strictly less than $\pi$ radians.

With a single unidirectional strictly-convex linear morph, move $u$ in $\Gamma_{s}^{\prime}$ to the point that is a convex combination of the positions of $x, y$, and $v$ with coefficients $\left(\frac{1-\xi}{2}, \frac{1-\xi}{2}, \xi\right)$; denote by $\Gamma_{s}^{\prime \prime}$ the drawing of $G$ obtained from this morph. Analogously, let $\left\langle\Gamma_{t}^{\prime}, \Gamma_{t}^{\prime \prime}\right\rangle$ be a unidirectional strictly-convex linear morph, where the point at which $u$ is placed in $\Gamma_{t}^{\prime \prime}$ is a convex combination of the positions of $x, y$, and $v$ with coefficients $\left(\frac{1-\xi}{2}, \frac{1-\xi}{2}, \xi\right)$.

For each $0 \leq j \leq \ell-1, \Gamma_{j}$ and $\Gamma_{j+1}$ are left-to-right equivalent strictly-convex level planar drawings of the hierarchical-st strictly-convex graph $\left(G, \boldsymbol{d}_{\boldsymbol{j}}, L_{j}, \gamma_{j}\right)$, where $\boldsymbol{d}_{\boldsymbol{j}}$ is an oriented straight line orthogonal to the direction of morph $\left\langle\Lambda_{j}, \Lambda_{j+1}\right\rangle, L_{j}$ is the set of lines through vertices of $G$ orthogonal to $\boldsymbol{d}_{\boldsymbol{j}}$, and $\gamma_{j}$ maps each vertex of $G$ to the line in $L_{j}$ through it. In particular, $\Gamma_{j}$ and $\Gamma_{j+1}$ are strictly-convex drawings of $G$ since $\Lambda_{j}$ and $\Lambda_{j+1}$ are strictly-convex drawings of $G^{\prime \prime}$ and by the choice of $\xi$; further, every face of $G$ is an st-face in $\Gamma_{j}$ and $\Gamma_{j+1}$ by Lemmata 3 and 4; moreover, $u$ moves parallel to the other vertices since $\left\langle\Lambda_{j}, \Lambda_{j+1}\right\rangle$ is unidirectional and since the points at which $u$ is placed in $\Gamma_{j}$ and $\Gamma_{j+1}$ are convex combinations of the positions of $x, y$, and $v$ with the same coefficients. By Lemma 7, $\left\langle\Gamma_{j}, \Gamma_{j+1}\right\rangle$ is strictly-convex and unidirectional. Hence, $\left\langle\Gamma_{s}, \Gamma_{s}^{\prime}, \Gamma_{s}^{\prime \prime},=\Gamma_{0}, \Gamma_{1}, \ldots, \Gamma_{\ell}=\Gamma_{t}^{\prime \prime}, \Gamma_{t}^{\prime}, \Gamma_{t}\right\rangle$ is a unidirectional strictly-convex morph between $\Gamma_{s}$ and $\Gamma_{t}$ with $\ell+4 \leq 2 n+2 m$ morphing steps.

