Bounding Helly Numbers via Betti Numbers*

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Abstract

We show that very weak topological assumptions are enough to ensure the existence of a Helly-type theorem. More precisely, we show that for any non-negative integers b and d there exists an integer $h(b,d)$ such that the following holds. If $\mathcal{F}$ is a finite family of subsets of $\mathbb{R}^d$ such that $\tilde\beta_i\left(\bigcap\mathcal{G}\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2\rceil - 1$ then $\mathcal{F}$ has Helly number at most $h(b,d)$. Here $\tilde\beta_i$ denotes the reduced $\mathbb{Z}_2$-Betti numbers (with singular homology). These topological conditions are sharp: not controlling any of these $\lceil d/2\rceil$ first Betti numbers allow for families with unbounded Helly number.

Our proofs combine homological non-embeddability results with a Ramsey-based approach to build, given an arbitrary simplicial complex K, some well-behaved chain map $C_*(K) \to C_*(\mathbb{R}^d)$. Both techniques are of independent interest.

1998 ACM Subject Classification I.3.5 Computational Geometry and Object Modeling, G.2 Discrete Mathematics

Keywords and phrases Helly-type theorem, Ramsey's theorem, Embedding of simplicial complexes, Homological almost-embedding, Betti numbers

Digital Object Identifier 10.4230/LIPIcs.SOCG.2015.507

Dedicated to the memory of Jiří Matoušek,
wonderful teacher, mentor, collaborator, and friend.

1 Introduction

Helly's classical theorem [13], a cornerstone of convex geometry, asserts that if a finite family of convex subsets of $\mathbb{R}^d$ has the property that any $d+1$ of the sets have a point in common then the whole family must have a point in common. Stated in the contrapositive, if $\mathcal{F}$ is a finite family of convex subsets of $\mathbb{R}^d$ with empty intersection then $\mathcal{F}$ contains a sub-family $\mathcal{G}$ of size at most $d+1$ that already has empty intersection. This inspired the definition of the *Helly number* of a family $\mathcal{F}$ of arbitrary sets. If $\mathcal{F}$ has empty intersection then its Helly

* PP, ZP and MT were partially supported by the Charles University Grant GAUK 421511. ZP was partially supported by the Charles University Grant SVV-2014-260103. ZP and MT were partially supported by the ERC Advanced Grant No. 267165 and by the project CE-ITI (GACR P202/12/G061) of the Czech Science Foundation. UW was partially supported by the Swiss National Science Foundation (grants SNSF-200020-138230 and SNSF-PP00P2-138948). Part of this work was done when XG was affiliated with INRIA Nancy Grand-Est and when MT was affiliated with Institutionen för matematik, Kungliga Tekniska Högskolan, then IST Austria.

number is defined as the size of the largest sub-family $\mathcal{G} \subseteq \mathcal{F}$ with the following properties: $\mathcal{G}$ has empty intersection and any proper sub-family of $\mathcal{G}$ has nonempty intersection; if $\mathcal{F}$ has nonempty intersection then its Helly number is, by convention, 1. With this terminology, Helly's theorem simply states that any finite family of convex sets in $\mathbb{R}^d$ has Helly number at most $d+1$.

In the spirit of Helly's theorem, bounds on Helly numbers, typically independent of the cardinality of the family, were given for a variety of situations in discrete geometry (such bounds are often referred to as *Helly-type theorems*); we refer to the surveys [9, 30, 27] for an overview of the abundant literature on this topic. Part of the interest for Helly numbers in discrete and computational geometry also stems from their interpretation in optimization problems. In short, a crucial step in applying the framework of generalized linear programming [1] to a geometric problem is to bound the size of so-called *feasible basis*; such bounds are Helly numbers in disguise. We come back to this question when we discuss some consequences of our main result.

Problem statement and results. The classical questions on Helly numbers are of two types, existential and quantitative: identify conditions under which Helly numbers can be bounded uniformly, and obtain sharp bounds. In this paper, we focus on the existential question and give the following new homological sufficient condition for bounding Helly numbers. Note that we consider homology with coefficients over $\mathbb{Z}_2$, denote by $\tilde{\beta}_i(X)$ the ith reduced Betti number (over $\mathbb{Z}_2$) of a space X, and use the notation $\bigcap \mathcal{F} := \bigcap_{U \in \mathcal{F}} U$ as a shorthand for the intersection of a family of sets.

► **Theorem 1.** *For any non-negative integers b and d there exists an integer $h(b,d)$ such that the following holds. If $\mathcal{F}$ is a finite family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i\left(\bigcap \mathcal{G}\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2 \rceil - 1$ then $\mathcal{F}$ has Helly number at most $h(b,d)$.*

Our proof hinges on a general principle, which we learned from Matoušek [19] but already underlies the classical proof of Helly's theorem from Radon's lemma, to derive Helly-type theorems from results of non-embeddability of certain simplicial complexes. The novelty of our approach is to examine these non-embeddability arguments from a homological point of view. This turns out to be a surprisingly effective idea, as homological analogues of embeddings appear to be much richer and easier to build than their homotopic counterparts. More precisely, our proof of Theorem 1 builds on two contributions of independent interest:

- We reformulate some non-embeddability results in homological terms. We obtain a homological analogue of the Van Kampen-Flores Theorem (Corollary 7) and, as a side-product, a homological version of Radon's lemma (Lemma 8). This is part of a systematic effort to translate various homotopy technique to a more tractable homology setting. It builds on, and extends, previous work on homological minors [29].
- By working with homology rather than homotopy, we can generalize a technique of Matoušek [19] that uses Ramsey's theorem to find embedded structures.

Theorem 1 is "qualitatively sharp", in the sense that all (reduced) Betti numbers $\tilde{\beta}_i$ with $0 \leq i \leq \lceil d/2 \rceil - 1$ need to be bounded to obtain a bounded Helly number. To see this, fix some k with $0 \leq k \leq \lceil d/2 \rceil - 1$. For n arbitrarily large, consider a geometric realization in $\mathbb{R}^d$ of the k-skeleton of the $(n-1)$-dimensional simplex (see [18, Section 1.6]); more specifically, let $V = \{v_1, \ldots, v_n\}$ be a set of points in general position in $\mathbb{R}^d$ (for instance, n points on the moment curve) and consider all geometric simplices $\sigma_A := \mathrm{conv}(A)$ spanned by subsets $A \subseteq V$ of cardinality $|A| \leq k+1$. By general position, $\sigma_A \cap \sigma_B = \sigma_{A \cap B}$, so this yields indeed a geometric realization.

For $1 \leq j \leq n$, let U_j be the union of all the simplices not containing the vertex v_j. We set $\mathcal{F} = \{U_1, \ldots, U_n\}$. Then, $\bigcap \mathcal{F} = \emptyset$, and for any proper sub-family $\mathcal{G} \subsetneq \mathcal{F}$, the intersection $\bigcap \mathcal{G}$ is either $\mathbb{R}^d$ (if $\mathcal{G} = \emptyset$) or (homeomorphic to) the k-dimensional skeleton of a $(n-1-|\mathcal{G}|)$-dimensional simplex. Thus, the Helly number of $\mathcal{F}$ equals n. Moreover, the k-skeleton $\Delta_{m-1}^{(k)}$ of an $(m-1)$-dimensional simplex has reduced Betti numbers $\tilde{\beta}_i = 0$ for $i \neq k$ and $\tilde{\beta}_k = \binom{m-1}{k+1}$. Thus, we can indeed obtain arbitrarily large Helly number as soon as at least one $\tilde{\beta}_k$ is unbounded. In particular, setting $k = 0$ yields the lower bound $h(b, d) \geq b + 1$.

