An Optimal Algorithm for the Separating Common Tangents of Two Polygons

Mikkel Abrahamsen*

Department of Computer Science, University of Copenhagen
Universitetsparken 5
DK-2100 Copanhagen Ø
Denmark
miab@di.ku.dk

―― **Abstract** ――

We describe an algorithm for computing the separating common tangents of two simple polygons using linear time and only constant workspace. A tangent of a polygon is a line touching the polygon such that all of the polygon lies to the same side of the line. A separating common tangent of two polygons is a tangent of both polygons where the polygons are lying on different sides of the tangent. Each polygon is given as a read-only array of its corners. If a separating common tangent does not exist, the algorithm reports that. Otherwise, two corners defining a separating common tangent are returned. The algorithm is simple and implies an optimal algorithm for deciding if the convex hulls of two polygons are disjoint or not. This was not known to be possible in linear time and constant workspace prior to this paper.

An outer common tangent is a tangent of both polygons where the polygons are on the same side of the tangent. In the case where the convex hulls of the polygons are disjoint, we give an algorithm for computing the outer common tangents in linear time using constant workspace.

1   Introduction

The problem of computing common tangents of two given polygons has received some attention in the case where the polygons are convex. For instance, it is necessary to compute outer common tangents of disjoint convex polygons in the classic divide-and-conquer algorithm for the convex hull of a set of n points in the plane by Preparata and Hong [12]. They give a naïve linear time algorithm for outer common tangents since that suffices for an $O(n \log n)$ time convex hull algorithm. The problem is also considered in various dynamic convex hull algorithms [5, 8, 11]. Overmars and van Leeuwen [11] give an $O(\log n)$ time algorithm for computing an outer common tangent of two disjoint convex polygons when a separating line is known, where each polygon has at most n corners. Kirkpatrick and Snoeyink [9] give an $O(\log n)$ time algorithm for the same problem, but without using a separating line. Guibas et al. [7] give an $\Omega(\log^2 n)$ lower bound on the time required to compute an outer common tangent of two intersecting convex polygons, even if it is known that they intersect in at most two points. They also describe an algorithm achieving that bound.

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Touissaint [13] considers the problem of computing separating common tangents of convex polygons and notes that the problem occurs in problems related to visibility, collision avoidance, range fitting, etc. He gives a linear time algorithm. Guibas et al. [7] give an $O(\log n)$ time algorithm for the same problem.

All the here mentioned works make use of the convexity of the polygons. If the polygons are not convex, one can use a linear time algorithm to compute the convex hulls before computing the tangents [6, 10]. However, if the polygons are given in read-only memory, it requires $\Omega(n)$ extra bits to store the convex hulls. In this paper, we also obtain linear time while using only constant workspace, i.e. $O(\log n)$ bits. For the outer common tangents, we require the convex hulls of the polygons to be disjoint. There has been some recent interest in constant workspace algorithms for geometric problems, see for instance [1, 2, 3, 4].

The problem of computing separating common tangents is of special interest because these only exist when the convex hulls of the polygons are disjoint, and our algorithm detects if they are not. Thus, we also provide an optimal algorithm for deciding if the convex hulls of two polygons are disjoint or not. This was to the best of our knowledge not known to be possible in linear time and constant workspace prior to our work.

1.1 Notation and some basic definitions

Given two points a and b in the plane, the closed line segment with endpoints a and b is written ab. When $a \neq b$, the line containing a and b which is infinite in both directions is written $\mathcal{L}(a, b)$.

Define the dot product of two points $x = (x_0, x_1)$ and $y = (y_0, y_1)$ as $x \cdot y = x_0 y_0 + x_1 y_1$, and let $x^\perp = (-x_1, x_0)$ be the counterclockwise rotation of x by the angle $\pi/2$. Now, for three points a, b, and c, we define $\mathcal{T}(a, b, c) = \mathrm{sgn}((b - a)^\perp \cdot (c - b))$, where sgn is the sign function. $\mathcal{T}(a, b, c)$ is 1 if c is to the left of the directed line from a to b, 0 if a, b, and c are collinear, and -1 if c is to the right of the directed line from a to b. We see that

$$\mathcal{T}(a, b, c) = \mathcal{T}(b, c, a) = \mathcal{T}(c, a, b) = -\mathcal{T}(c, b, a) = -\mathcal{T}(b, a, c) = -\mathcal{T}(a, c, b).$$

We also note that if a' and b' are on the line $\mathcal{L}(a, b)$ and appear in the same order as a and b, i.e., $(b - a) \cdot (b' - a') > 0$, then $\mathcal{T}(a, b, c) = \mathcal{T}(a', b', c)$ for every point c.

The *left half-plane* $\mathrm{LHP}(a, b)$ is the closed half plane with boundary $\mathcal{L}(a, b)$ lying to the left of directed line from a to b, i.e., all the points c such that $\mathcal{T}(a, b, c) \geq 0$. The *right half-plane* $\mathrm{RHP}(a, b)$ is just $\mathrm{LHP}(b, a)$.

Assume for the rest of this paper that P_0 and P_1 are two simple polygons in the plane with n_0 and n_1 corners, respectively, where P_k is defined by its corners $p_k[0], p_k[1], \ldots, p_k[n_k - 1]$ in clockwise or counterclockwise order, $k = 0, 1$. Indices of the corners are considered modulo n_k, so that $p_k[i]$ and $p_k[j]$ are the same corner when $i \equiv j \pmod{n_k}$.