Case 3 is very similar to Case 2, hence we only sketch the algorithm here. Let $G^{\prime \prime}$ be the graph obtained from $G^{\prime}$ by replacing paths $\left(x_{u}, u, y_{u}\right)$ and $\left(x_{v}, v, y_{v}\right)$ with edges $\left(x_{u}, y_{u}\right)$ and $\left(x_{v}, y_{v}\right)$, respectively, where $x_{u}$ and $y_{u}\left(x_{v}\right.$ and $\left.y_{v}\right)$ are the only neighbors of $u$ (resp. $v$ ) in $G^{\prime} ;\left(G^{\prime \prime}, \boldsymbol{d}_{s}, L_{s}^{\prime \prime}, \gamma_{s}^{\prime \prime}\right)$ and $\left(G^{\prime \prime}, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime \prime}, \gamma_{t}^{\prime \prime}\right)$ are hierarchical-st strictly-convex graphs, where $L_{s}^{\prime \prime}=L_{s}^{\prime} \backslash\left\{\gamma_{s}^{\prime}(u), \gamma_{s}^{\prime}(v)\right\}, L_{t}^{\prime \prime}=L_{t}^{\prime} \backslash\left\{\gamma_{t}^{\prime}(u), \gamma_{t}^{\prime}(v)\right\}, \gamma_{s}^{\prime \prime}(z)=\gamma_{s}^{\prime}(z)$ for each vertex $z$ in $G^{\prime \prime}$, and $\gamma_{t}^{\prime \prime}(z)=\gamma_{t}^{\prime}(z)$ for each vertex $z$ in $G^{\prime \prime}$. By Theorem $6,\left(G^{\prime \prime}, \boldsymbol{d}_{\boldsymbol{s}}, L_{s}^{\prime \prime}, \gamma_{s}^{\prime \prime}\right)$ and $\left(G^{\prime \prime}, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime \prime}, \gamma_{t}^{\prime \prime}\right)$ admit strictly-convex level planar drawings $\Lambda_{s}^{\prime \prime}$ and $\Lambda_{t}^{\prime \prime}$, respectively. We modify $\Lambda_{s}^{\prime \prime}$ into a strictly-convex level planar drawing $\Gamma_{s}^{\prime}$ of $\left(G, \boldsymbol{d}_{\boldsymbol{s}}, L_{s}^{\prime}, \gamma_{s}^{\prime}\right)$ by inserting $u(v)$ on $\gamma_{s}^{\prime}(u)$ (resp. $\left.\gamma_{s}^{\prime}(v)\right)$ at distance $\epsilon>0$ from the intersection point of $\gamma_{s}^{\prime}(u)$ with segment $\overline{x_{u} y_{u}}$ (of $\gamma_{s}^{\prime}(v)$
with segment $\left.\overline{x_{v} y_{v}}\right)$ in the interior of the facial cycle of $G^{\prime \prime}$ such that the facial cycle $C_{u v}$ of $G^{\prime}$ is a subdivision of $C_{u v}^{\prime \prime}$. Analogously, we modify $\Lambda_{t}^{\prime \prime}$ into a strictly-convex level planar drawing $\Gamma_{t}^{\prime}$ of $\left(G, \boldsymbol{d}_{\boldsymbol{t}}, L_{t}^{\prime}, \gamma_{t}^{\prime}\right)$. Drawings $\Gamma_{s}$ and $\Gamma_{s}^{\prime}\left(\Gamma_{t}\right.$ and $\left.\Gamma_{t}^{\prime}\right)$ are left-to-right equivalent.

Inductively construct a unidirectional strictly-convex morph $\left\langle\Lambda_{s}^{\prime \prime}=\Lambda_{0}, \ldots, \Lambda_{\ell}=\Lambda_{t}^{\prime \prime}\right\rangle$ with $\ell \leq 2(n-2)+2(m-1)$ morphing steps. Let $\xi>0$ be sufficiently small so that for every $0 \leq j \leq \ell$, inserting $u(v)$ in $\Lambda_{j}$ at a convex combination of the positions of $x_{u}, y_{u}, x_{v}$, and $y_{v}$ with coefficients $\left(\frac{1-\xi}{2}, \frac{1-\xi}{2}, \frac{\xi}{2}, \frac{\xi}{2}\right.$ ) (resp. $\left(\frac{\xi}{2}, \frac{\xi}{2}, \frac{1-\xi}{2}, \frac{1-\xi}{2}\right)$ ), removing edges $\left(x_{u}, y_{u}\right)$ and $\left(x_{v}, y_{v}\right)$, and inserting edges $\left(x_{u}, u\right),\left(y_{u}, u\right),\left(x_{v}, v\right),\left(y_{v}, v\right)$, and $(u, v)$ results in a strictly-convex drawing $\Gamma_{j}$ of $G$. With a unidirectional strictly-convex linear morph $\left\langle\Gamma_{s}^{\prime}, \Gamma_{s}^{\prime \prime}\right\rangle$, move $u$ in $\Gamma_{s}^{\prime}$ to the point that is a convex combination of the positions of $x_{u}, y_{u}$, $x_{v}$, and $y_{v}$ with coefficients $\left(\frac{1-\xi}{2}, \frac{1-\xi}{2}, \frac{\xi}{2}, \frac{\xi}{2}\right)$. With a unidirectional strictly-convex linear morph $\left\langle\Gamma_{s}^{\prime \prime}, \Gamma_{s}^{\prime \prime \prime}\right\rangle$, move $v$ in $\Gamma_{s}^{\prime \prime}$ to the point that is a convex combination of the positions of $x_{u}, y_{u}, x_{v}$, and $y_{v}$ with coefficients $\left(\frac{\xi}{2}, \frac{\xi}{2}, \frac{1-\xi}{2}, \frac{1-\xi}{2}\right)$. Define morph $\left\langle\Gamma_{t}^{\prime}, \Gamma_{t}^{\prime \prime}, \Gamma_{t}^{\prime \prime \prime}\right\rangle$ analogously. For each $0 \leq j \leq \ell-1, \Gamma_{j}$ and $\Gamma_{j+1}$ are left-to-right equivalent strictly-convex level planar drawings of the hierarchical-st strictly-convex graph $\left(G, \boldsymbol{d}_{\boldsymbol{j}}, L_{j}, \gamma_{j}\right)$, where $\boldsymbol{d}_{\boldsymbol{j}}$ is an oriented line orthogonal to the direction of morph $\left\langle\Lambda_{j}, \Lambda_{j+1}\right\rangle, L_{j}$ is the set of lines through vertices of $G$ and orthogonal to $\boldsymbol{d}_{\boldsymbol{j}}$, and $\gamma_{j}$ maps each vertex of $G$ to the line in $L_{j}$ through it. By Lemma $7,\left\langle\Gamma_{s}, \Gamma_{s}^{\prime}, \Gamma_{s}^{\prime \prime}, \Gamma_{s}^{\prime \prime \prime}=\Gamma_{0}, \ldots, \Gamma_{\ell}=\Gamma_{t}^{\prime \prime \prime}, \Gamma_{t}^{\prime \prime}, \Gamma_{t}^{\prime}, \Gamma_{t}\right\rangle$ is a unidirectional strictly-convex morph between $\Gamma_{s}$ and $\Gamma_{t}$ with $\ell+6 \leq 2 n+2 m$ morphing steps. We get the following.