Relation to previous work. The study of *topological conditions* (as opposed to more geometric ones like convexity) ensuring bounded Helly numbers started with *Helly's topological theorem* [14] (see also [8] for a modern version of the proof), which states that a finite family of open subsets of $\mathbb{R}^d$ has Helly number at most $d + 1$ if the intersection of any sub-family of at most d members of the family is either empty or a *homology cell*.[1] This includes the case of finite open *good cover*[2] in $\mathbb{R}^d$, where the same bound follows easily from the classical *Nerve theorem* [6, 5].

The "good cover" condition was subsequently relaxed by Matoušek [19] who showed that it is sufficient to control the low-dimensional homotopy of intersections: for any non-negative integers b and d there exists a constant $c(b, d)$ such that any finite family of subsets of $\mathbb{R}^d$ in which every sub-family intersects in at most b connected components, each $(\lceil d/2 \rceil - 1)$-*connected*,[3] has Helly number at most $c(b, d)$.

By Hurewicz' Theorem and the Universal Coefficient Theorem [12, Theorem 4.37 and Corollary 3A.6], a k-connected space X satisfies $\tilde{\beta}_i(X) = 0$ for all $i \leq k$. Thus, our condition indeed relaxes Matoušek's, in two ways: by using $\mathbb{Z}_2$-homology instead of the homotopy-theoretic assumptions of k-connectedness[4], and by allowing an arbitrary fixed bound b instead of $b = 0$.

Quantitatively, the bound on $h(b, d)$ that we obtain is very large as it follows from successive applications of Ramsey's theorems. However, as far as only the existence of uniform bounds is concerned, Theorem 1 not only generalizes Matoušek's result (which also uses Ramsey's theorem), but also subsumes a series of Helly-type theorems due to Amenta [2], Kalai and Meshulam [16], Colin de Verdière et al. [7], and Montejano [21]. Note that for results that hold in rather general ambient spaces, e.g. [16, 7, 21], Theorem 1 only subsumes the case of $\mathbb{R}^d$.

Our method also proves a bound of $d + 1$ on the Helly number of any family $\mathcal{F}$ such that $\tilde{\beta}_i \left(\bigcap \mathcal{G} \right) = 0$ for all $i \leq d$ and all $\mathcal{G} \subsetneq \mathcal{F}$ (see Corollary 10), which generalizes Helly's

[1] By definition, a homology cell is a topological space X all of whose (reduced, singular, integer coefficient) homology groups are trivial, as is the case if $X = \mathbb{R}^d$ or X is a single point. Here and in what follows, we refer the reader to standard textbooks like [12, 22] for further topological background and various topological notions that we leave undefined.

[2] An open good cover is a finite family of open subsets of $\mathbb{R}^d$ such that the intersection of any sub-family of at most d members is either empty or is contractible (and hence, in particular, a homology cell).

[3] We recall that a topological space X is k-connected, for some integer $k \geq 0$, if every continuous map $S^i \to X$ from the i-dimensional sphere to X, $0 \leq i \leq k$, can be extended to a map $D^{i+1} \to X$ from the $(i+1)$-dimensional disk to X.

[4] We also remark that our condition can be verified algorithmically since Betti numbers are easily computable, at least for sufficiently nice spaces that can be represented by finite simplicial complexes, say. By contrast, it is algorithmically undecidable whether a given 2-dimensional simplicial complex is 1-connected, see, e.g., the survey [26].

topological theorem as the sets of $\mathcal{F}$ are, for instance, not assumed to be open.[5] Under the weaker assumption that $\tilde{\beta}_i\left(\bigcap \mathcal{G}\right) = 0$ for all subfamilies $\mathcal{G} \subsetneq \mathcal{F}$ but only for $i \leq \lceil d/2 \rceil - 1$, our method still yields a bound of $d+2$ on the Helly number (see Corollary 9). In both cases the bounds are tight.

Note that Theorem 1 is similar, in spirit, to some of the general relations between the growth of Betti numbers and *fractional* Helly theorems conjectured by Kalai and Meshulam [15, Conjectures 6 and 7]. Kalai and Meshulam, in their conjectures, allow a polynomial growth of the Betti numbers in $|\bigcap \mathcal{G}|$. We remark that Theorem 1 is also sharp in the sense that even a linear growth of Betti number, already in $\mathbb{R}^1$, may yield unbounded Helly numbers.

Indeed, consider a positive integer n and open intervals $I_i := (i - 1.1; i + 0.1)$ for $i \in [n]$. Let $X_i := [0, n] \setminus X_i$. The intersection of all X_i is empty but the intersection of any proper subfamily is nonempty. In addition, the intersection of k such X_i can be obtained from $[0, n]$ by removing at most k open intervals, thus the reduced Betti numbers of such intersection are bounded by k.

In particular, the conjectures of Kalai and Meshulam cannot be strengthened to include Theorem 1.

Further consequences. The main strength of our result is to show that very weak assumptions on families of sets are enough to guarantee a bounded Helly number. A first natural application is as a tool to identify concrete situations in which Helly numbers are bounded. Let us give an example which, to the best of our knowledge, is not covered by any other Helly-type theorem appearing in the literature.

By an *affine k-sphere* in $\mathbb{R}^d$ for $0 \leq k \leq d-1$ we simply mean a geometric sphere of arbitrary center and radius inside some affine $(k+1)$-space of $\mathbb{R}^d$. An *affine sphere* is an affine k-sphere for some $k \in \{0, \ldots, d-1\}$. Theorem 1 implies that the Helly number of an arbitrary family of affine spheres in $\mathbb{R}^d$ is bounded since an arbitrary intersection of affine spheres is an empty set, singleton, or an affine sphere, all of them having bounded Betti numbers. A careful analysis can of course lead to a much better bound on the Helly number than the one given by Theorem 1; see for instance [17] for sharp bounds for the case of $(d-1)$-dimensional spheres in $\mathbb{R}^d$. However, note that Theorem 1 immediately reveals that the Helly number is bounded.

Theorem 1 also has consequences in the direction of optimization problems. Various optimization problems can be formulated as the minimization of some function $f : \mathbb{R}^d \to \mathbb{R}$ over some intersection $\bigcap_{i=1}^n C_i$ of subsets $C_1, C_2, \ldots, C_n$ of $\mathbb{R}^d$. If, for $t \in \mathbb{R}$, we let $L_t = f^{-1}\left((-\infty, t]\right)$ and $\mathcal{F}_t = \{C_1, C_2, \ldots, C_n, L_t\}$ then

$$\min_{x \in \bigcap_{i=1}^n C_i} f(x) = \min \left\{ t \in \mathbb{R} : \bigcap \mathcal{F}_t \neq \emptyset \right\}.$$

If the Helly number of the families $\mathcal{F}_t$ can be bounded *uniformly* in t by some constant h then there exists a subset of $h-1$ constraints $C_{i_1}, C_{i_2}, \ldots, C_{i_{h-1}}$ that suffice to define the minimum of f:

$$\min_{x \in \bigcap_{i=1}^n C_i} f(x) = \min_{x \in \bigcap_{j=1}^{h-1} C_{i_j}} f(x).$$

[5] In the original proof, this assumption is crucial and used to ensure that the union of the sets must have trivial homology in dimensions larger than d; this may fail if the sets are not open.