We assume that the corners are in general position in the sense that P_0 and P_1 have no common corners and the combined set of corners $\bigcup_{k=0,1}\{p_k[0], \ldots, p_k[n_k - 1]\}$ contains no three collinear corners.

A *tangent* of P_k is a line ℓ such that ℓ and P_k are not disjoint and such that P_k is contained in one of the closed half-planes defined by ℓ. The line ℓ is a *common tangent* of P_0 and P_1 if it is a tangent of both P_0 and P_1. A common tangent is an *outer* common tangent if P_0 and P_1 are on the same side of the tangent, and otherwise the tangent is *separating*. See Figure 1.

For a simple polygon P, we let $\mathcal{H}(P)$ be the convex hull of P. The following lemma is a well-known fact about $\mathcal{H}(P)$.
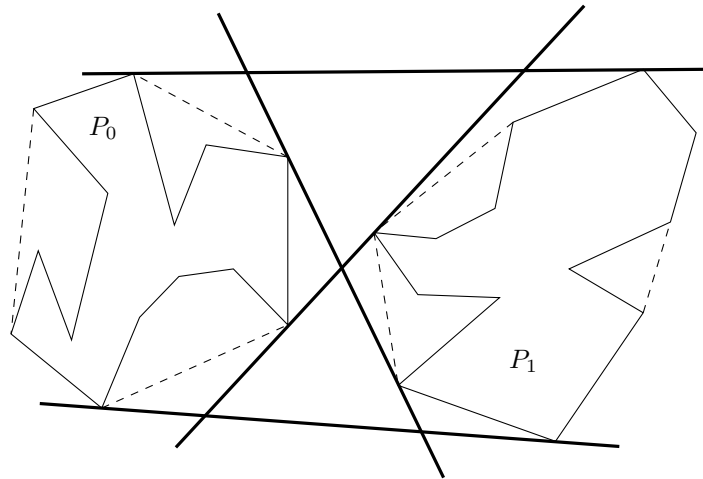
■ **Figure 1** Two polygons P_0 and P_1 and their four common tangents as thick lines. The edges of the convex hulls which are not edges of P_0 or P_1 are dashed.

▶ **Lemma 1.** *For a simple polygon P, $\mathcal{H}(P)$ is a convex polygon and the corners of $\mathcal{H}(P)$ appear in the same cyclic order as they do on P.*

The following lemma states folklore properties of tangents of polygons.

▶ **Lemma 2.** *A line is a tangent of a polygon P if and only if it is a tangent of $\mathcal{H}(P)$. Under our general position assumptions, the following holds: If one of $\mathcal{H}(P_0)$ and $\mathcal{H}(P_1)$ is completely contained in the other, there are no outer common tangents of P_0 and P_1. Otherwise, there are two or more. There are exactly two if P_0 and P_1 are disjoint. If $\mathcal{H}(P_0)$ and $\mathcal{H}(P_1)$ are not disjoint, there are no separating common tangents of P_0 and P_1. Otherwise, there are exactly two.*

2   Computing separating common tangents

In this section, we assume that the corners of P_0 and P_1 are both given in counterclockwise order. We prove that Algorithm 1 returns a pair of indices (s_0, s_1) such that the line $\mathcal{L}(p_0[s_0], p_1[s_1])$ is a separating common tangent with P_k contained in RHP$(p_{1-k}[s_{1-k}], p_k[s_k])$ for $k = 0, 1$. If the tangent does not exist, the algorithm returns NULL. The other separating common tangent can be found by a similar algorithm if the corners of the polygons are given in clockwise order and '$= 1$' is changed to '$= -1$' in lines 3 and 10.

The algorithm traverses the polygons in parallel one corner at a time using the indices t_0 and t_1. We say that the indices (s_0, s_1) define a *temporary line*, which is the line $\mathcal{L}(p_0[s_0], p_1[s_1])$. We update the indices s_0 and s_1 until the temporary line is the separating common tangent. At the beginning of an iteration of the loop at line 2, we traverse one corner $p_u[t_u]$ of P_u, $u = 0, 1$. If the corner happens to be on the wrong side of the intermediate line, we make the temporary line pass through that corner by updating s_u to t_u and we reset t_{1-u} to $s_{1-u} + 1$. The reason for resetting t_{1-u} is that a corner of P_{1-u} which was on the correct side of the old temporary line can be on the wrong side of the new line and thus needs be traversed again.

We show that if the temporary line is not a separating common tangent after each polygon has been traversed twice by the loop beginning at line 2, then the convex hulls of the polygons are not disjoint. Therefore, if a corner is found to be on the wrong side of the line defined by
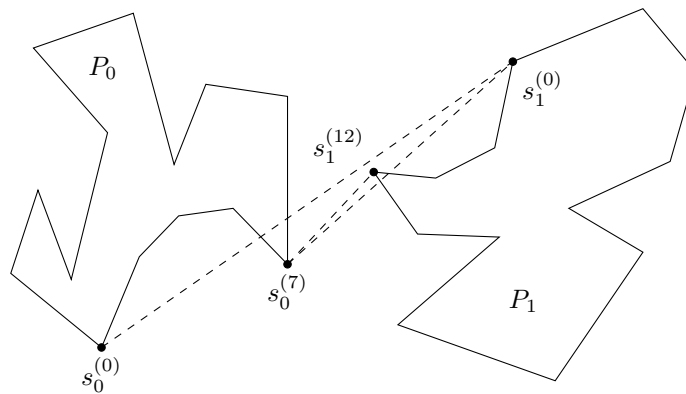
Figure 2 Algorithm 1 running on two polygons P_0 and P_1. The corners $p_k[s_k^{(i)}]$ are marked and labeled as $s_k^{(i)}$ for the initial values $s_k^{(0)}$ and after each iteration i where an update of s_k happens. The segments $p_0[s_0^{(i)}]p_1[s_1^{(i)}]$ on the temporary line are dashed.