- Theorem 9. There exists an algorithm to construct a strictly-convex unidirectional morph with $O(n)$ morphing steps between any two strictly-convex drawings of the same $n$-vertex plane graph.

A simple enhancement of the above described algorithm allows us to extend our results to (non-strictly) convex drawings of convex graphs. We have the following.

- Theorem 10. There exists an algorithm to construct a convex unidirectional morph with $O(n)$ morphing steps between any two convex drawings of the same n-vertex plane graph.

Proof sketch. First, with $O(n)$ unidirectional convex morphing steps we morph $\Gamma_{s}\left(\Gamma_{t}\right)$ into a convex drawing $\Gamma_{s}^{\prime}$ (resp. $\Gamma_{t}^{\prime}$ ) such that the polygon delimiting the outer face of $G$ is strictly-convex. This is done by moving, during each morphing step, all the internal vertices of a maximal path incident to the outer face whose internal vertices have degree two.

Second, we consider each maximal path $P=\left(u_{1}, \ldots, u_{k}\right)$ such that $u_{2}, \ldots, u_{k-1}$ are degree- 2 internal vertices of $G$; with a single linear morph in the direction of $\overline{u_{1} u_{k}}$, we move each of $u_{2}, \ldots, u_{k-1}$ in $\Gamma_{s}^{\prime}$ to the point which is a convex combination of the positions of $u_{1}$ and $u_{k}$ with the same coefficients as in $\Gamma_{t}^{\prime}$. Over all such paths $P$ this amounts to $O(n)$ unidirectional convex morphing steps; denote by $\Gamma_{s}^{\prime \prime}$ the resulting drawing of $G$.

Third, we replace each maximal path $P=\left(u_{1}, \ldots, u_{k}\right)$ such that $u_{2}, \ldots, u_{k-1}$ are degree- 2 internal vertices of $G$ with an edge $\left(u_{1}, u_{k}\right)$ in $G, \Gamma_{s}^{\prime \prime}$, and $\Gamma_{t}^{\prime}$; we obtain a strictly-convex graph $G^{\prime}$, and two strictly-convex drawings $\Lambda_{s}^{\prime \prime}$ and $\Lambda_{t}^{\prime}$ of $G^{\prime}$. We compute a strictly-convex unidirectional morph $\left\langle\Lambda_{s}^{\prime \prime}=\Lambda_{0}, \ldots, \Lambda_{\ell}=\Lambda_{t}^{\prime}\right\rangle$ with $\ell \in O(n)$ morphing steps as in Theorem 9. For each $0 \leq j \leq \ell$, we reinsert the internal vertices of each path $P=\left(u_{1}, \ldots, u_{k}\right)$ in $\Lambda_{j}$ at the points that are the convex combinations of the positions of $u_{1}$ and $u_{k}$ in $\Lambda_{j}$ with the same coefficients as in $\Gamma_{s}^{\prime \prime}$ and in $\Gamma_{t}^{\prime}$. Each morphing step $\left\langle\Gamma_{j}, \Gamma_{j+1}\right\rangle$ is convex and unidirectional, since $u_{i}$ moves between two points that are convex combinations of the positions of $u_{1}$ and $u_{k}$ with the same coefficients. Hence, $\left\langle\Gamma_{s}, \ldots, \Gamma_{s}^{\prime}, \ldots, \Gamma_{s}^{\prime \prime}=\Gamma_{0}, \Gamma_{1}, \ldots, \Gamma_{\ell}=\Gamma_{t}^{\prime}, \ldots, \Gamma_{t}\right\rangle$ is a unidirectional convex morph between $\Gamma_{s}$ and $\Gamma_{t}$ with $O(n)$ morphing steps.