A consequence of this observation, noted by Amenta [1], is that the minimum of f over $C_1 \cap C_2 \cap \ldots \cap C_n$ can[6] be computed in randomized $O(n)$ time by *generalized linear programming* [25]. Together with Theorem 1, this implies that an optimization problem of the above form can be solved in randomized linear time if it has the property that every intersection of some subset of the constraints with a level set of the function has bounded "topological complexity" (measured in terms of the sum of the first $\lceil d/2 \rceil$ Betti numbers). Let us emphasize that this linear-time bound holds in a real-RAM model of computation, where any constant-size subproblems can be solved in $O(1)$-time; it therefore concerns the *combinatorial difficulty* of the problem and says nothing about its *numerical difficulty*.

Organization, notation, etc. We prove Theorem 1 in three steps. We first set up our homological machinery in Section 2 (homological almost-embeddings, homological Van Kampen-Flores Theorem, and homological Radon lemma). We then present, in Section 3, variations of the technique that derives Helly-type theorems from non-embeddability. We finally introduce our refinement of this technique and the proof of Theorem 1 in Section 4. Due to space constraint, various proofs are only sketched and we refer to [11] for the full details.

We assume that the reader is familiar with basic topological notions and facts concerning simplicial complexes and singular and simplicial homology, as described in textbooks like [12, 22]. As remarked above, throughout this paper we will work with homology with $\mathbb{Z}_2$-coefficients unless explicitly stated otherwise. Moreover, while we will consider singular homology groups for topological spaces in general, for simplicial complexes we will work with simplicial homology groups. In particular, if X is a topological space then $C_*(X)$ will denote the singular chain complex of X, while if K is a simplicial complex, then $C_*(K)$ will denote the simplicial chain complex of K (both with $\mathbb{Z}_2$-coefficients).

We use the following notation. Let K be a (finite, abstract) simplicial complex. The *underlying topological space* of K is denoted by $|K|$. Moreover, we denote by $K^{(i)}$ the *i-dimensional skeleton* of K, i.e., the set of simplices of K of dimension at most i; in particular $K^{(0)}$ is the set of vertices of K. For an integer $n \geq 0$, let Δ_n denote the n-dimensional simplex.

Given a set X we let 2^X and $\binom{X}{k}$ denote, respectively, the set of all subsets of X (including the empty set) and the set of all k-element subsets of X. If $f : X \to Y$ is an arbitrary map between sets then we abuse the notation by writing $f(S)$ for $\{f(s) \mid s \in S\}$ for any $S \subseteq X$; that is, we implicitly extend f to a map from 2^X to 2^Y whenever convenient.

2 Homological Almost-Embeddings

In this section, we define *homological almost-embedding*, an analogue of topological embeddings on the level of chain maps, and show that certain simplicial complexes do not admit homological almost-embeddings in $\mathbb{R}^d$, in analogy to classical non-embeddability results due to Van Kampen and Flores.

Recall that an *embedding* of a finite simplicial complex K into $\mathbb{R}^d$ is simply an injective continuous map $|K| \to \mathbb{R}^d$. The fact that the complete graph on five vertices cannot be embedded in the plane has the following generalization.

[6] This requires f and $C_1, C_2, \ldots, C_n$ to be generic in the sense that the number of minima of f over $\cap_{i \in I} C_i$ is bounded uniformly for $I \subseteq \{1, 2, \ldots, n\}$.

▶ **Proposition 2** (Van Kampen [28], Flores [10]). *For $k \geq 0$, the complex $\Delta_{2k+2}^{(k)}$, the k-dimensional skeleton of the $(2k+2)$-dimensional simplex, does not embed in $\mathbb{R}^{2k}$.*

A basic tool for proving the non-embeddability of a simplicial complex is the so-called *Van Kampen obstruction*. Given a simplicial complex K, one can define, for each $d \geq 0$, a certain cohomology class $\mathfrak{o}^d(K)$ that resides in the cohomology group $H^d(\overline{K})$ of a certain auxiliary complex $\overline{K}$ (the quotient of the combinatorial deleted product by the natural $\mathbb{Z}_2$-action, see below); this cohomology class $\mathfrak{o}^d(K)$ is called the Van Kampen obstruction to embeddability into $\mathbb{R}^d$ because of the following fact:

▶ **Proposition 3** ([24, 31]). *A finite simplicial complex K with $\mathfrak{o}^d(K) \neq 0$ does not embed into $\mathbb{R}^d$.*

A slightly stronger conclusion actually holds: there is no *almost-embedding* $f\colon |K| \to \mathbb{R}^d$, i.e., no continuous map such that the images of disjoint simplices of K are disjoint. Proposition 2, and in fact the slightly stronger statement that $\Delta_{2k+2}^{(k)}$ does not admit an almost-embedding into $\mathbb{R}^{2k}$, then follows from the next result (for a short proof see, for instance, [20, Example 3.5]).

▶ **Proposition 4** ([28, 10]). *For every $k \geq 0$, $\mathfrak{o}^{2k}\left(\Delta_{2k+2}^{(k)}\right) \neq 0$.*

A close examination of the standard proof of Proposition 3 reveals that it is based on (co)homological arguments so that maps can be replaced by suitable chain maps at every step.[7] The appropriate analogue of an almost-embedding is the following:

▶ **Definition 5.** Let K be a simplicial complex, and consider a chain map[8] $\gamma\colon C_*(K) \to C_*(\mathbb{R}^d)$ from the simplicial chains in K to singular chains in $\mathbb{R}^d$.

(i) The chain map γ is called *nontrivial*[9] if the image of every vertex of K is a finite set of points in $\mathbb{R}^d$ (a 0-chain) of *odd* cardinality.

(ii) The chain map γ is called a *homological almost-embedding* of a simplicial complex K in $\mathbb{R}^d$ if it is nontrivial and if, additionally, the following holds: whenever σ and τ are disjoint simplices of K, their image chains $\gamma(\sigma)$ and $\gamma(\tau)$ have disjoint supports, where the support of a chain is the union of (the images of) the singular simplices with nonzero coefficient in that chain.

Definition 5 generalizes classical homotopic notions. Indeed, if $f\colon |K| \to \mathbb{R}^d$ is a continuous map then the induced chain map[10] $f_\sharp\colon C_*(K) \to C_*(\mathbb{R}^d)$ is nontrivial. Moreover, if f is an almost-embedding then the induced chain map is a homological almost-embedding. We can generalize Proposition 3 as follows:

▶ **Proposition 6.** *A finite simplicial complex K with $\mathfrak{o}^d(K) \neq 0$ has no homological almost-embedding in $\mathbb{R}^d$.*

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[7] This observation was already used in [29] to study the (non-)embeddability of certain simplicial complexes. What we call a *homological almost-embedding* in the present paper corresponds to the notion of a *homological minor* used in [29].

[8] We recall that a chain map $\gamma\colon C_* \to D_*$ between chain complexes is simply a sequence of homomorphisms $\gamma_n\colon C_n \to D_n$ that commute with the respective boundary operators, $\gamma_{n-1} \circ \partial_C = \partial_D \circ \gamma_n$.

[9] If we consider augmented chain complexes with chain groups also in dimension -1, then being nontrivial is equivalent to requiring that the generator of $C_{-1}(K) \cong \mathbb{Z}_2$ (this generator corresponds to the empty simplex in K) is mapped to the generator of $C_{-1}(\mathbb{R}^d) \cong \mathbb{Z}_2$.

[10] The induced chain map is defined as follows: We assume that we have fixed a total ordering of the vertices of K. For a p-simplex σ of K, the ordering of the vertices induces a homeomorphism $h_\sigma\colon |\Delta_p| \to |\sigma| \subseteq |K|$. The image $f_\sharp(\sigma)$ is defined as the singular p-simplex $f \circ h_\sigma$.