Algorithm 1: SeparatingCommonTangent(P_0, P_1)

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1 $s_0 \leftarrow 0; \quad t_0 \leftarrow 1; \quad s_1 \leftarrow 0; \quad t_1 \leftarrow 1; \quad u \leftarrow 0$

2 while $t_0 < 2n_0$ or $t_1 < 2n_1$

3   **if** $\mathcal{T}(p_{1-u}[s_{1-u}], p_u[s_u], p_u[t_u]) = 1$

4 $s_u \leftarrow t_u$

5 $t_{1-u} \leftarrow s_{1-u} + 1$

6 $t_u \leftarrow t_u + 1$

7 $u \leftarrow 1 - u$

8 for each $u \leftarrow \{0, 1\}$

9   **for each** $t \leftarrow \{0, \ldots, n_u - 1\}$

10   **if** $\mathcal{T}(p_{1-u}[s_{1-u}], p_u[s_u], p_u[t]) = 1$

11   **return** NULL

12 return (s_0, s_1)

(s_0, s_1) in the loop beginning at line 8, no separating common tangent can exist and NULL is returned. Let $s_k^{(i)}$ be the value of s_k after $i = 0, 1, \ldots$ iterations of the loop at line 2. We always have $s_k^{(0)} = 0$ due to the initialization of s_k. See Figure 2.

Assume that s_0 is updated in line 4 in iteration i. The point $p_0[s_0^{(i)}]$ is in the half-plane LHP$(p_1[s_1^{(i-1)}], p_0[s_0^{(i-1)}])$, but not on the line $\mathcal{L}(p_1[s_1^{(i-1)}], p_0[s_0^{(i-1)}])$. Therefore, we have the following observation.

▶ **Observation 3.** *When s_k is updated, the temporary line is rotated counterclockwise around s_{1-k} by an angle less than π.*

Assume in the following that the convex hulls of P_0 and P_1 are disjoint so that separating common tangents exist. Let (r_0, r_1) be the indices that define the separating common tangent such that P_k is contained in RHP$(p_{1-k}[r_{1-k}], p_k[r_k])$, i.e., (r_0, r_1) is the result we are going to prove that the algorithm returns.

Since $\mathcal{H}(P_k)$ is convex, the temporary line always divides $\mathcal{H}(P_k)$ into two convex parts. If we follow the temporary line from $p_{1-k}[s_{1-k}]$ in the direction towards $p_k[s_k]$, we enter

$\mathcal{H}(P_k)$ at some point x and thereafter leave $\mathcal{H}(P_k)$ again at some point y. We clearly have $x = y$ if and only if the temporary line is a tangent to $\mathcal{H}(P_k)$, since if $x = y$ and the line was no tangent, $\mathcal{H}(P_k)$ would only be a line segment. The part of the boundary of $\mathcal{H}(P_k)$ counterclockwise from x to y is in $\text{RHP}(p_{1-k}[s_{1-k}], p_k[s_k])$ whereas the part from y to x is on $\text{LHP}(p_{1-k}[s_{1-k}], p_k[s_k])$. We therefore have the following observation.

▶ **Observation 4.** *Let d be the index of the corner of $\mathcal{H}(P_k)$ strictly after y in counterclockwise order. There exists a corner $p_k[t]$ of P_k such that $\mathcal{T}(p_{1-k}[s_{1-k}], p_k[s_k], p_k[t]) = 1$ if and only if $\mathcal{T}(p_{1-k}[s_{1-k}], p_k[s_k], p_k[d]) = 1$.*

Let c_k be the index of the first corner of $\mathcal{H}(P_k)$ when following $\mathcal{H}(P_k)$ in counterclockwise order from y, $c_k = 0, \ldots, n_k - 1$. If y is itself a corner of $\mathcal{H}(P_k)$, we have $p_k[c_k] = y$. By observation 4 we see that $\mathcal{T}(p_{1-k}[s_{1-k}], p_k[s_k], p_k[c_k]) \geq 0$ with equality if and only if $p_k[c_k] = p_k[s_k] = y$. Let $c_k^{(0)}$ be c_k when only line 1 has been executed. Consider now the value of c_k after $i = 1, 2, \ldots$ iterations of the loop at line 2. Let $c_k^{(i)} = c_k$ and add n_k to $c_k^{(i)}$ until $c_k^{(i)} \geq c_k^{(i-1)}$. This gives a non-decreasing sequence of indices $c_k^{(0)}, c_k^{(1)}, \ldots$ of the first corner of $\mathcal{H}(P_k)$ in $\text{LHP}(p_{1-k}[s_{1-k}], p_k[s_k])$. Actually, we prove in the following that we need to add n_k to $c_k^{(i)}$ at most once before $c_k^{(i)} \geq c_k^{(i-1)}$. If $r_k < c_k^{(0)}$ we add n_k to r_k. Thus we have $0 = s_k^{(0)} \leq c_k^{(0)} \leq r_k < 2n_k$.