## References

1 S. Alamdari, P. Angelini, T. M. Chan, G. Di Battista, F. Frati, A. Lubiw, M. Patrignani, V. Roselli, S. Singla, and B. T. Wilkinson. Morphing planar graph drawings with a polynomial number of steps. In SODA, pages 1656-1667, 2013.
2 G. Aloupis, L. Barba, P. Carmi, V. Dujmovic, F. Frati, and P. Morin. Compatible connectivity-augmentation of planar disconnected graphs. In SODA, pages 1602-1615, 2015.

3 P. Angelini, G. Da Lozzo, G. Di Battista, F. Frati, M. Patrignani, and V. Roselli. Morphing planar graph drawings optimally. In ICALP, volume 8572 of $L N C S$, pages 126-137, 2014.
4 P. Angelini, F. Frati, M. Patrignani, and V. Roselli. Morphing planar graph drawings efficiently. In $G D$, volume 8242 of $L N C S$, pages 49-60, 2013.
5 I. Bárány and G. Rote. Strictly convex drawings of planar graphs. Documenta Mathematica, 11:369-391, 2006.
6 D. Barnette and B. Grünbaum. On Steinitz's theorem concerning convex 3-polytopes and on some properties of planar graphs. In Many Facets of Graph Theory, volume 110 of Lecture Notes in Mathematics, pages 27-40. Springer, 1969.
7 F. Barrera-Cruz, P. Haxell, and A. Lubiw. Morphing planar graph drawings with unidirectional moves. Mexican Conference on Discr. Math. and Comput. Geom., 2013.
8 N. Bonichon, S. Felsner, and M. Mosbah. Convex drawings of 3-connected plane graphs. Algorithmica, 47(4):399-420, 2007.
9 S. Cairns. Deformations of plane rectilinear complexes. Am. Math. Mon., 51:247-252, 1944.
10 N. Chiba, T. Yamanouchi, and T. Nishizeki. Linear algorithms for convex drawings of planar graphs. In Progress in Graph Theory, pages 153-173. Academic Press, New York, NY, 1984.
11 M. Chrobak and G. Kant. Convex grid drawings of 3-connected planar graphs. Int. J. Comput. Geometry Appl., 7(3):211-223, 1997.
12 C. Erten, S. G. Kobourov, and C. Pitta. Intersection-free morphing of planar graphs. In $G D$, volume 2912 of $L N C S$, pages 320-331, 2004.
13 C. Friedrich and P. Eades. Graph drawing in motion. J. Graph Alg. Ap., 6:353-370, 2002.
14 C. Gotsman and V. Surazhsky. Guaranteed intersection-free polygon morphing. Computers G Graphics, 25(1):67-75, 2001.
15 B. Grunbaum and G.C. Shephard. The geometry of planar graphs. Camb. Univ. Pr., 1981.
16 S. H. Hong and H. Nagamochi. Convex drawings of hierarchical planar graphs and clustered planar graphs. J. Discrete Algorithms, 8(3):282-295, 2010.
17 S. H. Hong and H. Nagamochi. A linear-time algorithm for symmetric convex drawings of internally triconnected plane graphs. Algorithmica, 58(2):433-460, 2010.
18 M. S. Rahman, S. I. Nakano, and T. Nishizeki. Rectangular grid drawings of plane graphs. Comput. Geom., 10(3):203-220, 1998.
19 M. S. Rahman, T. Nishizeki, and S. Ghosh. Rectangular drawings of planar graphs. J. of Algorithms, 50:62-78, 2004.
20 J. M. Schmidt. Contractions, removals, and certifying 3-connectivity in linear time. SIAM J. Comput., 42(2):494-535, 2013.

21 V. Surazhsky and C. Gotsman. Controllable morphing of compatible planar triangulations. ACM Trans. Graph, 20(4):203-231, 2001.
22 V. Surazhsky and C. Gotsman. Intrinsic morphing of compatible triangulations. Internat. J. of Shape Model., 9:191-201, 2003.

23 C. Thomassen. Planarity and duality of finite and infinite graphs. J. Comb. Theory, Ser. B, 29(2):244-271, 1980.
24 C. Thomassen. Deformations of plane graphs. J. Comb. Th. Ser. B, 34(3):244-257, 1983.
25 C. Thomassen. Plane representations of graphs. In J. A. Bondy and U. S. R. Murty, editors, Progress in Graph Theory, pages 43-69. Academic Press, New York, NY, 1984.


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