Sketch of proof: Like in the standard proof of Proposition 3, we construct given a homological almost-embedding of a complex K into $\mathbb{R}^d$ a non-trivial equivariant chain map from the (combinatorial) deleted product of that complex into $\mathbb{S}^{d-1}$, then into $\mathbb{S}^\infty$ through the inclusion $\mathbb{S}^{d-1} \to \mathbb{S}^\infty$. We can then interpret $\mathfrak{o}^d(K)$ in terms of the d-dimensional cohomology of $\mathbb{RP}^\infty$, the $\mathbb{Z}_2$ quotient of $\mathbb{S}^\infty$, and show that it should vanish. In one of the steps we need to replace (classical) equivariant homotopy with equivariant chain homotopy, which is somewhat technical. We refer to [11, Proposition 7] for a complete proof. ◀

As a consequence we obtain a homological analogue of the Van Kampen-Flores theorem:

▶ **Corollary 7.** *For $d \geq 0$, $\Delta_{d+2}^{(\lceil d/2 \rceil)}$ has no homological almost-embedding in $\mathbb{R}^d$.*

Proof. Propositions 4 and 6 together imply that for any $k \geq 0$, the k-skeleton $\Delta_{2k+2}^{(k)}$ of the $(2k+2)$-dimensional simplex has no homological almost-embedding in $\mathbb{R}^{2k}$. This proves the statement when d is even.

Assume that d is odd and write $d = 2k+1$. If K is a finite simplicial complex with $\mathfrak{o}^d(K) \neq 0$ and if CK is the cone over K then $\mathfrak{o}^{d+1}(CK) \neq 0$ (for a proof, see, for instance, [4, Lemma 8]). Since we know that $\mathfrak{o}^{2k}(\Delta_{2k+2}^{(k)}) \neq 0$ it follows that $\mathfrak{o}^{2k+1}(C\Delta_{2k+2}^{(k)}) \neq 0$. Consequently, $\mathfrak{o}^{2k+1}(\Delta_{2k+3}^{(k+1)}) \neq 0$ since $C\Delta_{2k+2}^{(k)}$ is a subcomplex of $\Delta_{2k+3}^{(k+1)}$ (and since there is an equivariant map from the deleted product of the subcomplex to the deleted product of the complex). Proposition 6 then implies that $\Delta_{2k+3}^{(k+1)}$ admits no homological almost-embedding in $\mathbb{R}^{2k+1}$. This proves the statement when d is odd. ◀

We also deduce a homological Radon lemma (note that $\partial\Delta_{d+1} = \Delta_{d+1}^{(d)}$); see [11, Lemma 10] for a proof.

▶ **Corollary 8.** *For $d \geq 0$, $\partial\Delta_{d+1}$ has no homological almost-embedding in $\mathbb{R}^d$.*

3 Helly-type theorems from non-embeddability

In this section, we review various applications, and formalize the ingredients, of a technique to prove Helly-type theorems from obstructions to embeddability. This technique was already present in the classical proof of Helly's convex theorem from Radon's lemma and was made more transparent by Matoušek [19].

3.1 Homotopic assumptions

Let $\mathcal{F} = \{U_1, U_2, \ldots, U_n\}$ denote a family of subsets of $\mathbb{R}^d$. We assume that $\mathcal{F}$ has empty intersection and that any proper subfamily of $\mathcal{F}$ has nonempty intersection. Our goal is to show how various conditions on the topology of the intersections of the subfamilies of $\mathcal{F}$ imply bounds on the cardinality of $\mathcal{F}$. For any (possibly empty) proper subset I of $[n] = \{1, 2, \ldots, n\}$ we write $U_{\overline{I}}$ for $\bigcap_{i \in [n] \setminus I} U_i$. We also put $U_{\overline{[n]}} = \mathbb{R}^d$.

Path-connected intersections in the plane. Consider the case where $d = 2$ and the intersections $\bigcap \mathcal{G}$ are path-connected for all subfamilies $\mathcal{G} \subsetneq \mathcal{F}$. Since every intersection of $n-1$ members of $\mathcal{F}$ is nonempty, we can pick, for every $i \in [n]$, a point p_i in $U_{\overline{\{i\}}}$. Moreover, as every intersection of $n-2$ members of $\mathcal{F}$ is connected, we can connect any pair of points p_i and p_j by an arc $s_{i,j}$ inside $U_{\overline{\{i,j\}}}$. We thus obtain a drawing of the complete graph on $[n]$ in the plane in a way that the edge between i and j is contained in $U_{\overline{\{i,j\}}}$ (see Figure 1). If $n \geq 5$ then the stronger form of non-planarity of K_5 implies that there exist two edges

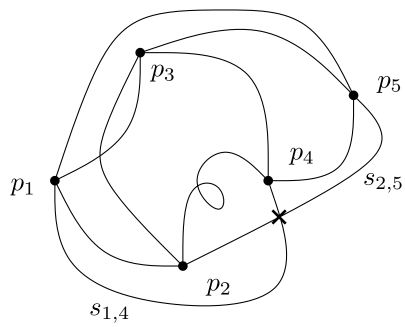


Figure 1 Two edges (arcs) with no common vertices intersect (in this case $s_{1,4}$ and $s_{2,5}$). The point in the intersection then belongs to all sets in $\mathcal{F}$.

$\{i, j\}$ and $\{k, \ell\}$ with no vertex in common and whose images intersect (see Proposition 3 and Lemma 4). Since $U_{\overline{\{i,j\}}} \cap U_{\overline{\{k,\ell\}}} = \bigcap \mathcal{F} = \emptyset$, this cannot happen and $\mathcal{F}$ has cardinality at most 4.

$\lceil d/2 \rceil$-connected intersections in $\mathbb{R}^d$. The previous argument generalizes to higher dimension as follows. Assume that the intersections $\bigcap \mathcal{G}$ are $\lceil d/2 \rceil$-connected[11] for all subfamilies $\mathcal{G} \subsetneq \mathcal{F}$. Then we can build by induction a function f from the $\lceil d/2 \rceil$-skeleton of Δ_{n-1} to $\mathbb{R}^d$ in a way that for any simplex σ, the image $f(\sigma)$ is contained in $U_{\overline{\sigma}}$. The previous case shows how to build such a function from the 1-skeleton of Δ_{n-1}. Assume that a function f from the ℓ-skeleton of Δ_{n-1} is built. For every $(\ell+1)$-simplex σ of Δ_{n-1}, for every facet τ of σ, we have $f(\tau) \subset U_{\overline{\tau}} \subseteq U_{\overline{\sigma}}$. Thus, the set

$$\bigcup_{\tau \text{ facet of } \sigma} f(\tau)$$

is the image of an ℓ-dimensional sphere contained in $U_{\overline{\sigma}}$, which has vanishing homotopy of dimension ℓ. We can extend f from this sphere to an $(\ell+1)$-dimensional ball so that the image is still contained in $U_{\overline{\sigma}}$. This way we extend f to the $(\ell+1)$-skeleton of Δ_{n-1}.

The Van Kampen-Flores theorem asserts that for any continuous function from $\Delta_{2k+2}^{(k)}$ to $\mathbb{R}^{2k}$ there exist two disjoint faces of $\Delta_{2k+2}^{(k)}$ whose images intersect (see Proposition 3 and Lemma 4). So, if $n \geq 2\lceil d/2 \rceil + 3$, then there exist two disjoint simplices σ and τ of $\Delta_{2\lceil d/2 \rceil+2}^{(\lceil d/2 \rceil)}$ such that $f(\sigma) \cap f(\tau)$ is nonempty. Since $f(\sigma) \cap f(\tau)$ is contained in $U_{\overline{\sigma}} \cap U_{\overline{\tau}} = \bigcap \mathcal{F} = \emptyset$, this is a contradiction and $\mathcal{F}$ has cardinality at most $2\lceil d/2 \rceil + 2$.