The following lemma intuitively says that the algorithm does not "jump over" the correct solution and it expresses the main idea in our proof of correctness.

▶ **Lemma 5.** *After each iteration $i = 0, 1, \ldots$ and for each $k = 0, 1$ we have*

$$0 \leq s_k^{(i)} \leq c_k^{(i)} \leq r_k < 2n_k.$$

Proof. We prove the lemma for $k = 0$. From the definition of r_0, we get that $0 = s_0^{(0)} \leq c_0^{(0)} \leq r_0 < 2n_0$. Since the sequence $s_0^{(0)}, s_0^{(1)}, \ldots$ is non-decreasing, the inequality $0 \leq s_k^{(i)}$ is true for every i.

Now, assume inductively that $s_0^{(i-1)} \leq c_0^{(i-1)} \leq r_0$ and consider what happens during iteration i. If neither s_0 nor s_1 is updated, the statement is trivially true from the induction hypothesis, so assume that an update happens.

By the *old temporary line* we mean the temporary line defined by $(s_0^{(i-1)}, s_1^{(i-1)})$ and the *new temporary line* is the one defined by $(s_0^{(i)}, s_1^{(i)})$. The old temporary line enters $\mathcal{H}(P_0)$ at some point x and exits at some point y when followed from $p_1[s_1^{(i-1)}]$. Likewise, let v be the point where the new temporary line exits $\mathcal{H}(P_0)$ when followed from $p_1[s_1^{(i)}]$. The point x exists since the convex hulls are disjoint.

Assume first that the variable u in the algorithm is 0, i.e., a corner of the polygon P_0 is traversed. In this case $s_1^{(i-1)} = s_1^{(i)}$.

We now prove $s_0^{(i)} \leq c_0^{(i)}$. Assume that $p_0[s_0^{(i-1)}] \neq p_0[c_0^{(i-1)}]$. The situation is depicted in Figure 3. In this case $\mathcal{T}(p_1[s_1^{(i-1)}], p_0[s_0^{(i-1)}], p_0[c_0^{(i-1)}]) = 1$. Hence, the update happens when $p_0[c_0^{(i-1)}]$ is traversed or earlier, so $s_0^{(i)} \leq c_0^{(i-1)} \leq c_0^{(i)}$. Assume now that $p_0[s_0^{(i-1)}] = p_0[c_0^{(i-1)}]$. We cannot have $c_0^{(i)} = c_0^{(i-1)}$ since $\mathcal{T}(p_1[s_1^{(i)}], p_0[s_0^{(i)}], p_0[c_0^{(i-1)}]) = -\mathcal{T}(p_1[s_1^{(i-1)}], p_0[s_0^{(i-1)}], p_0[s_0^{(i)}]) = -1$, therefore $c_0^{(i)} > c_0^{(i-1)}$. Consider the corner $p_0[c']$ on $\mathcal{H}(P_0)$ following $p_0[c_0^{(i-1)}]$ in counterclockwise order, $c' > c_0^{(i-1)}$. Due to the minimality of c', we have $c' \leq c_0^{(i)}$. By Observation 4, $\mathcal{T}(p_1[s_1^{(i-1)}], p_0[s_0^{(i-1)}], p_0[c']) = 1$. Therefore, s_0 must be updated when $p_0[c']$ is traversed or earlier, so $s_0^{(i)} \leq c' \leq c_0^{(i)}$.

For the inequality $c_0^{(i)} \leq r_0$, consider the new temporary line in the direction from $p_1[s_1^{(i-1)}]$ to $p_0[s_0^{(i)}]$. We prove that v is in the part of $\mathcal{H}(P_0)$ from y counterclockwise to r_0. The point $p_0[s_0^{(i)}]$ is in the polygon Q defined by the segment xy together with the part of
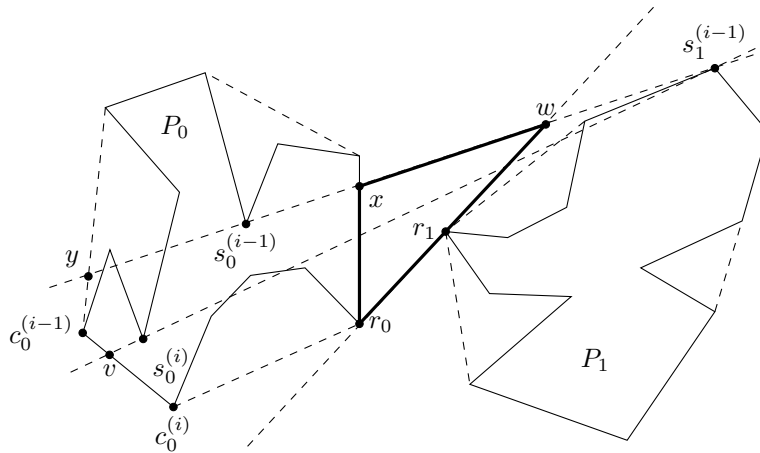
Figure 3 An update of s_0 happens in iteration i from $s_0^{(i-1)}$ to $s_0^{(i)}$ and $p_0[c_0]$ moves forward on $\mathcal{H}(P_0)$ from $p_0[c_0^{(i-1)}]$ to $p_0[c_0^{(i)}]$. The relevant corners are marked and labeled with their indices. The polygon $\mathcal{C}$ from the proof of Lemma 5 is drawn with thick lines.