By a more careful inspection of odd dimensions, the bound $2\lceil d/2 \rceil + 2$ can be improved to $d+2$. We skip this in the homotopic setting, but we will do so in the homological setting (which is stronger anyway); see Corollary 9 below.

Contractible intersections. Of course, the previous argument works with other non-embeddability results. For instance, if the intersections $\bigcap \mathcal{G}$ are contractible for all subfamilies then the induction yields a map f from the d-skeleton of Δ_{n-1} to $\mathbb{R}^d$ with the property that for any simplex σ, the image $f(\sigma)$ is contained in $U_{\overline{\sigma}}$. The topological Radon theorem [3] (see also [18, Theorem 5.1.2]) states that for any continuous function from Δ_{d+1} to $\mathbb{R}^d$ there exist two disjoint faces of Δ_{d+1} whose images intersect. So, if $n \geq d+2$ we again obtain

[11] Recall that a set is k-connected if it is connected and has vanishing homotopy in dimension 1 to k.

a contradiction (the existence of two disjoint simplices σ and τ such that $f(\sigma) \cap f(\tau) \neq \emptyset$ whereas $U_{\overline{\sigma}} \cap U_{\overline{\tau}} = \bigcap \mathcal{F} = \emptyset$), and the cardinality of $\mathcal{F}$ must be at most $d+1$.

3.2 From homotopy to homology

The previous reasoning can be transposed to homology as follows. Assume that for $i = 0, 1, \ldots, k-1$ and all subfamilies $\mathcal{G} \subsetneq \mathcal{F}$ we have $\tilde{\beta}_i(\bigcap \mathcal{G}) = 0$. We construct a nontrivial[12] chain map f from the simplicial chains of $\Delta_{n-1}^{(k)}$ to the singular chains of $\mathbb{R}^d$ by increasing dimension:

- For every $\{i\} \subset [n]$ we let $p_i \in U_{\overline{\{i\}}}$. This is possible since every intersection of $n-1$ members of $\mathcal{F}$ is nonempty. We then put $f(\{i\}) = p_i$ and extend it by linearity into a chain map from $\Delta_{n-1}^{(0)}$ to $\mathbb{R}^d$. Notice that f is nontrivial and that for any 0-simplex $\sigma \subseteq [n]$, the support of $f(\sigma)$ is contained in $U_{\overline{\sigma}}$.
- Now, assume, as an induction hypothesis, that there exists a nontrivial chain map f from the simplicial chains of $\Delta_{n-1}^{(\ell)}$ to the singular chains of $\mathbb{R}^d$ with the property that for any $(\leq \ell)$-simplex $\sigma \subseteq [n], \ell < k$, the support of $f(\sigma)$ is contained in $U_{\overline{\sigma}}$. Let σ be a $(\ell+1)$-simplex in $\Delta_{n-1}^{(\ell+1)}$. For every ℓ-dimensional face τ of σ, the support of $f(\tau)$ is contained in $U_{\overline{\tau}} \subseteq U_{\overline{\sigma}}$. It follows that the support of $f(\partial\sigma)$ is contained in $U_{\overline{\sigma}}$, which has trivial homology in dimension $\ell+1$. As a consequence, $f(\partial\sigma)$ is a boundary in $U_{\overline{\sigma}}$. We can therefore extend f to every simplex of dimension $\ell+1$ and then, by linearity, to a chain map from the simplicial chains of $\Delta_{n-1}^{(\ell+1)}$ to the singular chains of $\mathbb{R}^d$. This chain map remains nontrivial and, by construction, for any $(\leq \ell+1)$-simplex $\sigma \subseteq [n]$, the support of $f(\sigma)$ is contained in $U_{\overline{\sigma}}$.

If σ and τ are disjoint simplices of $\Delta_{n-1}^{(k)}$ then the intersection of the supports of $f(\sigma)$ and $f(\tau)$ is contained in $U_{\overline{\sigma}} \cap U_{\overline{\tau}} = \bigcap \mathcal{F} = \emptyset$ and these supports are disjoint. It follows that f is not only a nontrivial chain map, but also a homological almost-embedding in $\mathbb{R}^d$. We can then use obstructions to the existence of homological almost-embeddings to bound the cardinality of $\mathcal{F}$. Specifically, since we assumed that $\mathcal{F}$ has empty intersection and any proper subfamily of $\mathcal{F}$ has nonempty intersection, Corollary 7 implies:

▶ **Corollary 9.** *Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i(\bigcap \mathcal{G}) = 0$ for every $\mathcal{G} \subsetneq \mathcal{F}$ and $i = 0, 1, \ldots, \lceil d/2 \rceil - 1$. Then the Helly number of $\mathcal{F}$ is at most $d+2$.*

The homological Radon lemma (Lemma 8) yields:

▶ **Corollary 10.** *Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i(\bigcap \mathcal{G}) = 0$ for every $\mathcal{G} \subsetneq \mathcal{F}$ and $i = 0, 1, \ldots, d-1$. Then the Helly number of $\mathcal{F}$ is at most $d+1$.*

The examples showing, in the introduction, that Theorem 1 is qualitatively sharp can be modified to show that the previous Corollaries are also sharp in various ways.

First assume that for some values k, n there exists an embedding f of $\Delta_{n-1}^{(k)}$ into $\mathbb{R}^d$. Let K_i be the simplicial complex obtained by deleting the ith vertex of $\Delta_{n-1}^{(k)}$ (as well as all simplices using that vertex) and put $U_i := f(K_i)$. The family $\mathcal{F} = \{U_1, \ldots, U_n\}$ has Helly number exactly n, since it has empty intersection and all its proper subfamilies have nonempty intersection. Moreover, for every $\mathcal{G} \subseteq \mathcal{F}$, $\bigcap \mathcal{G}$ is the image through f of the k-skeleton of a simplex on $|\mathcal{F} \setminus \mathcal{G}|$ vertices, and therefore $\tilde{\beta}_i(\bigcap \mathcal{G}) = 0$ for every $\mathcal{G} \subseteq \mathcal{F}$ and $i = 0, \ldots, k-1$.

[12] See Definition 5.

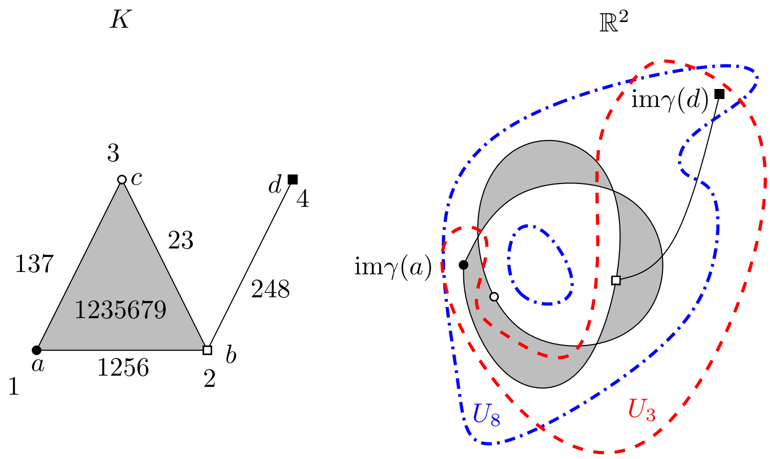


Figure 2 An example of a constrained map $\gamma\colon K \to \mathbb{R}^2$. A label at a face σ of K denotes $\Phi(\sigma)$. Note, for example, that the support of $\gamma(\{a, b, c\})$ needn't be a triangle since we work with chain maps. Constrains by Φ mean that a set U_i must contain cover images of all faces without label i. It is demonstrated by U_3 and U_8 for example.