$\mathcal{H}(P_0)$ from y counterclockwise to x. Therefore, the new temporary line enters and exits Q. It cannot exit through the segment xy, since the old and new temporary lines intersect at $p_1[s_1^{(i-1)}]$, which is in $\mathcal{H}(P_1)$. Therefore, v must be on the part of $\mathcal{H}(P_0)$ from y to x. If r_0 is on the part of $\mathcal{H}(P_0)$ from x counterclockwise to y, then v is on the part from y to r_0 as we wanted.

Otherwise, assume for contradiction that the points appear in the order $y, p_0[r_0], v, x$ counterclockwise along $\mathcal{H}(P_0)$, where $p_0[r_0] \neq v \neq x$. The endpoints of the segment $p_1[s_1^{(i-1)}]x$ are on different sides of the tangent defined by (r_0, r_1), so the segment intersects the tangent at a point w. The part of $\mathcal{H}(P_0)$ from $p_0[r_0]$ to x and the segments xw and $wp_0[r_0]$ form a simple polygon $\mathcal{C}$, see Figure 3 for an example. The new temporary line enters $\mathcal{C}$ at the point v, so it must leave $\mathcal{C}$ after v. The line cannot cross $\mathcal{H}(P_0)$ after v since $\mathcal{H}(P_0)$ is convex. It also cannot cross the segment xw at a point after v since the old and the new temporary line cross before v, namely at $p_1[s_1^{(i-1)}]$. The tangent defined by (r_0, r_1) and the new temporary line intersect before v since the endpoints of the segment $p_1[s_1^{(i-1)}]v$ are on different sides of the tangent. Therefore, the line cannot cross the segment $wp_0[r_0]$ at a point after v. Hence, the line cannot exit $\mathcal{C}$. That is a contradiction.

Therefore, v is on the part of $\mathcal{H}(P_0)$ from y to $p_0[r_0]$ and hence the first corner $p_0[c_0^{(i)}]$ of $\mathcal{H}(P_0)$ after v must be before or coincident with $p_0[r_0]$, so that $c_0^{(i)} \leq r_0$.

Assume now that $u = 1$ in the beginning of iteration i, i.e., a corner of the other polygon P_1 is traversed. In that case, we have $s_0^{(i)} = s_0^{(i-1)} \leq c_0^{(i-1)} \leq c_0^{(i)}$, and we need only prove $c_0^{(i)} \leq r_0$. Observation 3 gives that v is in the part of $\mathcal{H}(P_0)$ from y to x, since the new temporary line is obtained by rotating the old temporary line counterclockwise around $p_0[s_0^{(i-1)}]$ by an angle less than π. That v appears before $p_0[r_0]$ on $\mathcal{H}(P_0)$ counterclockwise from y follows from exactly the same arguments as in the case $u = 0$. This completes the proof.   ◄

▶ **Lemma 6.** *If the temporary line is different from the tangent defined by (r_0, r_1), then* $\mathcal{T}(p_0[s_0], p_1[s_1], p_1[r_1]) = 1$ *or* $\mathcal{T}(p_1[s_1], p_0[s_0], p_0[r_0]) = 1$.

Proof. Assume not. There are points of the temporary line on each side of the tangent because it is separating, so the temporary line and the tangent cross each other in a point a.

The point a is on the segment $p_0[r_0]p_1[r_1]$, since otherwise $p_0[r_0]$ and $p_1[r_1]$ would be on the same side of the temporary line, so $\mathcal{T}(p_0[s_0], p_1[s_1], p_1[r_1]) = 1$ or $\mathcal{T}(p_1[s_1], p_0[s_0], p_0[r_0]) = 1$. Choose a point d_R on the temporary line in $\text{RHP}(p_0[r_0], p_1[r_1])$ which is so far away from a that all intersections between the line and the polygons are on the same side of d_R as a. Choose d_L in a similar way in $\text{LHP}(p_0[r_0], p_1[r_1])$. We have $-1 = \mathcal{T}(p_0[r_0], p_1[r_1], d_R) = \mathcal{T}(p_0[r_0], a, d_R) = -\mathcal{T}(d_R, a, p_0[r_0])$, so the supports must appear in the order s_0, s_1 when traveling along the temporary line from d_R towards a for $\mathcal{T}(p_1[s_1], p_0[s_0], p_0[r_0]) \le 0$ to hold.

We also have that $p_0[s_0]$ is on the segment ad_L since $p_0[s_0] \in \text{LHP}(p_0[r_0], p_1[r_1])$ and $p_1[s_1]$ is on the segment ad_R since $p_1[s_1] \in \text{RHP}(p_0[r_0], p_1[r_1])$. Hence, the order of the supports from d_R towards a is s_1, s_0. That is a contradiction. ◀

We are now ready to prove that Algorithm 1 has the desired properties.

▶ **Theorem 7.** *If the polygons P_0 and P_1 have separating common tangents, Algorithm 1 returns a pair of indices (s_0, s_1) defining a separating common tangent such that P_k is contained in $RHP(p_{1-k}[s_{1-k}], p_k[s_k])$ for $k = 0, 1$. If no separating common tangents exist, the algorithm returns* `NULL`. *The algorithm runs in linear time and uses constant workspace.*

Proof. Assume first that separating common tangents do not exist. Then the test in line 10 makes the algorithm return `NULL` due to some corner $p_u[t]$ on the wrong side of the temporary line.

Assume now that separating common tangents do exist and that the temporary line is not the desired tangent. Without loss of generality, we may assume that $\mathcal{T}(p_1[s_1], p_0[s_0], p_0[r_0]) = 1$ by Lemma 6. Lemma 5 gives that $p_0[r_0]$ will be traversed if no other update of s_0 or s_1 happens. Therefore, an update happens before the loop at line 2 finishes. We conclude that when the loop finishes, the pair (s_0, s_1) defines the separating common tangent as stated.

When an update happens in iteration i of the loop at line 2, the sum $s_0 + s_1$ is increased by a value which is at least $\frac{i-j}{2}$, where $j \ge 0$ was the previous iteration where an update happened. Inductively, we see that the number of iterations is always at most $2(s_0 + s_1) + t_0 - s_0 + t_1 - s_1 \le 2(t_0 + t_1) \le 4(n_0 + n_1)$. ◀

3 Computing outer common tangents

In this section, we assume that two polygons P_0 and P_1 are given such that their convex hulls are disjoint. We assume that the corners $p_0[0], \ldots, p_0[n_0 - 1]$ of P_0 are given in counterclockwise order and the corners $p_1[0], \ldots, p_1[n_1 - 1]$ of P_1 are given in clockwise order. We say that the *orientation* of P_0 and P_1 is counterclockwise and clockwise, respectively. We prove that Algorithm 2 returns two indices (s_0, s_1) that define an outer common tangent such that P_0 and P_1 are both contained in $\text{RHP}(p_0[s_0], p_1[s_1])$.

As in the case of separating common tangents, we define $s_k^{(i)}$ as the value of s_k after $i = 0, 1, \ldots$ iterations of the loop at line 2 of Algorithm 2. See Figure 4. For this algorithm, we get a slightly different analogue to Observation 3:

▶ **Observation 8.** *When s_k is updated, the temporary line is rotated around s_{1-k} in the orientation of P_{1-k} by an angle less than π.*

Let y be the point where the temporary line enters $\mathcal{H}(P_k)$ when followed from $p_{1-k}[s_{1-k}]$ and x the point where it exits $\mathcal{H}(P_k)$. We have the following analogue of Observation 4.

▶ **Observation 9.** *Let d be the index of the corner of $\mathcal{H}(P_k)$ strictly after y following the orientation of P_k. There exists a corner $p_k[t]$ of P_k such that $\mathcal{T}(p_0[s_0], p_1[s_1], p_k[t]) = 1$ if and only if $\mathcal{T}(p_0[s_0], p_1[s_1], p_k[d]) = 1$.*
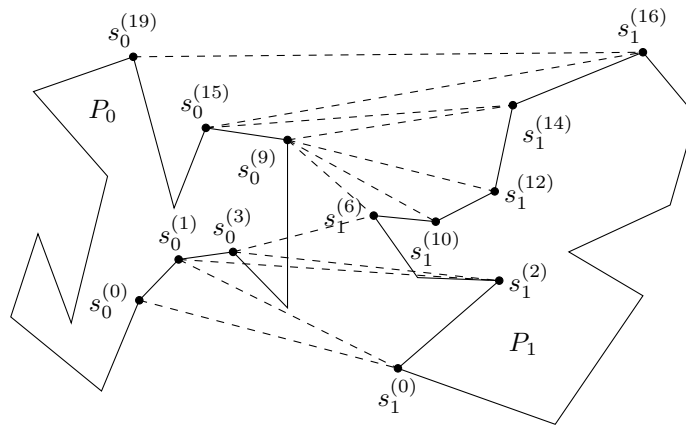
Figure 4 Algorithm 2 running on two polygons P_0 and P_1. The corners $p_k[s_k^{(i)}]$ are marked and labeled as $s_k^{(i)}$ for the initial values $s_k^{(0)}$ and after each iteration i where an update of s_k happens. The segments $p_0[s_0^{(i)}]p_1[s_1^{(i)}]$ on the temporary line are dashed.

Algorithm 2: OuterCommonTangent(P_0, P_1)

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1 $s_0 \leftarrow 0; \quad t_0 \leftarrow 1; \quad s_1 \leftarrow 0; \quad t_1 \leftarrow 1; \quad u \leftarrow 0$

2 **while** $t_0 < 2n_0$ or $t_1 < 2n_1$

3 $\quad$ **if** $\mathcal{T}(p_0[s_0], p_1[s_1], p_u[t_u]) = 1$

4 $\quad\quad$ $s_u \leftarrow t_u$

5 $\quad\quad$ $t_{1-u} \leftarrow s_{1-u} + 1$

6 $\quad$ $t_u \leftarrow t_u + 1$

7 $\quad$ $u \leftarrow 1 - u$

8 **return** (s_0, s_1)

Let c_k be the index of the first corner of $\mathcal{H}(P_k)$ after y following the orientation of P_k, where $p_k[c_k] = y$ if y is itself a corner of $\mathcal{H}(P_k)$. By Observation 9, we have $\mathcal{T}(p_0[s_0], p_1[s_1], p_k[c_k]) \geq 0$ with equality if and only if $p_k[c_k] = p_k[s_k] = y$. Define a non-decreasing sequence $c_k^{(0)}, c_k^{(1)}, \ldots$ of the value of c_k after $i = 0, 1, \ldots$ iterations as we did for separating tangents. Also, let the indices (r_0, r_1) define the outer common tangent that we want the algorithm to return such that $c_k^{(0)} \leq r_k < 2n_k$. We can now state the analogue to Lemma 5 for outer common tangents.