Such an embedding exists when $k = d$ and $n = d + 1$, as the d-dimensional simplex easily embeds into $\mathbb{R}^d$. Consequently, the bound of $d + 1$ is best possible under the assumptions of Corollary 10. Such an embedding also exists for $k = d - 1$ and $n = d + 2$, as we can first embed the $(d-1)$-skeleton of the d-simplex linearly, then add an extra vertex at the barycentre of the vertices of that simplex and embed the remaining faces linearly. This implies that if we relax the condition of Corollary 10 by only controlling the first $d - 2$ Betti numbers then the bound of $d + 1$ becomes false. It also implies that the bound of $d + 2$ is best possible under (a strengthening of) the assumptions of Corollary 9.

Constrained chain map. Let us formalize the technique illustrated by the previous example. We focus on the homological setting, as this is what we use to prove Theorem 1, but this can be easily transposed to homotopy.

As above, we have a family $\mathcal{F} = \{U_1, U_2, \ldots, U_n\}$ of subsets of $\mathbb{R}^d$ and we keep the notation for $U_{\overline{I}}$ introduced in the beginning of this section.

Let K be a simplicial complex and let $\gamma : C_*(K) \to C_*(\mathbb{R}^d)$ be a chain map from the simplicial chains of K to the singular chains of $\mathbb{R}^d$. We say that γ is *constrained by* $(\mathcal{F}, \Phi)$ if:

(i) Φ is a map from K to $2^{[n]}$ such that $\Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau)$ for all $\sigma, \tau \in K$ and $\Phi(\emptyset) = \emptyset$.

(ii) For any simplex $\sigma \in K$, the support of $\gamma(\sigma)$ is contained in $U_{\overline{\Phi(\sigma)}}$.

See Figure 2. We also say that a chain map γ from K is *constrained by* $\mathcal{F}$ if there exists a map Φ such that γ is constrained by $(\mathcal{F}, \Phi)$. In the above constructions, we simply set Φ to be the identity. As we already saw, constrained chain maps relate Helly numbers to homological almost-embeddings (see Definition 5) via the following observation:

▶ **Lemma 11.** *Let $\gamma : C_*(K) \to C_*(\mathbb{R}^d)$ be a nontrivial chain map constrained by $\mathcal{F}$. If $\bigcap \mathcal{F} = \emptyset$ then γ is a homological almost-embedding of K.*

Proof. Let $\Phi : K \to 2^{[n]}$ be such that γ is constrained by $(\mathcal{F}, \Phi)$. Since γ is nontrivial, it remains to check that disjoint simplices are mapped to chains with disjoint support. Let

σ and τ be two disjoint simplices of K. The supports of $\gamma(\sigma)$ and $\gamma(\tau)$ are contained, respectively, in $U_{\overline{\Phi(\sigma)}}$ and $U_{\overline{\Phi(\tau)}}$, and

$$U_{\overline{\Phi(\sigma)}} \cap U_{\overline{\Phi(\tau)}} = U_{\overline{\Phi(\sigma)\cap\Phi(\tau)}} = U_{\overline{\Phi(\sigma\cap\tau)}} = U_{\overline{\Phi(\emptyset)}} = U_{\overline{\emptyset}} = \bigcap \mathcal{F}.$$

Therefore, if $\bigcap \mathcal{F} = \emptyset$ then γ is a homological almost-embedding of K. ◀

3.3 Relaxing the connectivity assumption

In all the examples listed so far, the intersections $\bigcap \mathcal{G}$ must be connected. Matoušek [19] relaxed this condition into "having a bounded number of connected components", the assumptions then being on the topology of the components, by using Ramsey's theorem. The gist of our proof is to extend his idea to allow a bounded number of homology classes not only in the first dimension but in *any* dimension. Let us illustrate how Matoušek's idea works in dimension two:

▶ **Theorem 12** ([19, Theorem 2 with $d = 2$]). *For every positive integer b there is an integer $h(b)$ with the following property. If $\mathcal{F}$ is a finite family of subsets of $\mathbb{R}^2$ such that the intersection of any subfamily has at most b path-connected components, then the Helly number of $\mathcal{F}$ is at most $h(b)$.*

Let us fix b from above and assume that for any subfamily $\mathcal{G} \subsetneq \mathcal{F}$ the intersection $\bigcap \mathcal{G}$ consists of at most b path-connected components and that $\bigcap \mathcal{F} = \emptyset$. We start, as before, by picking for every $i \in [n]$, a point p_i in $U_{\overline{\{i\}}}$. This is possible as every intersection of $n-1$ members of $\mathcal{F}$ is nonempty. Now, if we consider some pair of indices $i, j \in [n]$, the points p_i and p_j are still in $U_{\overline{\{i,j\}}}$ but may lie in different connected components. It may thus not be possible to connect p_i to p_j *inside* $U_{\overline{\{i,j\}}}$. If we, however, consider $b+1$ indices $i_1, i_2, \ldots, i_{b+1}$ then all the points $p_{i_1}, p_{i_2}, \ldots, p_{i_{b+1}}$ are in $U_{\overline{\{i_1,i_2,\ldots,i_{b+1}\}}}$ which has at most b connected components, so at least one pair among of these points can be connected by a path inside $U_{\overline{\{i_1,i_2,\ldots,i_{b+1}\}}}$. Thus, while we may not get a drawing of the complete graph on n vertices we can still draw many edges.

To find many vertices among which every pair can be connected we will use the hypergraph version of the classical theorem of Ramsey:

▶ **Theorem 13** (Ramsey [23]). *For any x, y and z there is an integer $R_x(y,z)$ such that any x-uniform hypergraph on at least $R_x(y,z)$ vertices colored with at most y colors contains a subset of z vertices inducing a monochromatic sub-hypergraph.*

From the discussion above, for any $b+1$ indices $i_1 < i_2 < \ldots < i_{b+1}$ there exists a pair $\{k,\ell\} \in \binom{[b+1]}{2}$ such that p_{i_k} and p_{i_ℓ} can be connected inside $U_{\overline{\{i_1,i_2,\ldots,i_{b+1}\}}}$. Let us consider the $(b+1)$-uniform hypergraph on $[n]$ and color every set of indices $i_1 < i_2 < \ldots < i_{b+1}$ by one of the pairs in $\binom{[b+1]}{2}$ that can be connected inside $U_{\overline{\{i_1,i_2,\ldots,i_{b+1}\}}}$ (if more than one pair can be connected, we pick one arbitrarily). Let t be some integer to be fixed later. By Ramsey's theorem, if $n \geq R_{b+1}\left(\binom{b+1}{2}, t\right)$ then there exist a pair $\{k,\ell\} \in \binom{[b+1]}{2}$ and a subset $T \subseteq [n]$ of size t with the following property: for any $(b+1)$-element subset $S \subset T$, the points whose indices are the kth and ℓth indices of S can be connected inside $U_{\overline{S}}$.

Now, let us set $t = 5 + \binom{5}{2}(b-1) = 10b-5$. We claim that we can find five indices in T, denoted $i_1, i_2, \ldots, i_5$, and, for each pair $\{i_u, i_v\}$ among these five indices, some $(b+1)$-element subset $Q_{u,v} \subset T$ with the following properties:

(i) i_u and i_v are precisely in the kth and ℓth position in $Q_{u,v}$, and
(ii) for any $1 \leq u, v, u', v' \leq 5$, $\quad Q_{u,v} \cap Q_{u',v'} = \{i_u, i_v\} \cap \{i_{u'}, i_{v'}\}$.