▶ **Lemma 10.** *After each iteration $i = 0, 1, \ldots$ and for each $k = 0, 1$ we have*

$$0 \leq s_k^{(i)} \leq c_k^{(i)} \leq r_k < 2n_k.$$

Proof. Assume $k = 0$ and the induction hypothesis $s_0^{(i-1)} \leq c_0^{(i-1)} \leq r_0$. The inequality $s_0^{(i)} \leq c_0^{(i)}$ can be proven exactly as in the proof of lemma 5. Therefore, consider the inequality $c_0^{(i)} \leq r_0$ and assume that an update happens in iteration i.

Let the *old temporary line* and the *new temporary line* be the lines defined by the indices $(s_0^{(i-1)}, s_1^{(i-1)})$ and $(s_0^{(i)}, s_1^{(i)})$, respectively. Let y and x be the points where the old temporary line enters and exits $\mathcal{H}(P_0)$ followed from $p_1[s_1^{(i-1)}]$, respectively, and let v be the point where the new temporary line enters $\mathcal{H}(P_0)$. The points y and v exist since the convex hulls of P_0 and P_1 are disjoint.
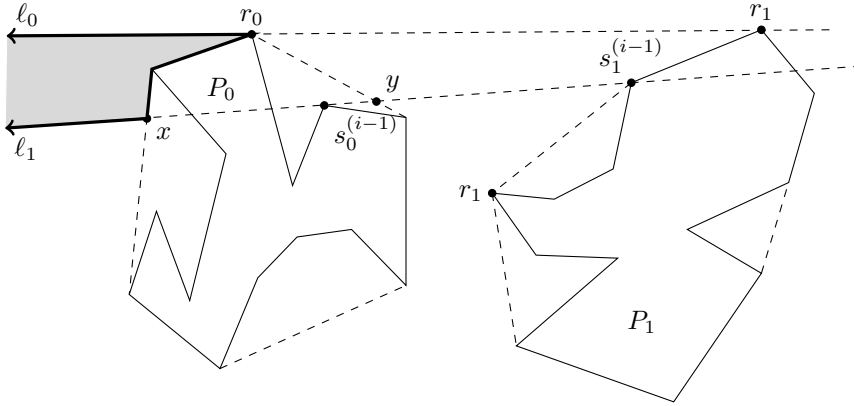
■ **Figure 5** The area $\mathcal{A}$ from the proof of Lemma 10 in grey. The relevant corners are marked and labeled with their indices.

Assume first that the variable u in the algorithm equals 0 when the update happens. We prove that v is in the part of $\mathcal{H}(P_0)$ from y to $p_0[r_0]$ following the orientation of P_0, which is counterclockwise. The point $p_0[s_0^{(i)}]$ is in the simple polygon Q bounded the part of $\mathcal{H}(P_0)$ from y counterclockwise to x and the segment xy. Therefore, the new temporary line must enter Q to get to $p_0[s_0^{(i)}]$. It cannot enter through xy, since the old and new temporary line cross at $p_1[s_1^{(i-1)}]$ which is not in $\mathcal{H}(P_k)$ by assumption. Therefore, it must enter through the part of $\mathcal{H}(P_0)$ from y to x, so v is in this part. If r_0 is not in the part of $\mathcal{H}(P_0)$ from y to x, it is clearly true that v is in the part from y to $p_0[r_0]$. Otherwise, assume for contradiction that the points appear on $\mathcal{H}(P_0)$ in the order $y, p_0[r_0], v, x$ and $p_0[r_0] \neq v \neq x$. Let ℓ_0 be the half-line starting at $p_0[r_0]$ following the tangent away from $p_1[r_1]$, and let ℓ_1 be the half-line starting at x following the old temporary line away from $p_1[s_1^{(i-1)}]$. The part of $\mathcal{H}(P_0)$ from $p_0[r_0]$ to x and the half-lines ℓ_0 and ℓ_1 define a possibly unbounded area $\mathcal{A}$ outside $\mathcal{H}(P_0)$, see Figure 5. We follow the new temporary line from $p_1[s_1^{(i-1)}]$ towards v. The point $p_1[s_1^{(i-1)}]$ is not in $\mathcal{A}$ and the new temporary line exits $\mathcal{A}$ at v since it enters $\mathcal{H}(P_0)$ at v, so it must enter $\mathcal{A}$ somewhere at a point on the segment $p_1[s_1^{(i-1)}]v$. It cannot enter through $\mathcal{H}(P_0)$ since $\mathcal{H}(P_0)$ is convex. It cannot enter through ℓ_0 since v and $p_1[s_1^{(i-1)}]$ are on the same side of the outer common tangent. It cannot enter through ℓ_1 since the old and new temporary line intersect in $p_1[s_1^{(i-1)}]$, which is not in $\mathcal{A}$. That is a contradiction, so v is on the part of $\mathcal{H}(P_0)$ from y to $p_0[r_0]$. Hence, the first corner after y is coincident with or before $p_0[r_1]$, i.e., $c_0^{(i)} \leq r_0$.