We first conclude the argument, assuming that we can obtain such indices and sets. Observe that from the construction of T, the i_u's and the $Q_{u,v}$'s we have the following property: for any $u, v \in [5]$, we can connect p_{i_u} and p_{i_v} inside $U_{\overline{Q_{u,v}}}$. This gives a drawing of K_5 in the plane. Since K_5 is not planar, there exist two edges with no vertex in common, say $\{u, v\}$ and $\{u', v'\}$, that cross. This intersection point must lie in

$$U_{\overline{Q_{u,v}}} \cap U_{\overline{Q_{u',v'}}} = U_{\overline{Q_{u,v} \cap Q_{u',v'}}} = U_{\overline{\{i_u, i_v\} \cap \{i_{u'}, i_{v'}\}}} = U_{\overline{\emptyset}} = \bigcap \mathcal{F} = \emptyset,$$

a contradiction. Hence the assumption that $n \geq R_{b+1}\left(\binom{b+1}{2}, t\right)$ is false and $\mathcal{F}$ has cardinality at most $R_{b+1}\left(\binom{b+1}{2}, 10b-5\right) - 1$, which is our $h(b)$.

The selection trick. It remains to derive the existence of the i_u's and the $Q_{u,v}$'s. It is perhaps better to demonstrate the method by a simple example to develop some intuition before we formalize it.

Example. Let us fix $b = 4$ and $\{k, \ell\} = \{2, 3\} \in \binom{[4+1]}{2}$. We first make a 'blueprint' for the construction inside the rational numbers. For any two indices $u, v \in [5]$ we form a totally ordered set $Q'_{u,v} \subseteq \mathbb{Q}$ of size $b+1 = 5$ by adding three rational numbers (different from $1, \ldots, 5$) to the set $\{u, v\}$ in such a way that u appears at the second and v at the third position of $Q'_{u,v}$. For example, we can set $Q'_{1,4}$ to be $\{0.5; 1; 4; 4.7; 5.13\}$. Apart from this we require that we add a different set of rational numbers for each $\{u, v\}$. Thus $Q'_{u,v} \cap Q'_{u',v'} = \{u, v\} \cap \{u', v'\}$. Our blueprint now appears inside the set $T' := \bigcup_{1 \leq u < v \leq 5} Q'_{u,v}$; note that both this set T' and the set T in which we search for the sets $Q_{u,v}$ have 35 elements. To obtain the required indices i_u and sets $Q_{u,v}$ it remains to consider the unique strictly increasing bijection $\pi_0 \colon T' \to T$ and set $i_u := \pi_0(u)$ and $Q_{u,v} := \pi_0(Q'_{u,v})$.

The general case. Let us now formalize the generalization of this trick that we will use to prove Theorem 1. Let Q be a subset of $[w]$. If $e_1 < e_2 < \ldots < e_w$ are the elements of a totally ordered set W then we call $\{e_i : i \in Q\}$ the *subset selected by Q in W*.

▶ **Lemma 14.** *Let $1 \leq q \leq w$ be integers and let Q be a subset of $[w]$ of size q. Let Y and Z be two finite totally ordered sets and let $A_1, A_2, \ldots, A_r$ be q-element subsets of Y. If $|Z| \geq |Y| + r(w-q)$, then there exist an injection $\pi : Y \to Z$ and r subsets $W_1, W_2, \ldots, W_r \in \binom{Z}{w}$ such that for every $i \in [r]$, Q selects $\pi(A_i)$ in W_i. We can further require that $W_i \cap W_j = \pi(A_i \cap A_j)$ for any two $i, j \in [r]$, $i \neq j$.*

We refer to [11, Lemma 24] for a proof.

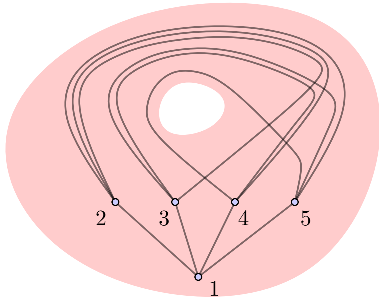
4 Constrained chain maps and Helly number

We now generalize the technique presented in Section 3 to obtain Helly-type theorems from non-embeddability results. We will construct constrained chain maps for arbitrary complexes. As above, $\mathcal{F} = \{U_1, U_2, \ldots, U_n\}$ denotes a family of subsets of $\mathbb{R}^d$ and for $I \subseteq [n]$ we keep the notation $U_{\overline{I}}$ as used in the previous section. Note that although so far we only used the *reduced* Betti numbers $\tilde{\beta}$, in this section it will be convenient to work with *standard* (non-reduced) Betti numbers β, starting with the following proposition.

▶ **Proposition 15.** *For any finite simplicial complex K and non-negative integer b there exists a constant $h_K(b)$ such that the following holds. For any finite family $\mathcal{F}$ of at least $h_K(b)$ subsets of $\mathbb{R}^d$ such that $\bigcap \mathcal{G} \neq \emptyset$ and $\beta_i(\cap \mathcal{G}) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and any $0 \leq i < \dim K$, there exists a nontrivial chain map $\gamma : C_*(K) \to C_*(\mathbb{R}^d)$ that is constrained by $\mathcal{F}$.*

Sketch of proof. We proceed by induction on the dimension k of K. The case $k=0$ is straightforward. The case $k=1$ is still slightly easier than the general case. It is a translation of Matoušek's [19] approach, as summarized in Section 3.3 for the 2-dimensional case, in the language of constrained chain maps.

In the general case, we start with the $(k-1)$-skeleton of a (possibly large) simplicial complex and need to find enough $(k-1)$-cycles that are boundaries to build the k-skeleton of K. The difficulty of finding those boundaries is illustrated by the following observation: for arbitrarily large n, there exist maps from the complete graph on n vertices into an annulus such that no triangle is a boundary (see the figure on the right for an example with $n=5$). We may thus have to examine complicated cycles when searching for boundaries, and the general case therefore requires new ingredients.



Let s denote some large enough integer, depending on the $(k-1)$-skeleton $K^{(k-1)}$. Assuming $h_k(b)$ is large enough, there exists, by the induction assumption, a nontrivial chain map $\gamma' \colon C_*(\Delta_s^{(k-1)}) \to C_*(\mathbb{R}^d)$ constrained by $\mathcal{F}$. One may hope to use Ramsey's theorem to find a copy of $K^{(k-1)}$ inside $\Delta_s^{(k-1)}$ on which γ' can be extended to a constrained chain map on all of K. This turns out to be impossible: while Ramsey's theorem may guarantee that the boundaries of the k-dimensional simplices all have the same homology class, we cannot prevent that common homology class to be non-zero!

We overcome this issue by considering the barycentric subdivision $\operatorname{sd} K$ of K and finding a suitable injection β of $(\operatorname{sd} K)^{(k-1)}$ into $\Delta_s^{(k-1)}$. We then consider the chain maps

$$C_*\left(K^{(k-1)}\right) \xrightarrow{\alpha} C_*\left((\operatorname{sd} K)^{(k-1)}\right) \xrightarrow{\beta_\sharp} C_*\left(\Delta_s^{(k-1)}\right) \xrightarrow{\gamma'} C_*(\mathbb{R}^d)$$

where α is a natural chain map corresponding to the subdivision and $\beta_\sharp$ is the chain map induced by β. We set $\gamma = \gamma' \circ \beta_\sharp \circ \alpha$. We use Ramsey's theorem to set β in such a way that the boundaries of the k-dimensional simplices all have the same homology class under γ. Again, Ramsey's theorem can only ensure that through $\gamma' \circ \beta_\sharp$ the boundaries of the k-simplices of $\operatorname{sd} K$ have the same, possibly non-trivial, homology class. But since every k-simplex of K is the sum of an *even* number of simplices of $\operatorname{sd} K$, and we compute homology over $\mathbb{Z}_2$, this is good enough, and γ can be extended to K.