Assume now that $u = 1$ in the beginning of iteration i so that a corner of the polygon P_1 is traversed. Observation 8 gives that v is on the part of $\mathcal{H}(P_0)$ from y counterclockwise to x. It follows that v appears before $p_0[r_0]$ on $\mathcal{H}(P_0)$ counterclockwise from y from exactly the same arguments as in the case $u = 0$. ◀

We have the following equivalent of Lemma 6 which, however, has a different proof.

▶ **Lemma 11.** *If the temporary line is different from the tangent defined by (r_0, r_1), then $\mathcal{T}(p_0[s_0], p_1[s_1], p_0[r_0]) = 1$ or $\mathcal{T}(p_0[s_0], p_1[s_1], p_1[r_1]) = 1$.*

Proof. Assume not. The points $p_0[s_0]$ and $p_1[s_1]$ are both in $\mathrm{RHP}(p_0[r_0], p_1[r_1])$. Therefore, the temporary line cannot be parallel with the tangent, since in that case we would have $\mathcal{T}(p_0[s_0], p_1[s_1], p_0[r_0]) = 1$. Let a be the intersection point between the tangent and the temporary line. The point a cannot be in the interior of the segment $p_0[r_0]p_1[r_1]$,
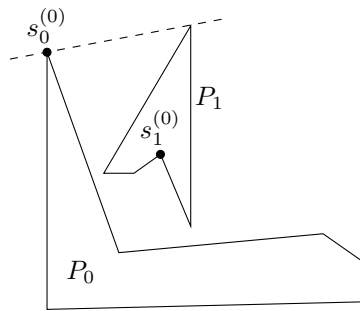
Figure 6 Two polygons P_0 and P_1 where Algorithm 2 does not work for the initial values of s_0 and s_1 as shown. The correct tangent is drawn as a dashed line.

since in that case, $p_0[r_0]$ and $p_1[r_1]$ would be on different sides of the temporary line, so $\mathcal{T}(p_0[s_0], p_1[s_1], p_0[r_0]) = 1$ or $\mathcal{T}(p_0[s_0], p_1[s_1], p_1[r_1]) = 1$. Assume without loss of generality that a is on the half-line from $p_0[r_0]$ going away from $p_1[r_1]$. Also assume that $p_0[s_0] \neq a$, since otherwise $p_0[s_0] = a = p_0[r_0]$ and $-1 = \mathcal{T}(p_0[r_0], p_1[r_1], p_1[s_1]) = -\mathcal{T}(p_0[s_0], p_1[s_1], p_1[r_1])$. Now, $1 = \mathcal{T}(p_1[r_1], p_0[r_0], p_0[s_0]) = \mathcal{T}(p_1[r_1], a, p_0[s_0]) = -\mathcal{T}(p_0[s_0], a, p_1[r_1])$. This forces $p_1[s_1]$ to be on the segment $p_0[s_0]a$.

From a, the orders of the points are $p_1[s_1], p_0[s_0]$ and $p_0[r_0], p_1[r_1]$ along the temporary line and the tangent, respectively. The points $ap_1[s_1]p_0[r_0]$ form a triangle Δ_0 and $ap_0[s_0]p_1[r_1]$ form a larger triangle Δ_1 containing Δ_0. The part $\Delta_1 \setminus \Delta_0$ of Δ_1 not in Δ_0 is therefore a quadrilateral $p_0[s_0]p_1[s_1]p_0[r_0]p_1[r_1]$ with all inner angles less than π, so the diagonals $p_0[s_0]p_0[r_0]$ and $p_1[s_1]p_1[r_1]$ cross each other. Hence, the convex hulls of P_0 and P_1 are not disjoint. ◀

We can now prove the stated properties of Algorithm 2 in much the same way as the proof of Theorem 7.

▶ **Theorem 12.** *If the polygons P_0 and P_1 have disjoint convex hulls, Algorithm 2 returns a pair of indices (s_0, s_1) defining an outer common tangent such that P_0 and P_1 are contained in $RHP(s_0, s_1)$. The algorithm runs in linear time and uses constant workspace.*

4 Concluding Remarks

We have described an algorithm for computing the separating common tangents of two simple polygons in linear time using constant workspace. We have also described an algorithm for computing outer common tangents using linear time and constant workspace when the convex hulls of the polygons are disjoint. Figure 6 shows an example where Algorithm 2 does not work when applied to two disjoint polygons with overlapping convex hulls. In fact, if there was no bound on the values t_0 and t_1 in the loop at line 2, the algorithm would update s_0 and s_1 infinitely often and never find the correct tangent. An obvious improvement is to find an equally fast and space efficient algorithm which does not require the convex hulls to be disjoint. An algorithm for computing an outer common tangent of two polygons, when such one exists, also decides if one convex hull is completely contained in the other. Together with the algorithm for separating common tangents presented in Section 2, we would have an optimal algorithm for deciding the complete relationship between the convex hulls: if one is contained in the other, and if not, whether they are disjoint or not. However, keeping in mind that it is harder to compute an outer common tangent of intersecting convex polygons

than of disjoint ones [7], it would not be surprising if it was also harder to compute an outer common tangent of general simple polygons than simple polygons with disjoint convex hulls when only constant workspace is available.

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