This outline brushes under the rug some technical difficulties raised by the use of barycentric subdivision; we refer the interested reader to [11, Proposition 25] for full details. ◀

The case $K = \Delta_{2k+2}^{(k)}$, with $k = \lceil d/2 \rceil$, of Proposition 15 finally implies Theorem 1.

Proof of Theorem 1. Let b and d be fixed integers, let $k = \lceil d/2 \rceil$ and let $K = \Delta_{2k+2}^{(k)}$. Let $h_K(b+1)$ denote the constant from Proposition 15 (we plug in $b+1$ because we need to switch between reduced and non-reduced Betti numbers). Let $\mathcal{F}$ be a finite family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i\left(\bigcap \mathcal{G}\right) \le b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \le i \le \dim K = \lceil d/2 \rceil - 1$, in particular $\beta_i\left(\bigcap \mathcal{G}\right) \le b+1$ for such $\mathcal{G}$. Let $\mathcal{F}^*$ denote an inclusion-minimal sub-family of $\mathcal{F}$ with empty intersection: $\bigcap \mathcal{F}^* = \emptyset$ and $\bigcap(\mathcal{F}^* \setminus \{U\}) \neq \emptyset$ for any $U \in \mathcal{F}^*$. If $\mathcal{F}^*$ has size at least $h_K(b+1)$, it satisfies the assumptions of Proposition 15 and there exists a nontrivial chain map from K that is constrained by $\mathcal{F}^*$. Since $\mathcal{F}^*$ has empty intersection, this chain map is a homological almost-embedding by Lemma 11. However, no such homological almost-embedding exists by Corollary 7, so $\mathcal{F}^*$ must have size at most $h_K(b+1)-1$. As a consequence, the Helly number of $\mathcal{F}$ is bounded and the statement of Theorem 1 holds with $h(b,d) = h_K(b+1)-1$. ◀

Acknowledgments. We would like to express our immense gratitude to Jiří Matoušek, not only for raising the problem addressed in the present paper and valuable discussions about it, but, much more generally, for the privilege of having known him, as our teacher, mentor, collaborator, and friend. Through his tremendous depth and insight, and the generosity with which he shared them, he greatly influenced all of us.

We further thank Jürgen Eckhoff for helpful comments on a preliminary version of the paper, and Andreas Holmsen and Gil Kalai for providing us with useful references.

References

1 N. Amenta. Helly-type theorems and generalized linear programming. *Discrete & Computational Geometry*, 12:241–261, 1994.
2 N. Amenta. A short proof of an interesting Helly-type theorem. *Discrete & Computational Geometry*, 15:423–427, 1996.
3 E. G. Bajmóczy and I. Bárány. On a common generalization of Borsuk's and Radon's theorem. *Acta Math. Acad. Sci. Hungar.*, 34(3-4):347–350, 1979.
4 M. Bestvina, M. Kapovich, and B. Kleiner. Van Kampen's embedding obstruction for discrete groups. *Invent. Math.*, 150(2):219–235, 2002.
5 A. Björner. Nerves, fibers and homotopy groups. *Journal of Combinatorial Theory, Series A*, 102(1):88–93, 2003.
6 K. Borsuk. On the imbedding of systems of compacta in simplicial complexes. *Fundamenta Mathematicae*, 35:217–234, 1948.
7 É. Colin de Verdière, G. Ginot, and X. Goaoc. Multinerves and Helly numbers of acyclic families. In *Proceedings of the 2012 symposium on Computational Geometry*, SoCG'12, pages 209–218, 2012. `http://arxiv.org/abs/1101.6006`.
8 H. Debrunner. Helly type theorems derived from basic singular homology. *American Mathematical Monthly*, 77:375–380, 1970.
9 J. Eckhoff. Helly, Radon and Carathéodory type theorems. In P.M. Gruber and J.M. Wills, editors, *Handbook of Convex Geometry*, pages 389–448. North Holland, 1993.
10 A. I. Flores. Über die Existenz n-dimensionaler Komplexe, die nicht in den $\mathbb{R}^{2n}$ topologisch einbettbar sind. *Ergeb. Math. Kolloqu.*, 5:17–24, 1933.
11 X. Goaoc, P. Paták, Z. Patáková, M. Tancer, and U. Wagner. Bounding Helly numbers via Betti numbers. `http://arxiv.org/abs/1310.4613`.
12 A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, UK, 2002.
13 E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresbericht Deutsch. Math. Verein.*, 32:175–176, 1923.
14 E. Helly. Über Systeme von abgeschlossenen Mengen mit gemeinschaftlichen Punkten. *Monaths. Math. und Physik*, 37:281–302, 1930.
15 G. Kalai. Combinatorial expectations from commutative algebra. In I. Peeva and V. Welker, editors, *Combinatorial Commutative Algebra*, volume 1(3), pages 1729–1734. Oberwolfach Reports, 2004.
16 G. Kalai and R. Meshulam. Leray numbers of projections and a topological Helly-type theorem. *Journal of Topology*, 1:551–556, 2008.

17 H. Maehara. Helly-type theorems for spheres. *Discrete & Computational Geometry*, 4(1):279–285, 1989.
18 J. Matoušek. *Using the Borsuk-Ulam Theorem.* Springer-Verlag, Berlin, 2003.
19 J. Matoušek. A Helly-type theorem for unions of convex sets. *Discrete & Computational Geometry*, 18:1–12, 1997.
20 S. A. Melikhov. The van Kampen obstruction and its relatives. *Proc. Steklov Inst. Math.*, 266(1):142–176, 2009.
21 L. Montejano. A new topological Helly theorem and some transversal results. *Discrete & Computational Geometry*, 52(2):390–398, 2014.
22 J. R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, Menlo Park, CA, 1984.
23 F. P. Ramsey. On a problem in formal logic. *Proc. London Math. Soc.*, 30:264—-286, 1929.
24 A. Shapiro. Obstructions to the imbedding of a complex in a euclidean space. I. The first obstruction. *Ann. of Math. (2)*, 66:256–269, 1957.
25 M. Sharir and E. Welzl. A combinatorial bound for linear programming and related problems. In *Proc. 9th Sympos. on Theo. Aspects of Comp. Science*, pages 569–579, 1992.
26 R. I. Soare. Computability theory and differential geometry. *Bull. Symbolic Logic*, 10(4):457–486, 2004.
27 M. Tancer. Intersection patterns of convex sets via simplicial complexes: A survey. In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 521–540. Springer New York, 2013.
28 E. R. van Kampen. Komplexe in euklidischen Räumen. *Abh. Math. Sem. Univ. Hamburg*, 9:72–78, 1932.
29 U. Wagner. Minors in random and expanding hypergraphs. In *Proceedings of the 27th Annual Symposium on Computational Geometry (SoCG)*, pages 351—360, 2011.
30 R. Wenger. Helly-type theorems and geometric transversals. In Jacob E. Goodman and Joseph O'Rourke, editors, *Handbook of Discrete & Computational Geometry*, chapter 4, pages 73–96. CRC Press LLC, Boca Raton, FL, 2nd edition, 2004.
31 W.-T. Wu. On the realization of complexes in euclidean spaces. I, II, III. *Acta Math. Sinica (English transl. of I and III in Sci. Sinica)*, 5:505–552, 1955.